Chapter 2

Generation of analytic semigroups by elliptic operators

2.1 Assumptions and formulation of the boundary value problem

In this chapter Ω will denote either $\mathbf{R}^n$ or an open subset of $\mathbf{R}^n$ ($n \geq 2$) with sufficiently smooth boundary $\partial\Omega$. For any $x \in \partial\Omega$ we denote by $\nu(x)$ the exterior unit normal vector to $\partial\Omega$ at $x \in \partial\Omega$.
We shall consider the linear second order differential operator $\mathcal{A}(x, D)$ with real coefficients operating on complex valued functions $u(x)$ defined in the domain Ω

$$\begin{aligned} \mathcal{A}(x, D) &= \sum_{i,j=1}^{n} D_i(a_{ij}(x)D_j) + \sum_{i=1}^{n} b_i(x)D_i + c(x) \\ &= \operatorname{div}(A \cdot D) + B \cdot D + c. \end{aligned} \tag{2.1}$$

The leading part of $\mathcal{A}(x, D)$ is denoted by $\mathcal{A}^0(x, D)$:

$$\mathcal{A}^0(x, D) = \sum_{i,j} a_{ij}(x)D_iD_j.$$

In what follows we assume the following conditions.

SMOOTHNESS CONDITION ON Ω: Ω is uniformly regular of class C^2. (2.2)

SMOOTHNESS CONDITION ON $\mathcal{A}$:

$$a_{ij} = a_{ji} \in C_b^1(\overline{\Omega}) \quad \text{and} \quad b_i, c \in L^\infty(\Omega). \tag{2.3}$$

ELLIPTICITY CONDITION ON Ω: $\mathcal{A}$ is uniformly μ-elliptic in Ω, i.e., there exists a constant $\mu \geq 1$ such that for any $x \in \overline{\Omega}$ and $\xi \in \mathbf{R}^n$

$$\mu^{-1}|\xi|^2 \leq \mathcal{A}^0(x,\xi) \leq \mu|\xi|^2, \tag{2.4}$$

Moreover if $\Omega \neq \mathbf{R}^n$, we consider some boundary conditions. These conditions are expressed by a linear first order differential operator with real coefficients defined for $x \in \partial\Omega$:

$$\mathcal{B}(x,D) = \sum_{i=1}^{n} \beta_i(x) D_i + \gamma(x) \tag{2.5}$$

We assume the following.
SMOOTHNESS CONDITION ON $\mathcal{B}$:

$$\beta_i, \gamma \in UC_b^1(\overline{\Omega}), \tag{2.6}$$

i.e., β, γ are differentiable on $\partial\Omega$ and the derivatives are all bounded and uniformly continuous on $\partial\Omega$ and the uniform nontangentiality condition

$$\inf_{x\in\partial\Omega} \left| \sum_{i=1}^{n} \beta_i(x)\nu_i(x) \right| > 0 \tag{2.7}$$

holds.

In the sequel the Agmon-Douglis-Nirenberg a priori estimates will be very useful. They hold for operators with complex valued coefficients for which (2.3) holds and uniform ellipticity consists in requiring that there exists a constant $\mu \geq 1$ such that for any $x \in \overline{\Omega}$ and $\xi \in \mathbf{R}^n$

$$\mu^{-1}|\xi|^2 \leq |\mathcal{A}^0(x,\xi)| \leq \mu|\xi|^2, \tag{2.8}$$

Due to the ellipticity of $\mathcal{A}$, (2.8), we get that for every real vector $\xi = (\xi_1, \ldots, \xi_n) \neq 0$ and for every point $x \in \overline{\Omega}$ there holds $\mathcal{A}^0(x,\xi) \neq 0$. Hence in particular for every linearly independent real vectors ξ and η, the polynomial $\mathcal{A}^0(x, \xi + \tau\eta)$ of the variable τ has no real roots. We assume the following.
ROOT CONDITION: For every pair of linearly independent real vectors ξ, η the polynomial $\mathcal{A}^0(x, \xi + \tau\eta)$ of the variable τ has a unique root τ_1^+ with positive imaginary part.

It is easy to verify that if $n \geq 3$ all elliptic operators satisfy the Root Condition. Indeed in the case $\xi \perp \eta$, if we take for simplicity $\eta = e_n$, then $\mathcal{A}^0(x, \xi + \tau\eta) = \mathcal{A}^0(x, \xi', \tau\eta)$ with $\xi' = (\xi_1, \ldots, \xi_{n-1})$, $\xi' \neq 0$. We define the constant functions $f_\eta, g_\eta : \mathbf{R}^{n-1} \setminus \{0\} \to \mathbf{N}$ as follows

$$f_\eta(\xi') = \#\{\tau \in \mathbf{C} : \ \mathcal{A}^0(x, \xi + \tau\eta) = 0, \, \mathrm{Im}\,\tau > 0\}$$
$$g_\eta(\xi') = \#\{\tau \in \mathbf{C} : \ \mathcal{A}^0(x, \xi + \tau\eta) = 0), \, \mathrm{Im}\,\tau < 0\},$$

and we observe that since if τ is a root for ξ, η then $-\tau$ is a root for $-\xi$, $-\eta$ we deduce $f_\eta(\xi') = g_\eta(-\xi')$. Moreover, if $n \geq 3$ then $g_\eta(-\xi') = g_\eta(\xi')$. In fact, the points ξ' and $-\xi'$ can be joined by a smooth simple curve γ in $\mathbf{R}^{n-1} \setminus \{0\}$ (which is a connected set) and the roots of the polynomial $\tau \mapsto \mathcal{A}^0(x, \gamma(\cdot) + \tau\eta)$ are continuous functions along γ. If

g_η were not constant along γ, the imaginary part of some roots would change sign, hence it would vanish and give a real root, which is impossible. Therefore, $f_\eta(\xi')$, $g_\eta(\xi')$ and $g_\eta(-\xi')$ coincide everywhere on $\mathbf{R}^{n-1}\setminus\{0\}$ if $n \geq 3$. The general case can be recovered by the previous one. Indeed let $\xi, \eta \in \mathbf{R}^n \setminus \{0\}$ with ξ and η linearly independent; we can write $\xi = \xi' + \xi''\hat{\eta}$ with $\hat{\eta} = \frac{\eta}{|\eta|}$, $\xi' \neq 0$ and $\xi' \perp \hat{\eta}$, then $\mathcal{A}^0(\xi + \tau\eta) = \mathcal{A}^0(\xi' + \tau'\hat{\eta})$ with $\tau' = \xi'' + \tau|\eta|$ and $\xi' \perp \hat{\eta}$. Finally we observe that $f_\eta(\xi') = f_{\hat{\eta}}(\xi')$ and $g_\eta(\xi') = g_{\hat{\eta}}(\xi')$; thus repeating the argument above we conclude for two arbitrary linearly independent vectors ξ, η.
Moreover, we require that the boundary conditions are expressed as before by (2.5) with complex coefficients

$$\beta_i, \gamma \in UC_b^1(\overline{\Omega}; \mathbf{C}) \tag{2.9}$$

that must "complement" the differential equation. This condition called *complementing boundary condition* consists of an algebraic criterion involving the leading parts of $\mathcal{A}$ and $\mathcal{B}$.

COMPLEMENTING CONDITION (2.10)

Let x be an arbitrary point on $\partial\Omega$ and ν be the outward normal unit vector to $\partial\Omega$ at x. For each vector $\xi \neq 0$ tangential to $\partial\Omega$ at x, let τ_1^+ be the root of the polynomial $\mathcal{A}^0(x, \xi+\tau\nu)$ with positive imaginary part. Then the polynomial $\mathcal{B}^0(x, \xi + \tau\nu) = \langle \beta(x), \xi + \tau\nu \rangle$ has to be linearly independent modulo the polynomial $(\tau - \tau_1^+)$. This means that τ_1^+ cannot be solution of $\mathcal{B}^0(\xi + \tau\nu) = 0$ and it is obviously satisfied if (2.7) holds.

We notice that if the coefficients of $\mathcal{A}$ are real and satisfy

$$\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq \mu|\xi|^2 \qquad x \in \bar{\Omega},\ \xi \in \mathbf{R}^n$$

for some $\mu > 0$, then the Root Condition is immediately satisfied. Indeed in that case the polynomial in τ, $\mathcal{A}^0(\xi + \tau\nu)$ has not real roots, therefore it has two conjugate complex solutions.

Remark 2.1.1. The reason why we have considered complex valued coefficients and introduced assumption (2.8) is the fact that we shall use the Agmon-Douglis-Nirenberg estimates (2.13) and (2.14) with $\mathcal{A}$ replaced by the operator $\mathcal{A} + e^{i\theta}D_{tt}$ in $n+1$ variables (x, t), with $\theta \in [-\pi/2, \pi/2]$, which satisfies (2.8) and the Root Condition too.

2.2 Basic estimates for elliptic equations

The aim of this chapter is to prove that under the assumptions listed in Section 2.1, the realizations of $\mathcal{A}$ with homogeneous boundary conditions $\mathcal{B}u = 0$ in $\partial\Omega$, are sectorial operators in suitable Banach spaces. As a result they generate analytic semigroups in those spaces (see Proposition 1.2.3).

A sufficient condition for the sectoriality of an operator is given in Proposition 1.2.7. Here we first need some existence and uniqueness results for elliptic boundary value problems of the type

$$\begin{cases} \lambda u - \mathcal{A}(\cdot, D)u = f & \text{in } \Omega \\ \mathcal{B}(\cdot, D)u = 0 & \text{in } \partial\Omega \end{cases} \tag{2.11}$$

and then some resolvent estimate like (1.8).
Now we recall the a priori estimates due to Agmon, Douglis and Nirenberg that hold for operators with complex coefficients satisfying hypothesis of Section 2.1 in $\mathbf{R}^n$ as well as in regular domains. For a complete analysis of these estimates we refer to [2] and [3]. We recall them in the following theorem in a way that will be used later. We set

$$M = \max\{\|a_{ij}\|_{1,\infty}, \|b_i\|_\infty, \|c\|_\infty\}. \tag{2.12}$$

Theorem 2.2.1. *(Agmon-Douglis-Nirenberg)*

(i) Let $\mathcal{A}(x, D)$ be defined as in (2.1). Suppose that $a_{ij}, b_i, c : \mathbf{R}^n \to \mathbf{C}$ satisfy hypotheses (2.3), (2.8) and the Root Condition. Then for every $p \in (1, +\infty)$ there exists a strictly positive constant C depending only on p, n, μ and M such that for every $u \in W^{2,p}(\mathbf{R}^n)$

$$\|u\|_{W^{2,p}(\mathbf{R}^n)} \le C \left(\|u\|_{L^p(\mathbf{R}^n)} + \|\mathcal{A}(\cdot, D)u\|_{L^p(\mathbf{R}^n)} \right). \tag{2.13}$$

(ii) Let Ω be an open set in $\mathbf{R}^n$ with uniformly C^2 boundary, and $\mathcal{A}(x, D)$ defined by (2.1). Suppose that $a_{ij}, b_i, c : \Omega \to \mathbf{C}$ satisfy hypotheses (2.3), (2.8) and the Root Condition. Let in addition β_i, γ satisfy (2.9) and the complementing condition. For every $u \in W^{2,p}(\Omega)$, with $1 < p < \infty$, set $f = \mathcal{A}(\cdot, D)u$, $g = \mathcal{B}(\cdot, D)u_{|\partial\Omega}$. Then there is $C_1 = C_1(p, n, \mu, M, \Omega) > 0$ such that

$$\|u\|_{W^{2,p}(\Omega)} \le C_1 \left(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} + \|g_1\|_{W^{1,p}(\Omega)} \right). \tag{2.14}$$

where g_1 is any $W^{1,p}$ extension of g to Ω.

Observe that the estimates in Theorem 2.2.1 are not true for $p = 1$ and $p = \infty$. For this reason the theory of $L^p(\Omega)$, $1 < p < \infty$ cannot be rearranged to the cases L^1 or L^∞. For $p = \infty$ this difficulty has been overcome by K. Masuda and H.B. Stewart (see [42], [43]) using the classical L^p theory and by passing to the limit in the L^p estimates in a suitable way.
One of the ways to solve the case $p = 1$ consists in using duality from L^∞.
This chapter is organized as follows: in Section 2.2.1 we discuss the generation in L^p, $1 < p < \infty$ for an elliptic operator of second order with homogeneous non tangential boundary conditions. Using these results we study the same problem in $L^\infty(\Omega)$. Finally in Section 2.5 we confine our attention to a particular boundary operator and we prove sectoriality for the realization in L^1 of the operator $\mathcal{A}$ with the homogeneous boundary condition there specified.

2.2.1 Analytic semigroups in $L^p(\mathbf{R}^n)$, $1 < p < \infty$

First suppose $\Omega = \mathbf{R}^n$ and consider the realization of $\mathcal{A}$ in $L^p(\mathbf{R}^n)$. Define

$$D(A_p) = W^{2,p}(\mathbf{R}^n), \quad A_p u = \mathcal{A}(\cdot, D)u, \quad u \in D(A_p), \tag{2.15}$$

We start by the simplest case when $a_{ij} = \delta_{ij}$ $b_i, c = 0$. In this way the operator in (2.1) reduces to the Laplace operator:

$$\Delta = \sum_{i=1}^{n} D_{ii}.$$

By (i) of the Theorem 2.2.1, it follows that the operator Δ with domain $W^{2,p}(\mathbf{R}^n)$ is closed.

Theorem 2.2.2. *Let $1 < p < \infty$ and consider the operator Δ with domain given by $W^{2,p}(\mathbf{R}^n)$. Then, there exist $\frac{\pi}{2} < \vartheta_0 < \pi$ and $M_\vartheta > 0$ depending on p such that $\rho(\Delta) \supset \Sigma_\vartheta = \{\lambda \in \mathbf{C};\ \lambda \neq 0,\ |\arg\lambda| < \vartheta\}$ and the estimate*

$$\|(\lambda - \Delta)^{-1}\|_{\mathcal{L}(L^p(\mathbf{R}^n))} \leq \frac{M_\vartheta}{|\lambda|} \tag{2.16}$$

holds for $\lambda \in \Sigma_\vartheta$ for any $\vartheta < \vartheta_0$.

Proof. First we consider the case $p \geq 2$. For $u \in C_0^\infty(\mathbf{R}^n)$, we put $u^* := \bar{u}|u|^{p-2}$ where $\bar{u}$ denotes the complex conjugate of u. Since the function $f(z) = \bar{z}|z|^{p-2}$ is continuously differentiable, $u^* \in C_0^1(\mathbf{R}^n)$. By the chain rule we obtain

$$D_h u^* = |u|^{p-2} D_h \bar{u} + (p-2)|u|^{p-4}\bar{u}\mathrm{Re}\,(\bar{u}D_h u).$$

Integration by parts yields

$$\begin{aligned}
-\int_{\mathbf{R}^n} \Delta u \cdot u^* &= -\int_{\mathbf{R}^n} \sum_{h=1}^{n} (D_{hh}u)\bar{u}|u|^{p-2} \\
&= \int_{\mathbf{R}^n} \sum_{h=1}^{n} D_h u D_h(\bar{u}|u|^{p-2}) \\
&= \int_{\mathbf{R}^n} \sum_{h=1}^{n} (|u|^{p-2} D_h u D_h \bar{u} \\
&\quad + (p-2)|u|^{p-4}\bar{u}D_h u \mathrm{Re}\,(\bar{u}D_h u)).
\end{aligned}$$

Since

$$\mathrm{Re}\,(|u|^2 D_h u D_h \bar{u}) = (\mathrm{Re}\,(\bar{u}D_h u))^2 + (\mathrm{Im}\,(\bar{u}D_h u))^2,$$

then

$$\begin{aligned}
-\mathrm{Re} \int_{\mathbf{R}^n} \Delta u \cdot u^* = (p-1) \int_{\mathbf{R}^n} |u|^{p-4} \sum_{h=1}^{n} (\mathrm{Re}\,(\bar{u}D_h u))^2 \\
+ \int_{\mathbf{R}^n} |u|^{p-4} \sum_{h=1}^{n} \mathrm{Im}\,(\bar{u}D_h u))^2 =: (p-1)A^2 + B^2 \geq 0
\end{aligned} \tag{2.17}$$

and

$$-\text{Im}\int_{\mathbf{R}^n}\Delta u\cdot u^*=(p-2)\int_{\mathbf{R}^n}|u|^{p-4}\sum_{h=1}^{n}\text{Im}\,(\bar{u}D_hu)\text{Re}\,(\bar{u}D_hu).$$

Now, using the Cauchy- Schwartz inequality we obtain

$$\int_{\mathbf{R}^n}|u|^{p-4}\Big|\sum_{h=1}^{n}\text{Im}\,(\bar{u}D_hu)\text{Re}\,(\bar{u}D_hu)\Big|\leq$$
$$\int_{\mathbf{R}^n}|u|^{\frac{p-4}{2}}\Big|\text{Re}\,(\bar{u}Du))\Big|\;|u|^{\frac{p-4}{2}}\Big|\text{Im}\,(\bar{u}Du))\Big|\leq$$
$$\Big(\int_{\mathbf{R}^n}|u|^{p-4}\sum_{h=1}^{n}(\text{Re}\,(\bar{u}D_hu)^2\Big)^{\frac{1}{2}}\Big(\int_{\mathbf{R}^n}|u|^{p-4}\sum_{h=1}^{n}(\text{Im}\,(\bar{u}D_hu)^2\Big)^{\frac{1}{2}}=AB$$

and so

$$\Big|\text{Im}\int_{\mathbf{R}^n}\Delta u\cdot u^*\Big|\leq|p-2|AB. \tag{2.18}$$

If $1<p<2$, we get the same estimates (2.17) and (2.18) by approximation, using the functions $u^*=\bar{u}(|u|^2+\delta)^{\frac{p-2}{2}}$ and letting $\delta\to 0$.
Now we look for the smallest positive γ_0 such that

$$|p-2|AB\leq\gamma_0[(p-1)A^2+B^2]$$

for all A,B. Since for such γ_0 we have that

$$\gamma_0(p-1)\frac{A^2}{B^2}-|p-2|\frac{A}{B}+\gamma_0\geq 0$$

for all A,B, then $(p-2)^2-4(p-1)\gamma_0^2\leq 0$ and so

$$\gamma_0\geq\frac{|p-2|}{2\sqrt{p-1}}.$$

Setting $\displaystyle\int_{\mathbf{R}^n}\Delta u\cdot u^*\,dx=:x+iy$, we have obtained

$$\begin{cases} x\leq 0\\ |y|\leq\gamma|x|\end{cases} \tag{2.19}$$

for $\gamma\geq\gamma_0(p)$. Define $\vartheta_0=\pi-\arctan\gamma_0$, $\vartheta<\vartheta_0$ and prove that $\rho(\Delta)\supset\Sigma_\vartheta$.
Let $\vartheta<\vartheta_0$ and consider $\lambda\in\Sigma_\vartheta$ and $u\in C_0^\infty(\mathbf{R}^n)$, with $\|u\|_{L^p(\mathbf{R}^n)}=1$, so that $\|u^*\|_{L^{p'}(\mathbf{R}^n)}=1=\langle u,u^*\rangle_{L^p,L^{p'}}$. Then, by (2.19) we get $\langle\Delta u,u^*\rangle_{L^p,L^{p'}}\in\mathbf{C}\setminus\Sigma_{\vartheta_0}$, hence

$$\begin{aligned}\|\lambda u-\Delta u\|_{L^p(\mathbf{R}^n)}&\geq|\langle\lambda u-\Delta u,u^*\rangle_{L^p,L^{p'}}|=|\lambda-\langle\Delta u,u^*\rangle_{L^p,L^{p'}}|\\&\geq\text{dist}(\lambda,\mathbf{C}\setminus\Sigma_{\vartheta_0})\geq C_\vartheta|\lambda|.\end{aligned}$$

By density, we deduce

$$\|\lambda u-\Delta u\|_{L^p(\mathbf{R}^n)}\geq C_\vartheta|\lambda|\|u\|_{L^p(\mathbf{R}^n)} \tag{2.20}$$

for all $u \in W^{2,p}(\mathbf{R}^n)$. Now, using the Fourier transform we prove that $\lambda \in \rho(\Delta)$. The injectivity of $\lambda - \Delta$ follows from (2.20). By (2.13) and using inequality (2.20) we have

$$\begin{aligned}\|u\|_{W^{2,p}(\mathbf{R}^n)} &\leq c(\|u\|_{L^p(\mathbf{R}^n)} + \|\Delta u\|_{L^p(\mathbf{R}^n)}) \\ &\leq c(\|u\|_{L^p(\mathbf{R}^n)} + |\lambda|\|u\|_{L^p(\mathbf{R}^n)} + \|\lambda u - \Delta u\|_{L^p(\mathbf{R}^n)}) \\ &= c((1+|\lambda|)\|u\|_{L^p(\mathbf{R}^n)} + \|\lambda u - \Delta u\|_{L^p(\mathbf{R}^n)}) \\ &\leq C\|\lambda u - \Delta u\|_{L^p(\mathbf{R}^n)} \end{aligned} \tag{2.21}$$

where the constant C depends on p, ϑ, λ. Now, inequality (2.21) and the closedness of Δ in $W^{2,p}(\mathbf{R}^n)$ imply that $(\lambda - \Delta)(W^{2,p}(\mathbf{R}^n))$ is closed in $L^p(\mathbf{R}^n)$. We have only to prove that $(\lambda - \Delta)(W^{2,p}(\mathbf{R}^n))$ is dense in $L^p(\mathbf{R}^n)$.
Consider the space $\mathcal{S}(\mathbf{R}^n)$ which is dense in $L^p(\mathbf{R}^n)$ and prove that

$$\forall f \in \mathcal{S}(\mathbf{R}^n) \ \exists u \in W^{2,p}(\mathbf{R}^n) \text{ such that } (\lambda - \Delta)u = f$$

Now, the solution in $W^{2,p}(\mathbf{R}^n)$ of $\lambda u - \Delta u = f$ is the function $u \in \mathcal{S}(\mathbf{R}^n)$ whose Fourier transform is

$$\hat{u} = \frac{\hat{f}}{\lambda + |\xi|^2}$$

This shows that

$$(\lambda - \Delta)(W^{2,p}(\mathbf{R}^n)) \supseteq \mathcal{S}(\mathbf{R}^n)$$

hence it is dense in $L^p(\mathbf{R}^n)$. □

The previous theorem implies that the realization of Δ in $L^p(\mathbf{R}^n)$ is a sectorial operator.

Corollary 2.2.3. *Let $1 < p < \infty$ and $\lambda \in \mathbf{C}$ with $\operatorname{Re} \lambda > 0$. Then for every $f \in L^p(\mathbf{R}^n)$ there exists a unique $u \in W^{2,p}(\mathbf{R}^n)$ such that*

$$(\lambda - \Delta)u = f. \tag{2.22}$$

Moreover

$$|\lambda|\|u\|_{L^p(\mathbf{R}^n)} + |\lambda|^{\frac{1}{2}}\|Du\|_{L^p(\mathbf{R}^n)} + \|D^2u\|_{L^p(\mathbf{R}^n)} \leq c\|f\|_{L^p(\mathbf{R}^n)} \tag{2.23}$$

where c depends on n, p.

Proof. The result can be easily obtained from the previous one. By the estimate (2.20) and (2.21) we deduce

$$|\lambda|\|u\|_{L^p(\mathbf{R}^n)} \leq C_\theta^{-1}\|f\|_{L^p(\mathbf{R}^n)} \tag{2.24}$$

$$\|D^2u\|_{L^p(\mathbf{R}^n)} \leq C\|f\|_{L^p(\mathbf{R}^n)} \tag{2.25}$$

and finally using the interpolation estimate (A.1)

$$\|\nabla u\|_{L^p(\mathbf{R}^n)} \leq c\|D^2u\|^{\frac{1}{2}}_{L^p(\mathbf{R}^n)}\|u\|^{\frac{1}{2}}_{L^p(\mathbf{R}^n)} \leq C|\lambda|^{-1/2}\|f\|_{L^p(\mathbf{R}^n)}. \tag{2.26}$$

Summing up (2.24),(2.26), (2.25) and redefining the constant we get the claim. □
Actually for what concerns the existence and the uniqueness of the solution of (2.22) in $\mathbf{R}^n$ we state the following theorem (see for example [44] for details).

Theorem 2.2.4. *Let $f \in L^p(\mathbf{R}^n)$, then for every $\lambda \notin (-\infty, 0]$ there exists $u \in W^{2,p}(\mathbf{R}^n)$ such that $\lambda u - \Delta u = f$ and the estimate*

$$\|u\|_{W^{2,p}(\mathbf{R}^n)} \le c(n,\lambda)\|f\|_{L^p(\mathbf{R}^n)}$$

holds.

In the following proposition we extend (2.23) to a more general operator than the Laplacian.

Proposition 2.2.5. *Let $1 < p < \infty$. Then, there exist $\omega_0 \in \mathbf{R}$, $M_p > 0$ depending on n, p, μ, M such that if $\operatorname{Re}\lambda \ge \omega_0$, then for every $u \in W^{2,p}(\mathbf{R}^n)$ we have*

$$|\lambda|\|u\|_{L^p(\mathbf{R}^n)} + |\lambda|^{\frac{1}{2}}\|Du\|_{L^p(\mathbf{R}^n)} + \|D^2u\|_{L^p(\mathbf{R}^n)} \le M_p\|\lambda u - \mathcal{A}(\cdot, D)u\|_{L^p(\mathbf{R}^n)} \tag{2.27}$$

Proof. Let $\mathcal{E}$ the operator in $n+1$ variables defined by

$$\mathcal{E}(x,t,D) = \mathcal{A}(x,D) + e^{i\theta}D_{tt} \tag{2.28}$$

with $\theta \in [-\pi/2, \pi/2]$. It satisfies the uniform ellipticity condition (2.8) with constant $\mu_{\mathcal{E}} = \mu\sqrt{2}$. Indeed, it is obvious that $|\mathcal{A}^0(x,\xi) + e^{i\theta}\eta^2| \le \mu(|\xi|^2 + \eta^2) \le \mu\sqrt{2}(|\xi|^2 + \eta^2)$; for the converse inequality, we look for $\mu_{\mathcal{E}} > 1$ such that

$$|\mathcal{A}^0(x,\xi) + e^{i\theta}\eta^2| \ge \mu_{\mathcal{E}}^{-1}(|\xi|^2 + \eta^2) \tag{2.29}$$

for all $x \in \bar{\Omega}$, $(\xi,\eta) \in \mathbf{R}^n \times \mathbf{R}$ and for every $\theta \in [-\pi/2, \pi/2]$. We observe that

$$\begin{aligned} |\mathcal{A}^0(x,\xi) + e^{i\theta}\eta^2| &= \left[(\langle A\xi,\xi\rangle + \eta^2\cos\theta)^2 + \eta^4\sin^2\theta\right]^{1/2} \\ &= \left[(\langle A\xi,\xi\rangle)^2 + \eta^4 + 2\eta^2\langle A\xi,\xi\rangle\cos\theta\right]^{1/2} \\ &\ge (\frac{1}{\mu^2}|\xi|^4 + \eta^4)^{1/2} \end{aligned}$$

Since we look for a $\mu_{\mathcal{E}}$ such that (2.29) holds, if

$$(\frac{1}{\mu^2}|\xi|^4 + \eta^4)^{1/2} \ge \mu_{\mathcal{E}}^{-1}(|\xi|^2 + \eta^2)$$

or equivalently using that $2|\xi|^2\eta^2 \le |\xi|^4 + \eta^4$

$$\frac{2}{\mu_{\mathcal{E}}^2}(|\xi|^4 + \eta^4) \le \frac{1}{\mu^2}|\xi|^4 + \eta^4 \tag{2.30}$$

holds for all $(\xi,\eta) \in \mathbf{R}^n \times \mathbf{R}$ we conclude. Now, it is easy to see that if $\mu_{\mathcal{E}}$ satisfies

$$\begin{cases} \dfrac{2}{\mu_{\mathcal{E}}^2} - \dfrac{1}{\mu^2} \le 0 \\ \dfrac{2}{\mu_{\mathcal{E}}^2} - 1 \le 0 \end{cases}$$

that is if $\mu_{\mathcal{E}} \geq \mu\sqrt{2}$, then (2.30) is proved.
Let $\eta \in C_c^\infty(\mathbf{R})$ be such that $\eta \equiv 1$ in $[-\frac{1}{2}, \frac{1}{2}]$ and $\operatorname{supp}\eta \subseteq [-1,1]$. For every $u \in W^{2,p}(\mathbf{R}^n)$ and $r > 0$ we set

$$v(t,x) = \eta(t)e^{irt}u(x) \qquad t \in \mathbf{R},\, x \in \mathbf{R}^n. \tag{2.31}$$

Then

$$\mathcal{E}\, v = \eta(t)e^{irt}(\mathcal{A} - e^{i\theta}r^2)u + e^{i(\theta+rt)}(\eta'' + 2ir\eta')u.$$

Now, we can prove (2.27). Estimate (2.13), applied to the function v implies that there exists $C = C(n,p,\mu,M)$ such that

$$\begin{aligned}
\|v\|_{W^{2,p}(\mathbf{R}^{n+1})} &\leq C\left[\|v\|_{L^p(\mathbf{R}^{n+1})} + \|\mathcal{E}v\|_{L^p(\mathbf{R}^{n+1})}\right] \\
&\leq C\big[\|u\|_{L^p(\mathbf{R}^n)} \\
&\quad + \|\eta e^{irt}(\mathcal{A} - e^{i\theta}r^2)u + e^{i(\theta+rt)}(\eta'' + 2ir\eta')u\|_{L^p(\mathbf{R}^{n+1})}\big] \\
&\leq C\left[\|u\|_{L^p(\mathbf{R}^n)} + \|(\mathcal{A} - e^{i\theta}r^2)u\|_{L^p(\mathbf{R}^n)} + (1+2r)\|u\|_{L^p(\mathbf{R}^n)}\right] \\
&\leq C\left[(1+r)\|u\|_{L^p(\mathbf{R}^n)} + \|(\mathcal{A} - e^{i\theta}r^2)u\|_{L^p(\mathbf{R}^n)}\right].
\end{aligned} \tag{2.32}$$

On the other hand, since $\eta \equiv 1$ in $[-\frac{1}{2}, \frac{1}{2}]$, then

$$\begin{aligned}
\|v\|^p_{W^{2,p}(\mathbf{R}^{n+1})} \geq &\|v\|^p_{W^{2,p}(\mathbf{R}^n \times]-\frac{1}{2},\frac{1}{2}[)} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\mathbf{R}^n} \sum_{|\alpha|\leq 2} |D^\alpha(e^{irt}u(x))|^p dxdt = \\
&= \int_{\mathbf{R}^n} \Big[(1 + r^p + r^{2p})|u|^p + (1+2r^p)\sum_{j=1}^n |D_j u|^p + \sum_{j,k=1}^n |D_{jk}u|^p\Big] dx \\
&\geq r^{2p}\|u\|^p_{L^p(\mathbf{R}^n)} + r^p\|Du\|^p_{L^p(\mathbf{R}^n)} + \|D^2u\|^p_{L^p(\mathbf{R}^n)}.
\end{aligned} \tag{2.33}$$

Taking into account (2.32), it follows

$$\begin{aligned}
&r^2\|u\|_{L^p(\mathbf{R}^n)} + r\|Du\|_{L^p(\mathbf{R}^n)} + \|D^2u\|_{L^p(\mathbf{R}^n)} \\
&\quad \leq 3\|v\|_{W^{2,p}(\mathbf{R^{n+1}})} \leq 3C\left[(1+r)\|u\|_{L^p(\mathbf{R}^n)} + \|(\mathcal{A} - e^{i\theta}r^2)u\|_{L^p(\mathbf{R}^n)}\right]
\end{aligned} \tag{2.34}$$

where C is like in (2.32). We can select r sufficiently large such that $r^2 - 3C(1+r) \geq \frac{r^2}{2}$ we get

$$\frac{1}{2}r^2\|u\|_{L^p(\mathbf{R}^n)} + r\|Du\|_{L^p(\mathbf{R}^n)} + \|D^2u\|_{L^p(\mathbf{R}^n)} \leq C\|(\mathcal{A} - e^{i\theta}r^2)u\|_{L^p(\mathbf{R}^n)} \tag{2.35}$$

Taking $\lambda = e^{i\theta}r^2$ we get (2.27) with $M_p = 6C$. □

Now, by using the continuity method (see Theorem 1.5.3) we are able to prove existence and uniqueness for the solution of

$$\lambda u - \mathcal{A}u = f \in L^p(\mathbf{R}^n)$$

for $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda$ large enough.

Theorem 2.2.6. *Let $1 < p < \infty$. There exist $\tilde{\omega}_0 \in \mathbf{R}$, $C > 0$ depending on n, p, μ, M such that if $\operatorname{Re}\lambda \geq \tilde{\omega}_0$, then for every $f \in L^p(\mathbf{R}^n)$*

$$\lambda u - \mathcal{A}u = f$$

has a unique solution $u \in W^{2,p}(\mathbf{R}^n)$ and the following estimates hold

$$\|(\lambda - A_p)^{-1}\|_{\mathcal{L}(L^p(\mathbf{R}^n))} \leq \frac{C}{|\lambda|}; \tag{2.36}$$

$$\|\nabla(\lambda - A_p)^{-1}\|_{\mathcal{L}(L^p(\mathbf{R}^n))} \leq \frac{C}{|\lambda|^{\frac{1}{2}}}; \tag{2.37}$$

$$\|D^2(\lambda - A_p)^{-1}\|_{\mathcal{L}(L^p(\mathbf{R}^n))} \leq C. \tag{2.38}$$

Proof. We consider the Banach spaces

$$X = W^{2,p}(\mathbf{R}^n), \qquad Y = L^p(\mathbf{R}^n)$$

and the operators

$$L_0 = \lambda - \Delta, \quad L_1 = \lambda - \mathcal{A}, \quad L_t = \lambda - \mathcal{A}_t := \lambda - [(1-t)\Delta + t\mathcal{A}].$$

We can observe that $\mathcal{A}_t$ satisfies (2.4) with $\mu_t \geq \mu$ and the constant in (2.12) for $\mathcal{A}_t$, $M_t \leq (1 \vee M)$.
Moreover, by Corollary 2.2.3 we know that the operator L_0 is invertible for $\operatorname{Re}\lambda > 0$, and by the Proposition 2.2.5 applied to the operator $\mathcal{A}_t := (1-t)\Delta + t\mathcal{A}$ we get that there exist $\omega_0 \in \mathbf{R}$ and M_p depending only on n, p, μ, M, λ such that for every $\operatorname{Re}\lambda \geq \omega_0$ and $t \in [0,1]$,

$$\|u\|_{W^{2,p}(\mathbf{R}^n)} \leq M_p \|(\lambda - \mathcal{A}_t)u\|_{L^p(\mathbf{R}^n)}.$$

Since the hypotheses of Theorem 1.5.3 are satisfied we get the invertibility of the operator $L_1 = \lambda - \mathcal{A}$ for $\operatorname{Re}\lambda \geq \tilde{\omega}_0 := \sup\{\omega_0, 0\}$.
The estimates (2.36), (2.37) and (2.38), are immediate consequences of Proposition 2.2.5.
□

In view of Theorem 2.2.6 and Proposition 1.2.7 we have shown that the operator A_p defined in (2.15) is sectorial.

2.2.2 L^p-estimates on domains

In this section Ω will be either a smooth open subset of $\mathbf{R}^n$ or the half space $\mathbf{R}^n_+$. We suppose that $\mathcal{A}$, $\mathcal{B}$ satisfy assumption of Section 2.1. In this case we define

$$\begin{aligned} D(A_p^B) =& \{u \in W^{2,p}(\Omega);\ \mathcal{B}(\cdot, D)u = 0 \text{ in } \partial\Omega\}, \\ A_p^B u =& \mathcal{A}(\cdot, D)u,\ u \in D(A_p^B). \end{aligned} \tag{2.39}$$

A_p^B is the realization in $L^p(\Omega)$ of $\mathcal{A}(\cdot, D)$ with homogeneous oblique boundary condition. In order to prove that A_p^B is sectorial we prove that its resolvent set contains a complex half plane and the resolvent estimate (1.3) holds.
Here also we start with the simplest case of the Laplacian in the half space $\mathbf{R}^n_+$. The crucial points are

(i) to show an a-priori estimate for A_p^B,

(ii) to solve the Neumann problem in $\mathbf{R}^n_+$.

By means of the continuity method we deduce existence and uniqueness in $\mathbf{R}^n_+$ for the problem related to $\mathcal{A}$ with a boundary operator like $\mathcal{B}$. Finally, using the regularity of the boundary $\partial\Omega$, we deduce an analogous result in the domain Ω.

We need to prove an estimate like (2.27) for the resolvent of the operator A_p^B as next proposition states.

Proposition 2.2.7. *Let Ω be an open set with uniformly C^2 boundary. Then there exist $\omega_1 \in \mathbf{R}$, $M_p > 0$, depending on n, p, μ, M, Ω, such that if $\operatorname{Re}\lambda \geq \omega_1$, then for every $u \in W^{2,p}(\Omega)$ we have, setting $g = \mathcal{B}(\cdot, D)u_{|\partial\Omega}$,*

$$\begin{aligned} &|\lambda|\|u\|_{L^p(\Omega)} + |\lambda|^{\frac{1}{2}}\|Du\|_{L^p(\Omega)} + \|D^2u\|_{L^p(\Omega)} \leq \\ &\qquad M_p\|\lambda u - \mathcal{A}(\cdot, D)\|_{L^p(\Omega)} + |\lambda|^{1/2}\|g_1\|_{L^p(\Omega)} + \|Dg_1\|_{L^p(\Omega)} \end{aligned} \tag{2.40}$$

where g_1 is any extension of g belonging to $W^{1,p}(\Omega)$.

Proof. The proof of this result can be obtained as in Proposition 2.2.5, using now estimate (2.14) instead of (2.13) in $\Omega \times \mathbf{R}$. Let g_1 be any regular extension to Ω of the trace $(\mathcal{B}(\cdot, D)u)_{|\partial\Omega}$. Then (2.32) has to be replaced by

$$\begin{aligned} \|v\|_{W^{2,p}(\Omega\times\mathbf{R})} &\leq C_1\big(\|v\|_{L^p(\Omega\times\mathbf{R})} + \|\mathcal{E}v\|_{L^p(\Omega\times\mathbf{R})} + \|\eta e^{irt}g_1\|_{W^{1,p}(\Omega\times\mathbf{R})}\big) \\ &\leq C\big((r+1)\|u\|_{L^p(\Omega)} + \|(\mathcal{A} - e^{i\theta}r^2)u\|_{L^p(\Omega)} \\ &\quad + (r+1)\|g_1\|_{L^p(\Omega)} + \|Dg_1\|_{L^p(\Omega)}\big), \end{aligned} \tag{2.41}$$

where $C = C(n, p, \mu, M)$. Accordingly, (2.34) has to be replaced by

$$\begin{aligned} &r^2\|u\|_{L^p(\Omega)} + r\|Du\|_{L^p(\Omega)} + \|D^2u\|_{L^p(\Omega)} \\ &\quad \leq 3\|v\|_{W^{2,p}(\Omega\times\mathbf{R})} \leq 3C\big[(1+r)\|u\|_{L^p(\Omega)} + \|(A - e^{i\theta}r^2)u\|_{L^p(\Omega)} \\ &\quad + (r+1)\|g_1\|_{L^p(\Omega)} + \|Dg_1\|_{L^p(\Omega)}\big] \end{aligned} \tag{2.42}$$

As before taking $\lambda = e^{i\theta}r^2$ with r sufficiently large such that $3C(1+r) \leq \frac{r^2}{2}$ we get (2.40). □

Proposition 2.2.8. *Let $1 < p < \infty$. Then there exists $\omega_2 \in \mathbf{R}$ depending on n, p, such that if $\operatorname{Re}\lambda > \omega_2$ and $f \in L^p(\mathbf{R}^n_+)$ the problem*

$$\begin{cases} \lambda u - \Delta u = f & \text{in } \mathbf{R}^n_+ \\ \dfrac{\partial u}{\partial x_n} = 0 & \text{in } \partial\mathbf{R}^n_+ \end{cases} \tag{2.43}$$

has a unique solution $u \in W^{2,p}(\mathbf{R}^n_+)$. *Moreover there exists a constant* $c(\lambda) = c(n, p, \lambda)$ *such that*

$$\|u\|_{W^{2,p}(\mathbf{R}^n_+)} \le c(\lambda)\|f\|_{L^p(\mathbf{R}^n_+)}. \tag{2.44}$$

Proof. Uniqueness and (2.44) are consequences of Proposition 2.2.7. Concerning the existence, we consider the even extension of f with respect to the last variable

$$\tilde{f}(x', x_n) = \begin{cases} f(x', x_n) & x_n \ge 0 \\ f(x', -x_n) & x_n < 0 \end{cases}$$

By Theorem 2.2.2, for $\operatorname{Re}\lambda > 0$ there exists a unique solution $\tilde{u} \in W^{2,p}(\mathbf{R}^n)$ such that $\lambda\tilde{u} - \Delta\tilde{u} = \tilde{f}$. Now, it is easy to verify that the function $u(x', x_n) := \tilde{u}(x', -x_n)$ solves the elliptic problem $\lambda u - \Delta u = \tilde{f}$ in $\mathbf{R}^n$, and, by uniqueness, $u = \tilde{u}$, that is, $\tilde{u}$ is even in x_n and so $\frac{\partial \tilde{u}}{\partial x_n}(x', 0) = 0$. Therefore for $\operatorname{Re}\lambda > \sup\{\omega_1, 0\} =: \omega_2$, the restriction of $\tilde{u}$ in $\mathbf{R}^n_+$ is the unique solution of (2.43).

□

The following theorem extends results of existence and uniqueness of problem (2.43) to a problem where $\mathcal{A}$ replaces the Laplacian and more general oblique boundary conditions are considered.

Theorem 2.2.9. *Let* $1 < p < \infty$. *We assume that* $\beta_i, \gamma \in UC^1_b(\mathbf{R}^n_+)$ *and that the uniform non tangentiality condition*

$$\inf_{x \in \partial\mathbf{R}^n_+} |\langle \beta(x), e_n \rangle| > 0 \tag{2.45}$$

holds. Then there exists $\omega_3 \in \mathbf{R}$ *depending on* n, p, μ *such that for every* $f \in L^p(\mathbf{R}^n_+)$ *and* $\operatorname{Re}\lambda > \omega_3$ *the problem*

$$\begin{cases} \lambda u - \mathcal{A}u = f & \text{in } \mathbf{R}^n_+ \\ \dfrac{\partial u}{\partial \beta} + \gamma u = 0 & \text{in } \partial\mathbf{R}^n_+ \end{cases} \tag{2.46}$$

has a unique solution $u \in W^{2,p}(\mathbf{R}^n_+)$.

Proof. We set

$$X = W^{2,p}(\mathbf{R}^n_+) \qquad Y = L^p(\mathbf{R}^n_+) \times W^{1,p}(\mathbf{R}^{n-1})$$

and consider the operators $L_s : X \to Y$ so defined

$$L_s u := \Big(\lambda u - [(1-s)\Delta u + s\mathcal{A}u], (1-s)\frac{\partial u}{\partial \nu} + s(\gamma u + \frac{\partial u}{\partial \beta})\Big), \quad s \in [0,1],$$

where ν is the exterior unit normal vector to the domain, that is $\nu = -e_n$. We notice that

$$(1-s)\frac{\partial u}{\partial \nu} + s\frac{\partial u}{\partial \beta} = \frac{\partial u}{\partial \tau}$$

with $\tau = (1-s)\nu + s\beta$ satisfies (2.45) independently of s. Moreover $A_s = (1-s)\Delta u + s\mathcal{A}u$ satisfies (2.4) with $\mu_s \geq \mu$ and $M_s \leq (1 \vee M)$, therefore we can ignore the dependence of those constants by s. Hence in (2.40) the constant M_p can be chosen independently by s and the estimate

$$\|L_s u\|_Y \geq M_p^{-1} \|u\|_X$$

holds for every $s \in [0,1]$. By Proposition 2.2.8, L_0 is surjective, therefore by Theorem 1.5.3, L_1 is surjective too. □

The hypothesis of smoothness of the domain suggests to go back by means of local charts to balls or half balls of $\mathbf{R}^n$ and to apply the results obtained before in order to get the same result in Ω as the next theorem states.

Theorem 2.2.10. *Let Ω, $\mathcal{A}$ and $\mathcal{B}$ be as in* (2.1)-(2.7)*. Then there exists ω_4 depending on n, p, μ, Ω such that if $\operatorname{Re}\lambda \geq \omega_4$ and $f \in L^p(\Omega)$, the problem*

$$\begin{cases} \lambda u - \mathcal{A}(\cdot, D)u = f & \text{in } \Omega \\ \mathcal{B}(\cdot, D)u = 0 & \text{in } \partial\Omega \end{cases} \tag{2.47}$$

has a unique solution $u \in W^{2,p}(\Omega)$. Moreover there exists $C = C(n, p, \mu, M, \Omega) > 0$ such that

$$|\lambda| \|u\|_{L^p(\Omega)} + |\lambda|^{\frac{1}{2}} \|Du\|_{L^p(\Omega)} + \|D^2 u\|_{L^p(\Omega)} \leq C \|f\|_{L^p(\Omega)}. \tag{2.48}$$

Proof. Observe that if we prove the existence of a solution of (2.47) then uniqueness and estimate (2.48) follow immediately from Proposition 2.2.7. Indeed the estimate

$$|\lambda| \|u\|_{L^p(\Omega)} \leq M_1 \|\lambda u - \mathcal{A}u\|_{L^p(\Omega)}$$

yields the injectivity of $\lambda - A_p^B$. Thus, we have only to prove the surjectivity of the operator $\lambda - A_p^B$.
By the regularity of the boundary $\partial\Omega$ we can consider a partition of unity $\{(\eta_h^2, U_h)\}_{h \in \mathbf{N}}$ such that $\operatorname{supp} \eta_h \subset U_h$, $\sum_{h=0}^{\infty} \eta_h^2(x) = 1$ for every $x \in \overline{\Omega}$, $0 \leq \eta_h \leq 1$ and $\|\eta_h\|_{W^{2,\infty}} \leq c_\eta$ for every $h \in \mathbf{N}$. Moreover let $(U_h)_{h \in \mathbf{N}}$ be such that $U_0 \subset\subset \Omega$, U_h for $h \geq 1$ is a ball such that $\{U_h\}_{h \geq 1}$ is a covering of $\partial\Omega$ and $\{U_h\}_{h \in \mathbf{N}}$ is a covering of Ω with bounded overlapping, that is, there is $\kappa > 0$ such that

$$\sum_{h \in \mathbf{N}} \chi_{U_h}(x) \leq \kappa, \quad \forall x \in \overline{\Omega}. \tag{2.49}$$

Moreover there exist coordinate transformations $\varphi_h : U_h \to B(0,1)$, C^2 diffeomorphisms, such that

$$\varphi_h(\overline{U_h} \cap \Omega) = B^+(0,1)$$
$$\varphi_h(\overline{U_h} \cap \partial\Omega) = B(0,1) \cap \{x_n = 0\}.$$

Moreover, all the coordinate transformations φ_h and their inverses are supposed to have uniformly bounded derivatives up to the second order,

$$\sup_{h \in \mathbf{N}} \sum_{1 \leq |\alpha| \leq 2} \left(\|D^\alpha \varphi_h\|_\infty + \|D^\alpha \varphi_h^{-1}\|_\infty \right) \leq c \tag{2.50}$$

Let $f \in L^p(\Omega)$; then we can write $f = \sum_{h=0}^{\infty} \eta_h^2 f$. We notice that $\eta_0 f \in L^p(\mathbf{R}^n)$, $\mathrm{supp}\,(\eta_0 f) \subseteq U_0$. Thus if we extend a_{ij}, b_i and c to the whole space $\mathbf{R}^n$ in such a way that their qualitative properties are preserved, to the extension $\tilde{\mathcal{A}}$ we can apply the Theorem 2.2.6. Hence there exists $\tilde{\omega}_0 \in \mathbf{R}$ such that for $\mathrm{Re}\,\lambda \geq \tilde{\omega}_0$ the operator $\lambda - \tilde{\mathcal{A}}$ is invertible in $L^p(\mathbf{R}^n)$. Therefore if $R(\lambda) : L^p(\mathbf{R}^n) \to W^{2,p}(\mathbf{R}^n)$ denotes the resolvent of the operator $\tilde{A}_p$ in $\mathbf{R}^n$, we can define

$$R_0(\lambda) f := \eta_0 R(\lambda)(\eta_0 f).$$

Then supp $R_0(\lambda) f \subseteq U_0$ and $R_0(\lambda) : L^p(\Omega) \to W^{2,p}(\Omega)$ and

$$\begin{aligned}(\lambda - \mathcal{A}) R_0(\lambda) f &= (\lambda - \mathcal{A})(\eta_0 R(\lambda)(\eta_0 f)) \\ &= \eta_0(\lambda - \mathcal{A}) R(\lambda)(\eta_0 f) + ((\lambda - \mathcal{A})\eta_0 I + \eta_0(\lambda - \mathcal{A}))\,(R(\lambda)(\eta_0 f)) \\ &= \eta_0^2 f + [\lambda - \mathcal{A}, \eta_0] R(\lambda)(\eta_0 f)\end{aligned}$$

where $[X, Y] = XY - YX$ is the commutator of X and Y. Letting

$$S_{\eta_0}(\lambda) := [\lambda - \mathcal{A}, \eta_0 I] R(\lambda) \eta_0$$

we can write

$$(\lambda - \mathcal{A}) R_0(\lambda) f = \eta_0^2 f + S_{\eta_0}(\lambda) f.$$

It is immediate to verify that $[\lambda - \mathcal{A}, \eta_0 I] g = -[\mathcal{A}, \eta_0 I] g$. Moreover

$$-[\mathcal{A}, \eta_0 I] g = -2 \sum_{h,k=1}^{N} a_{hk} D_h \eta_0 D_k g - g(\sum_{i,j=1}^{n} (D_i(a_{ij} D_j \eta_0) + b_i D_i \eta_0)$$

If we define $B_0 = [\lambda - \mathcal{A}, \eta_0 I]$, we observe that B_0 is at most a first order differential operator whose coefficients depend on those of $\mathcal{A}$ and the function η_0. We have

$$\|B_0 g\|_{L^p(\Omega)} \leq C(M, c_\eta) \|g\|_{W^{1,p}(\Omega)}. \tag{2.51}$$

Hence, using (2.51) and estimates (2.36), (2.37), we get

$$\begin{aligned}\|S_{\eta_0}(\lambda) f\|_{L^p(\Omega)} &= \|B_0 (\lambda - \mathcal{A})^{-1}(\eta_0 f)\|_{L^p(\Omega)} \\ &\leq C(M, c_\eta) \|(\lambda - \mathcal{A})^{-1}(\eta_0 f)\|_{W^{1,p}(\Omega)} \\ &\leq \frac{C}{\sqrt{|\lambda|}} \|\eta_0 f\|_{L^p(\Omega)}\end{aligned} \tag{2.52}$$

where $C = C(n, p, \mu, M, c_\eta, \Omega)$ e $\mathrm{Re}\,\lambda \geq \tilde{\omega}_0$. So for $S_{\eta_0}(\lambda)$ we get the following estimate

$$\|S_{\eta_0}(\lambda)\|_{\mathcal{L}(L^p(\Omega))} \leq C|\lambda|^{-1/2}.$$

Now, we consider the case $h \geq 1$. Let

$$v_h(y) := (\eta_h f)(\varphi_h^{-1}(y)) =: T_h(\eta_h f)(y)$$

then $v_h \in W^{2,p}(\mathbf{R}^n_+)$. We denote by $\hat{\mathcal{A}}_h$ the operator in $\mathbf{R}^n_+$ determined by the change of variables given by φ_h

$$\hat{\mathcal{A}}_h w := \mathrm{div}(\hat{A}_h\, Dw) + \langle \hat{B}_h, Dw \rangle + \hat{c}_h w \tag{2.53}$$

defined by the coefficients (here for $\hat{\mathcal{A}}$ and its coefficients we omit the index h to simplify the notations)

$$\begin{aligned}
\hat{A}_h(y) :=& \big(D\varphi_h\big)(\varphi_h^{-1}(y)) \cdot A(\varphi_h^{-1}(y)) \cdot \big(D\varphi_h\big)^t(\varphi_h^{-1}(y)) \\
(\hat{B}_h(y))_l :=& \mathrm{Tr}\Big[\big(D\varphi_h\big)(\varphi_h^{-1}(y)) \cdot A(\varphi_h^{-1}(y)) \cdot H^l(\varphi_h^{-1}(y)) \cdot \big(D\varphi_h^{-1}\big)^t(y)\Big] \\
&+ \mathrm{Tr}\Big[\big(D\varphi_h\big)(\varphi_h^{-1}(y)) \cdot G^j(y)\Big]\big(D\varphi_h\big)^t_{jl}(\varphi_h^{-1}(y)) - \frac{\partial}{\partial y_j}\Big[\hat{a}_{jl}(y)\Big] \\
&+ \Big[\big(D\varphi_h\big)(\varphi_h^{-1}(y)) \cdot B(\varphi_h^{-1}(y))\Big]_l \\
\hat{c}_h(y) :=& c(\varphi_h^{-1}(y))
\end{aligned}$$

where $H^l_{ki} = D^2_{ki}(\varphi_h)_l$ and $G^j_{ki} = D_k a_{ij}(\varphi_h^{-1}(y))$. We remark that $\mathcal{A}(\eta_h u)(x) = \hat{\mathcal{A}}_h v_h(y)$. For what concerns the boundary condition we get

$$\begin{aligned}
\mathcal{B}(\eta_h u)(x) &= \beta(x) \cdot D(\eta_h u)(x) + \gamma(x)(\eta_h u)(x) \\
&= \Big[\big(D\varphi_h\big)(\varphi_h^{-1}(y)) \cdot \beta(\varphi_h^{-1}(y))\Big](Dv_h)(y) \cdot D(\eta_h u)(x) + \gamma(\varphi_h^{-1}(y))v_h(y) \\
&= \frac{\partial v_h}{\partial \hat{\beta}}(y) + \hat{\gamma} v_h(y) = \hat{\mathcal{B}}_h v_h(y)
\end{aligned}$$

where $\hat{\beta}(y) = \Big[\big(D\varphi_h\big)(\varphi_h^{-1}(y)) \cdot B(\varphi_h^{-1}(y))\Big]$ and $D\varphi_h$ denotes the Jacobian matrix of φ_h and $\hat{\gamma}(y) = \gamma(\varphi_h^{-1}(y))$. Now, since β is not tangent to $\partial\Omega$, $\hat{\beta}$ is not tangent to $\mathbf{R}^n_+$. We define

$$R_h(\lambda)f := T_h^{-1}\Big(T_h(\eta_h)(\lambda - \hat{\mathcal{A}}_h)^{-1}T_h(\eta_h f)\Big),$$

where $(\lambda - \hat{\mathcal{A}}_h)^{-1}$ is the resolvent of $\hat{\mathcal{A}}_h$ in $\mathbf{R}^n_+$ with the boundary condition $\hat{\mathcal{B}}_h v_h = 0$. Then $R_h(\lambda) : L^p(\Omega) \to W^{2,p}(\Omega)$ with $\mathcal{B}R_h(\lambda)f = 0$ in $\partial\Omega$ and $\mathrm{supp}\,(R_h(\lambda)f) \subset U_h$. We get

$$(\lambda - \mathcal{A})R_h(\lambda)f = \eta_h^2 f + S_{\eta_h}(\lambda)f$$

where $S_{\eta_h}(\lambda) = T_h^{-1}\Big([\lambda - \hat{\mathcal{A}}_h, T_h(\eta_h)](\lambda - \hat{\mathcal{A}}_h)^{-1}(T_h(\eta_h f))\Big)$.
As before for $\mathrm{Re}\,\lambda$ sufficiently large

$$\|S_{\eta_h}(\lambda)f\|_{L^p(\Omega)} \leq c(n, p, \mu, M, \Omega, c_\eta)|\lambda|^{-1/2}\|\eta_h f\|_{L^p(\Omega)} \tag{2.54}$$

Finally, letting

$$V(\lambda) = \sum_{h\in\mathbf{N}\cup\{0\}} R_h(\lambda) : L^p(\Omega) \to W^{2,p}(\Omega)$$

observe that $\mathcal{B}V(\lambda)f = 0$ in $\partial\Omega$ and

$$(\lambda - \mathcal{A})V(\lambda)f = \sum_{h=0}^{\infty} \eta_h^2 f + \sum_{h=0}^{\infty} S_{\eta_h}(\lambda)f = f + \sum_{h=0}^{\infty} S_{\eta_h}(\lambda)f.$$

Hence

$$(\lambda - \mathcal{A})V(\lambda) : L^p(\Omega) \to L^p(\Omega) \qquad \text{and} \qquad (\lambda - \mathcal{A})V(\lambda) = I + \sum_{h=0}^{\infty} S_{\eta_h}(\lambda)$$

Now, let observe that we can select λ with $\operatorname{Re}\lambda$ sufficiently large such that

$$\|\sum_{h=0}^{\infty} S_{\eta_h}(\lambda)\|_{\mathcal{L}(L^p(\Omega))} \le \frac{1}{2}, \tag{2.55}$$

indeed, since each S_{η_h} has support contained in U_h and the covering $\{U_i\}_i$ has bounded overlapping (2.49), then

$$\begin{aligned}\|\sum_{h=0}^{\infty} S_{\eta_h}(\lambda)f\|_{L^p(\Omega)} &\le \sum_{i=0}^{\infty}\int_{U_i} |\sum_{h=0}^{\infty} S_{\eta_h}(\lambda)f|^p\,dx\\ &\le \frac{c}{\sqrt{|\lambda|}}\sum_{i=0}^{\infty}\int_{U_i}|f|^p\;dx \le \frac{c}{\sqrt{|\lambda|}}\|f\|_{L^p(\Omega)}\end{aligned}$$

where $c = c(M, c_\eta, \kappa, \Omega)$. Then, (2.55) ensures that for $\operatorname{Re}\lambda$ sufficiently large, the operator $I + \sum_{h=0}^{\infty} S_{\eta_h}(\lambda)$ is invertible in $L^p(\Omega)$ with inverse $W(\lambda) : L^p(\Omega) \to L^p(\Omega)$. Hence, since $(\lambda - \mathcal{A})V(\lambda)W(\lambda) = I$ in $L^p(\Omega)$ and $u = V(\lambda)W(\lambda)f \in W^{2,p}(\Omega)$ is the solution of (2.47) for $\operatorname{Re}\lambda$ large enough. □

2.3 Generation of analytic semigroup in $L^\infty(\Omega)$ and in the space $C(\overline{\Omega})$

Henceforth Ω will be a domain with uniformly C^2 boundary and we set, for $x_0 \in \mathbf{R}^n$ and $r > 0$,

$$\Omega_{x_0,r} = \Omega \cap B(x_0, r). \tag{2.56}$$

Our aim is to prove that the realization A^B_∞ of $\mathcal{A}$ in L^∞ with homogeneous oblique boundary conditions as in (2.5)-(2.7) is a sectorial operator. In order to reach this we need that $\rho(A^B_\infty)$ contains an half plane and that an estimate like $|\lambda|\|u\|_{L^\infty(\Omega)} \le c\|\lambda u - \mathcal{A}u\|_{L^\infty(\Omega)}$ hold for $\operatorname{Re}\lambda$ large, $\lambda \in \rho(A^B_\infty)$. An important tool for the proof of the resolvent estimate in L^∞ is given by the following lemma in which a Caccioppoli type inequality in the L^p norm is stated.

Lemma 2.3.1. *Let $p > 1$ and $u \in W^{2,p}_{loc}(\Omega)$. For every λ with $\operatorname{Re}\lambda \ge \omega_1$ (ω_1 is given in Proposition (2.2.7)), set $f = \lambda u - \mathcal{A}u$ and $g = \mathcal{B}u_{|\partial\Omega}$. Then there exists C_1 depending only by n, p, μ, M and Ω such that for every $x_0 \in \overline{\Omega}$, $r \le 1$, $\alpha \ge 1$,*

$$\begin{aligned}&|\lambda|\|u\|_{L^p(\Omega_{x_0,r})} + |\lambda|^{\frac{1}{2}}\|Du\|_{L^p(\Omega_{x_0,r})} + \|D^2u\|_{L^p(\Omega_{x_0,r})}\\ &\quad\le C_1\Bigg\{\|f\|_{L^p(\Omega_{x_0,(\alpha+1)r})} + (|\lambda|^{1/2} + \frac{1}{\alpha r})\|g_1\|_{L^p(\Omega_{x_0,(\alpha+1)r})} + \|Dg_1\|_{L^p(\Omega_{x_0,(\alpha+1)r})}\\ &\qquad + \frac{1}{\alpha}\left[(\frac{1}{r^2} + \frac{|\lambda|^{1/2}}{r})\|u\|_{L^p(\Omega_{x_0,(\alpha+1)r})} + r^{-1}\|Du\|_{L^p(\Omega_{x_0,(\alpha+1)r})}\right]\Bigg\}\end{aligned} \tag{2.57}$$

where g_1 is any extension to $\overline{\Omega}$ of $\mathcal{B}u_{|\partial\Omega}$ of class $W^{1,p}_{loc}$.

Proof. Let $\theta_0 : \mathbf{R}^n \to \mathbf{R}$ be a smooth function such that $\theta_0 = 1$ in $B(0,r)$, $\operatorname{supp}\theta_0 \subset B(0,(\alpha+1)r)$ with

$$\|\theta_0\|_{L^\infty(\mathbf{R}^n)} + \alpha r\|D\theta_0\|_{L^\infty(\mathbf{R}^n)} + \alpha^2 r^2\|D^2\theta_0\|_{L^\infty(\mathbf{R}^n)} \le K$$

where K does not depend on α and r. We fix $x_0 \in \overline{\Omega}$, we set $\theta(x) = \theta_0(x - x_0)$. Define

$$v(x) = \theta(x)u(x), \quad x \in \Omega.$$

then v satisfies the following equation

$$\lambda v - \mathcal{A}(\cdot, D)v = \theta f - 2\sum_{i,j} a_{ij} D_i\theta D_j u - u\Big(\sum_{ij} D_i(a_{ij}D_j\theta) - \sum_{i=1}^n b_i D_i\theta\Big) =: f' \tag{2.58}$$

and the following boundary condition

$$\mathcal{B}v = \theta g + u\sum_{i=1}^n \beta_i D_i\theta \quad \text{in } \partial\Omega$$

Now, since $\operatorname{Re}\lambda \ge \omega_1$ and u and v coincide in $\Omega_{x_0,r}$, using Proposition 2.2.7 we get

$$\begin{aligned}
&|\lambda|\|u\|_{L^p(\Omega_{x_0,r})} + |\lambda|^{\frac{1}{2}}\|Du\|_{L^p(\Omega_{x_0,r})} + \|D^2u\|_{L^p(\Omega_{x_0,r})} \\
&\quad \le |\lambda|\|v\|_{L^p(\Omega)}|\lambda|^{\frac{1}{2}}\|Dv\|_{L^p(\Omega)} + \|D^2u\|_{L^p(\Omega)} \\
&\quad \le M_p(\|f'\|_{L^p(\Omega)} + |\lambda|^{1/2}\|\theta g_1 + u\sum_{i=1}^n \beta_i D_i\theta\|_{L^p(\Omega)} \\
&\qquad + \|D(\theta g_1) + D(u\sum_{i=1}^n \beta_i D_i\theta)\|_{L^p(\Omega)}).
\end{aligned} \tag{2.59}$$

Set $C_0 = \max_{i,j}\|a_{ij}\|_{W^{1,\infty}(\Omega)} + \max_i\|b_i\|_{L^\infty(\Omega)}$.
Then

$$\begin{aligned}
\|f'\|_{L^p(\Omega)} &\le \|f\|_{L^p(\Omega_{x_0,(\alpha+1)r})} + C_0K\Big(\frac{2}{\alpha r}\|Du\|_{L^p(\Omega_{x_0,(\alpha+1)r})} \\
&\quad + \frac{1}{\alpha^2r^2}\|u\|_{L^p(\Omega_{(x_0,(\alpha+1)r)})} + \frac{1}{\alpha r}\|u\|_{L^p(\Omega_{x_0,(\alpha+1)r})}\Big).
\end{aligned} \tag{2.60}$$

Moreover

$$\begin{aligned}
&|\lambda|^{1/2}\|u\sum_{i=1}^n \beta_i D_i\theta\|_{L^p(\Omega)} + \|D(u\sum_{i=1}^n \beta_i D_i\theta)\|_{L^p(\Omega)} \\
&\quad \le |\lambda|^{1/2}\sum_{i=1}^n \|\beta_i\|_\infty \frac{K}{\alpha r}\|u\|_{L^p(\Omega_{x_0,(\alpha+1)r})} \\
&\qquad + \sum_{i=1}^N \Big(\|D\beta_i\|_\infty\frac{K}{\alpha r} + \|\beta_i\|_\infty\frac{K}{\alpha^2r^2}\Big)\|u\|_{L^p(\Omega_{x_0,(\alpha+1)r})} \\
&\qquad + \sum_{i=1}^n \|\beta_i\|_\infty\frac{K}{\alpha r}\|Du\|_{L^p(\Omega_{x_0,(\alpha+1)r})} \\
&\quad \le \frac{CK}{\alpha}\Big[\Big(\frac{|\lambda|^{1/2}}{r} + \frac{2}{r^2}\Big)\|u\|_{L^p(\Omega_{x_0,(\alpha+1)r})} + \frac{1}{r}\|Du\|_{L^p(\Omega_{x_0,(\alpha+1)r})}\Big]
\end{aligned} \tag{2.61}$$

where $C=\sum_{i=1}^n \|\beta_i\|_{C^1(\overline{\Omega})}$, and

$$
\begin{aligned}
&|\lambda|^{1/2}\|\theta g_1\|_{L^p(\Omega)}+\|D(\theta g_1)\|_{L^p(\Omega)}\\
&\quad\le |\lambda|^{1/2}\|g_1\|_{L^p(\Omega_{x_0,(\alpha+1)r})}+\frac{K}{\alpha r}\|g_1\|_{L^p(\Omega_{x_0,(\alpha+1)r})}+\|Dg_1\|_{L^p(\Omega_{x_0,(\alpha+1)r})}
\end{aligned}
\tag{2.62}
$$

Taking into account that $r\le 1$ and $\alpha\ge 1$, replacing (2.60), (2.61) and (2.62) in (2.59) we get the claim. □

As a consequence we get the resolvent estimate as the following theorem states.

Theorem 2.3.2. *Let $p>n$. Then there exists $K>0$ depending on n,p,μ,M,Ω, such that for every $\lambda\in\mathbf{C}$ with $\mathrm{Re}\,\lambda\ge\Lambda_p^1=\omega_1\vee 1$ (ω_1 is given in Proposition 2.2.7) and for every $u\in C_b^1(\overline{\Omega})\cap W^{2,p}_{loc}(\Omega)$*

$$
\begin{aligned}
&|\lambda|\|u\|_{L^\infty(\Omega)}+|\lambda|^{1/2}\|Du\|_{L^\infty(\Omega)}+|\lambda|^{n/2p}\sup_{x_0\in\overline{\Omega}}\|D^2u\|_{L^p(\Omega_{x_0,|\lambda|^{-1/2}})}\\
&\quad\le K\Big(|\lambda|^{n/2p}\sup_{x_0\in\overline{\Omega}}\|\lambda u-\mathcal{A}u\|_{L^p(\Omega_{x_0,|\lambda|^{-1/2}})}\\
&\qquad+|\lambda|^{1/2}\|g_1\|_{L^\infty(\Omega)}+|\lambda|^{n/2p}\sup_{x_0\in\overline{\Omega}}\|Dg_1\|_{L^p(\Omega_{x_0,|\lambda|^{-1/2}})}\Big),
\end{aligned}
\tag{2.63}
$$

where g_1 is any extension of $g=\mathcal{B}u_{|\partial\Omega}$ belonging to $W^{1,p}_{loc}$. Moreover, there is $\tilde{K}>0$ such that if $\mathcal{A}u\in L^\infty(\Omega)$ and $\mathcal{B}u_{|\partial\Omega}\in C^1(\partial\Omega)$, then

$$
\begin{aligned}
&|\lambda|\|u\|_{L^\infty(\Omega)}+|\lambda|^{1/2}\|Du\|_{L^\infty(\Omega)}+|\lambda|^{n/2p}\sup_{x_0\in\overline{\Omega}}\|D^2u\|_{L^p(\Omega_{x_0,|\lambda|^{-1/2}})}\\
&\quad\le\tilde{K}\Big(\|\lambda u-\mathcal{A}u\|_{L^\infty(\Omega)}+|\lambda|^{1/2}\|\mathcal{B}u\|_{C(\partial\Omega)}+\|\mathcal{B}u\|_{C^1(\partial\Omega)}\Big).
\end{aligned}
\tag{2.64}
$$

Proof. Let $x_0\in\overline{\Omega}$, $|\lambda|\ge 1$, $\mathrm{Re}\,\lambda\ge\omega_1$ and $r=|\lambda|^{-\frac{1}{2}}$; then using the Sobolev inequality (i) of Theorem 1.5.2 we get

$$
\begin{aligned}
&|\lambda|\|u\|_{L^\infty(\Omega_{x_0,r})}+|\lambda|^{\frac12}\|Du\|_{L^\infty(\Omega_{x_0,r})}+|\lambda|^{\frac{n}{2p}}\|D^2u\|_{L^p(\Omega_{x_0,r})}\\
&\quad\le(2C+1)|\lambda|^{\frac{n}{2p}}\Big(|\lambda|\|u\|_{L^p(\Omega_{x_0,r})}+|\lambda|^{\frac12}\|Du\|_{L^p(\Omega_{x_0,r})}+\|D^2u\|_{L^p(\Omega_{x_0,r})}\Big).
\end{aligned}
$$

Now, using Lemma 2.3.1, we get, for every $\alpha\ge 1$,

$$
\begin{aligned}
&|\lambda|^{\frac{n}{2p}}\Big(|\lambda|\|u\|_{L^p(\Omega_{x_0,r})}+|\lambda|^{\frac12}\|Du\|_{L^p(\Omega_{x_0,r})}+\|D^2u\|_{L^p(\Omega_{x_0,r})}\Big)\\
&\quad\le C_1|\lambda|^{\frac{n}{2p}}[\|f\|_{L^p(\Omega_\alpha)}+|\lambda|^{1/2}\big(1+\frac{1}{\alpha}\big)\|g_1\|_{L^p(\Omega_\alpha)}\\
&\quad+\|Dg_1\|_{L^p(\Omega_\alpha)}+\frac{2}{\alpha}\big(|\lambda|\|u\|_{L^p(\Omega_\alpha)}+|\lambda|^{1/2}\|Du\|_{L^p(\Omega_\alpha)}\big)\\
&\quad\le C(|\lambda|^{\frac{n}{2p}}\|f\|_{L^p(\Omega_\alpha)}+\omega_n^{1/p}(\alpha+1)^{n/p}|\lambda|^{1/2}\|g_1\|_{L^\infty(\Omega_\alpha)}\\
&\quad+|\lambda|^{\frac{n}{2p}}\|Dg_1\|_{L^p(\Omega_\alpha)}+(\frac{\omega_n^{1/p}(\alpha+1)^{n/p}}{\alpha})\big(|\lambda|\|u\|_{L^\infty}(\Omega)+|\lambda|^{1/2}\|Du\|_{L^\infty(\Omega)}\big))
\end{aligned}
\tag{2.65}
$$

where $\Omega_\alpha = \Omega \cap B_\alpha(x_0) = \Omega \cap B(x_0, (\alpha+1)|\lambda|^{-1/2})$. Therefore

$$\begin{aligned}&|\lambda|\|u\|_{L^\infty(\Omega_{x_0,r})} + |\lambda|^{\frac{1}{2}}\|Du\|_{L^\infty(\Omega_{x_0,r})} + |\lambda|^{\frac{n}{2p}}\|D^2u\|_{L^p(\Omega_{x_0,r})}\\ &\quad\le C[|\lambda|^{\frac{n}{2p}}\|f\|_{L^p(\Omega_\alpha)} + \omega_n^{1/p}(\alpha+1)^{n/p}|\lambda|^{1/2}\|g_1\|_{L^\infty(\Omega_\alpha)}\\ &\quad+ |\lambda|^{\frac{n}{2p}}\|Dg_1\|_{L^p(\Omega_\alpha)} + (\frac{\omega_n^{1/p}(\alpha+1)^{n/p}}{\alpha})\big(|\lambda|\|u\|_{L^\infty(\Omega_\alpha)} + |\lambda|^{1/2}\|Du\|_{L^\infty(\Omega_\alpha)}\big)\end{aligned} \tag{2.66}$$

where C is a constant depending on p, n, μ, Ω. Taking the supremum over $x_0 \in \overline{\Omega}$ of the three addenda on the left hand side of (2.66) and summing up we get

$$\begin{aligned}&|\lambda|\|u\|_{L^\infty(\Omega)} + |\lambda|^{1/2}\|Du\|_{L^\infty(\Omega)} + |\lambda|^{\frac{n}{2p}}\sup_{x_0\in\overline{\Omega}}\|D^2u\|_{L^p(\Omega_{x_0,|\lambda|^{-1/2}})}\\ &\quad\le C\Big(|\lambda|^{\frac{n}{2p}}\sup_{x_0\in\overline{\Omega}}\|f\|_{L^p(\Omega_\alpha)} + \omega_n^{\frac{1}{p}}\frac{(\alpha+1)^{\frac{n}{p}}}{\alpha}(|\lambda|\|u\|_{L^\infty(\Omega)} + |\lambda|^{\frac{1}{2}}\|Du\|_{L^\infty(\Omega)})\\ &\quad+ \omega_n^{1/p}(\alpha+1)^{n/p}|\lambda|^{1/2}\|g_1\|_{L^\infty(\Omega)} + |\lambda|^{\frac{n}{2p}}\sup_{x_0\in\overline{\Omega}}\|Dg_1\|_{L^p(\Omega_\alpha)}\Big)\end{aligned}$$

Taking α sufficiently large in such a way that

$$C\omega_n^{\frac{1}{p}}\frac{(\alpha+1)^{\frac{n}{p}}}{\alpha} \le \frac{1}{2},$$

we obtain

$$\begin{aligned}&|\lambda|\|u\|_{L^\infty(\Omega)} + |\lambda|^{\frac{1}{2}}\|Du\|_{L^\infty(\Omega)} + |\lambda|^{\frac{n}{2p}}\sup_{x_0\in\overline{\Omega}}\|D^2u\|_{L^p(\Omega_{x_0,|\lambda|^{-1/2}})}\\ &\quad\le 2C(|\lambda|^{\frac{n}{2p}}\sup_{x_0\in\overline{\Omega}}\|f\|_{L^p(\Omega_\alpha)} + |\lambda|^{1/2}\|g_1\|_{L^\infty(\Omega)} + |\lambda|^{\frac{n}{2p}}\sup_{x_0\in\overline{\Omega}}\|Dg_1\|_{L^p(\Omega_\alpha)})\end{aligned}$$

Finally we can obtain (2.63) covering each ball $B_\alpha(x_0)$ with a finite number of balls with radius $|\lambda|^{-\frac{1}{2}}$.
To prove (2.64) we use (2.63), which implies

$$\begin{aligned}&|\lambda|\|u\|_{L^\infty(\Omega)} + |\lambda|^{1/2}\|Du\|_{L^\infty(\Omega)} + |\lambda|^{n/2p}\sup_{x_0\in\overline{\Omega}}\|D^2u\|_{L^p(\Omega_{x_0,|\lambda|^{-1/2}})}\\ &\quad\le K\big[\omega_n^{1/p}\Big(\|\lambda u - \mathcal{A}u\|_{L^\infty(\Omega)} + \|Dg_1\|_{L^\infty(\Omega)}\Big) + |\lambda|^{1/2}\|g_1\|_{L^\infty(\Omega)}\big]\end{aligned}$$

Finally, choosing $g_1 = E(\mathcal{B}u_{\partial\Omega})$, where $E \in \mathcal{L}(C(\partial\Omega), C(\overline{\Omega})) \cap \mathcal{L}(C^1(\partial\Omega), C^1(\overline{\Omega}))$ is an extension operator we get the claim. □

Next theorem, together with the resolvent estimate (2.64), is sufficient to prove the sectoriality of the realization of $\mathcal{A}$ in $L^\infty(\Omega)$ so defined

$$\begin{cases} D(A_\infty^B) = \{u \in \bigcap_{p\ge1} W^{2,p}_{loc}(\Omega);\quad u,\, \mathcal{A}u \in L^\infty(\Omega),\ \mathcal{B}u_{|\partial\Omega} = 0\},\\ A_\infty^B u = \mathcal{A}u.\end{cases}$$

Theorem 2.3.3. *The operator $A_\infty^B : D(A_\infty^B) \to L^\infty(\Omega)$ is sectorial. Moreover, $D(A_\infty^B) \subset C^{1,\alpha}(\overline{\Omega})$, for every $\alpha \in]0,1[$.*

Proof. Fix $p > n$. Let $\Lambda_0 = \inf_{p>n} \Lambda_p^1$; then we prove that the resolvent set of A_∞^B contains the half plane $\{\lambda \in \mathbf{C};\ \mathrm{Re}\,\lambda > \Lambda_0\}$. First we show that the $\rho(A_\infty^B)$ contains the half plane $\{\mathrm{Re}\,\lambda \ge \Lambda_p^1\}$. For any $f \in L^\infty(\Omega)$ and $k \in \mathbf{N}$, let ψ_k be a cut-off function such that

$$0 \le \psi_k \le 1, \quad \psi_k \equiv 1 \quad \text{in } B(0,k), \quad \psi_k \equiv 0 \quad \text{outside } B(0,2k).$$

We consider $f_k = \psi_k f$. Now, if $\mathrm{Re}\,\lambda > \Lambda_p^1$, then, by Theorem 2.2.10, the problem

$$\begin{cases} \lambda u_k - \mathcal{A} u_k = f_k & \text{in } \Omega \\ \mathcal{B} u_k = 0 & \text{in } \partial\Omega \end{cases} \tag{2.67}$$

has a unique solution $u_k \in W^{2,p}(\Omega)$ and $\|u_k\|_{W^{2,p}(\Omega)} \le C\|f_k\|_{L^p(\Omega)}$ where C is a constant depending on λ, n, p, M, Ω and μ. In particular, by the Sobolev embedding theorem (see Theorem 1.5.2), $u_k \in C_b^1(\overline{\Omega})$, therefore using (2.64) we get

$$\|u_k\|_{C^1(\Omega)} + \sup_{x_0 \in \overline{\Omega}} \|D^2 u_k\|_{L^p(\Omega_{x_0,|\lambda|^{-1/2}})} \le K(\lambda)\|f_k\|_{L^\infty(\Omega)} \le K(\lambda)\|f\|_{L^\infty(\Omega)}. \tag{2.68}$$

Therefore, $\{u_k\}_k$ is bounded in $C^1(\Omega)$, so that there exists a subsequence converging uniformly on each compact subset of $\overline{\Omega}$ to a function $u \in C(\overline{\Omega}) \cap Lip(\Omega)$ such that

$$\|u\|_{L^\infty(\Omega)} + [u]_{Lip(\Omega)} \le K(\lambda)\|f\|_{L^\infty(\Omega)}. \tag{2.69}$$

Now, we show that $u \in W^{2,p}_{loc}(\Omega)$ and that it solves

$$\begin{cases} \lambda u - \mathcal{A} u = f & \text{in } \Omega \\ \mathcal{B} u = 0 & \text{in } \partial\Omega \end{cases}$$

Let $B(x_0, R)$ be the closed ball with $x_0 \in \overline{\Omega}$ and $R \ge 4|\lambda|^{-1/2}$, then by (2.68) we know that $\{u_k\}_k$ is bounded in $W^{2,p}(\Omega_{x_0,R})$, so that the limit function u is in $W^{2,p}(\Omega_{x_0,R})$. Since x_0 and R are arbitrary, $u \in W^{2,p}_{loc}(\Omega)$. Moreover there exists a subsequence $\{u_{\phi(k)}\}_k$ converging to u in $W^{1,p}(\Omega_{x_0,R})$, and for h, k sufficiently large

$$\begin{cases} \lambda(u_{\phi(h)} - u_{\phi(k)}) - \mathcal{A}(u_{\phi(h)} - u_{\phi(k)}) = 0 & \text{in } \Omega_{x_0,R} \\ \mathcal{B}(u_{\phi(h)} - u_{\phi(k)}) = 0 & \text{in } \partial\Omega \cap B_{x_0,R} \end{cases}$$

Now, applying Lemma 2.3.1 to the function $u_{\phi(h)} - u_{\phi(k)}$, we get

$$\begin{aligned} \|u_{\phi(h)} - u_{\phi(k)}\|_{W^{2,p}(\Omega_{x_0,|\lambda|^{-1/2}})} &\le C(\lambda)\|u_{\phi(h)} - u_{\phi(k)}\|_{W^{1,p}(\Omega_{x_0,2|\lambda|^{-1/2}})} \\ &\le C(\lambda)\|u_{\phi(h)} - u_{\phi(k)}\|_{W^{1,p}(\Omega_{x_0,R})} \to 0 \qquad \text{as } h,k \to \infty. \end{aligned}$$

Covering $B(x_0, R/2)$ by a finite number of balls with radius $|\lambda|^{-1/2}$ we get that $\{u_{\phi(k)}\}_k$ converges in $W^{2,p}(\Omega_{x_0,R/2})$, so that, letting $k \to \infty$ in (2.67) we get $\lambda u - Au = f$ in $\Omega_{x_0,R/2}$.
Moreover since the trace operator $u \to u_{\partial\Gamma}$ is continuous from $W^{1,p}(\Gamma)$ to $L^p(\partial\Gamma, d\mathcal{H}^{n-1})$ for every open subset Γ of $\mathbf{R}^n$ with bounded Lipschitz boundary, then $\mathcal{B}$ is a linear and continuous operator from $W^{2,p}(\Omega_{x_0,R/2})$ to $L^p(\partial\Omega_{x_0,R/2})$, hence we get

$$\|\mathcal{B}(u_k - u)\|_{L^p(\partial\Omega \cap B(x_0,R/2))} \le c_1\|u_k - u\|_{W^{2,p}(\Omega_{x_0,R/2})},$$

where c_1 is a constant depending on Ω, R and by $\|\beta_i\|_{L^\infty(\Omega)}, \|\gamma\|_{L^\infty(\Omega)}$. Therefore we get $\mathcal{B}u = 0$ in $\partial\Omega \cap B(x_0, R/2)$. Since x_0 and R are arbitrary, then

$$\begin{cases} \lambda u - \mathcal{A}u = f & \text{in } \Omega \\ \mathcal{B}u = 0 & \text{in } \partial\Omega \end{cases}$$

Now, fixed any $q > n$ we can write (2.67) as follows

$$\Lambda_q u_k - \mathcal{A}u_k = (\Lambda_q - \lambda)u_k + f_k.$$

We observe that the right hand side is in $L^\infty(\Omega)$, and its sup norm is bounded by a constant independent of k. Repeating the above arguments we conclude that $u \in W^{2,q}_{\rm loc}(\Omega)$ for all $q > n$, so that $u \in D(A^B_\infty)$. Therefore $\rho(A^B_\infty) \supset \{\lambda \in \mathbf{C} :\ \mathrm{Re}\,\lambda > \Lambda^1_p\}$ for every $p > n$. Thus, from estimate (2.64) and Proposition 1.2.7 we conclude that A^B_∞ is sectorial. Now, let $u \in D(A^B_\infty)$, then by the Sobolev embedding u is continuously differentiable and its gradient is bounded: indeed, fixed $p > n$ and $f = \Lambda_p u - \mathcal{A}u$, by estimate (2.69) we get

$$\|Du\|_{L^\infty(\Omega)} \le c(\|u\|_{L^\infty(\Omega)} + \|Au\|_{L^\infty(\Omega)})$$

Moreover, choosing $p = n/(1-\alpha)$, using Theorem 1.5.2 (inequality (ii)) and (2.64) with $\lambda = \Lambda^1_p$ we get, for $i = 1, \ldots, n$,

$$|D_i u(x) - D_i u(y)| \le c|x-y|^\alpha(\|u\|_{L^\infty(\Omega)} + \|Au\|_{L^\infty(\Omega)})$$

for all $x, y \in \mathbf{R}^n$ such that $|x-y| \le (\Lambda^1_p)^{-1/2}$. On the other hand, if $|x-y| \ge (\Lambda^1_p)^{-1/2}$ then

$$\begin{aligned} \frac{|D_i u(x) - D_i u(y)|}{|x-y|^\alpha} &\le 2\|D_i u\|_{L^\infty(\Omega)}(\Lambda^1_p)^{\alpha/2} \\ &\le c(\|u\|_{L^\infty(\mathbf{R}^n)} + \|Au\|_{L^\infty(\mathbf{R}^n)}) \end{aligned}$$

Therefore, $D(A^B_\infty) \subset C^{1,\alpha}(\overline{\Omega})$ for $\alpha \in]0,1[$.

□

From Theorems 2.3.2 and 2.3.3 we get the following result.

Corollary 2.3.4. *Let Λ_0 be as in Theorem 2.3.3. Set*

$$\begin{cases} D(A^B_C) = \{u \in \bigcap_{p\ge1} W^{2,p}_{loc}(\Omega);\quad u,\, \mathcal{A}u \in C_b(\overline{\Omega}),\, \mathcal{B}u_{|\partial\Omega} = 0\}, \\ A^B_C u : D(A^B_C) \to C_b(\overline{\Omega}), \qquad A^B_C = \mathcal{A}u. \end{cases}$$

Then the resolvent set of A^B_C contains the half plane $\{\lambda \in \mathbf{C};\ \mathrm{Re}\,\lambda > \Lambda_0\}$, and A^B_C is sectorial.

Proof. Since $D(A^B_\infty) \subset C_b(\overline{\Omega})$, then $\rho(A^B_\infty) \subset \rho(A^B_C)$. Therefore $\rho(A^B_C)$ contains the half plane $\{\mathrm{Re}\,\lambda > \Lambda_0\}$. Estimate (2.64) and Proposition 1.2.7 prove that A^B_C is sectorial. □

2.4 Elliptic boundary value problems in some Sobolev spaces of negative order

In this section, as in the preceding one, we suppose that Ω is a domain with uniformly C^2 boundary $\partial\Omega$. Here our aim is to prove existence, uniqueness and some useful estimates for the solution of a boundary value problem for an elliptic operator $\mathcal{A}$ in suitable Sobolev spaces of negative order. Actually, we are interested in deducing L^1 norm estimates of the gradient of the resolvent of the realization of $\mathcal{A}$ in L^1 (see Theorem 2.5.3). This can be done by duality starting from the solution of the dual problem.
In this section we follow, with significant modifications, ideas from [47], [48]. Before stating the main result, let us introduce some notation.
Let $1 \le p < \infty$; we shall consider the Banach spaces $(W_0^{1,p}(\Omega))'$ and $(W^{1,p}(\Omega))'$ respectively denoted by $W^{-1,p'}(\Omega)$ and $W_*^{-1,p'}(\Omega)$ (we set $1' = \infty$). Each element $f \in W^{-1,p'}(\Omega)$ (resp. $f \in W_*^{-1,p'}(\Omega)$) admits a (not unique) $L^{p'}$ representation; that is, there exist $f_0, f_1, \ldots, f_n \in L^{p'}(\Omega)$ such that

$$\langle f, v\rangle_* = \int_\Omega f_0 v\, dx + \sum_{i=1}^n \int_\Omega f_i D_i v dx \tag{2.70}$$

for every $v \in W_0^{1,p'}(\Omega)$ (resp. $v \in W^{1,p'}(\Omega)$), where $\langle \cdot, \cdot \rangle_*$ denotes the duality between $W^{-1,p}$ and $W_0^{1,p'}$ (resp. $W_*^{-1,p}$ and $W^{1,p'}$), see [1, Theorem 3.8]. In order to indicate an $L^{p'}$ representation of f we often write

$$f = f_0 - \sum_{i=1}^n D_i f_i \tag{2.71}$$

where the equality has to be intended in the distributional sense specified in (2.70). Obviously $(W^{1,p}(\Omega))'$ is continuously embedded in $(W_0^{1,p}(\Omega))'$, and there is a natural embedding of $L^{p'}(\Omega)$ in $(W^{1,p}(\Omega))'$: we can identify any $L^{p'}$ function f_0 with the functional

$$v \mapsto \int_\Omega f_0(x) v(x)\, dx.$$

We can consider these spaces as Banach spaces endowed with either the norm induced by duality or the norm defined by

$$\inf\left\{\sum_{i=0}^n \|f_i\|_{L^{p'}(\Omega)},\ f_i \text{ satisfying } (2.70)\right\}.$$

In the following lemma we prove some useful estimates that hold in these spaces.

Lemma 2.4.1. *For each $p > n$ there exist two constants c_1, c_2 such that for each $x_0 \in \overline{\Omega}$, $r > 0$ and $u \in L^p(\Omega)$ with support in $\Omega_{x_0,r}$ (given in (2.56)),*

$$\|u\|_{W_*^{-1,p}(\Omega)} \le c_1 r \|u\|_{L^p(\Omega)} \tag{2.72}$$

$$\|u\|_{W_*^{-1,\infty}(\Omega)} \le c_2 r^{1-n/p} \|u\|_{L^p(\Omega)} \tag{2.73}$$

PROOF. Let $\varphi \in W^{1,p'}(\Omega)$ be such that $\|\varphi\|_{W^{1,p'}(\Omega)} \leq 1$. Then by Sobolev embedding $\varphi \in L^q(\Omega)$ with $q = (np')/(n-p')$ and $\|\varphi\|_{L^q(\Omega)} \leq c$ where c depends only on Ω. Hence

$$\|u\|_{W_*^{-1,p}(\Omega)} = \sup\left\{\int_\Omega u\varphi\, dx\,;\ \varphi \in W^{1,p'}(\Omega),\ \|\varphi\|_{W^{1,p'}(\Omega)} \leq 1\right\}$$

but the following estimate holds

$$\int_\Omega u\varphi\, dx \leq \|u\|_{L^{q'}(\Omega_{x_0,r})}\|\varphi\|_{L^q(\Omega)} \leq cr\|u\|_{L^p(\Omega)}$$

and (2.72) is proved. In a similar way one can prove (2.73). $\square$

Here, in order to obtain a precise estimate for the L^∞ norm of the solution of an elliptic boundary value problem in $W_*^{-1,\infty}(\Omega)$, we follow a procedure similar to the one used by Stewart in [42] and in [43] starting by $W_*^{-1,p}(\Omega)$, $1 < p < \infty$.

2.4.1 Formally adjoint boundary value problems

Let $\mathcal{A}$ and $\mathcal{B}$ be the operators defined respectively in (2.1) and in (2.5) satisfying (2.4) and (2.7). Let consider the elliptic problem (2.11); we are interested in the formulation of its formally adjoint boundary value problem, hence, (at this moment) we do not take care of the smoothness properties of the coefficients and we proceed by formal computations. We define the formally adjoint differential operator $\mathcal{A}^*$ of $\mathcal{A}$ by

$$\mathcal{A}^* = \sum_{i,j=1}^n D_j(a_{ij}^* D_i) + \sum_{j=1}^n b_j^* D_j + c^* \tag{2.74}$$

with

$$a_{ij}^* = a_{ij} \quad b_i^* = -b_i \quad c^* = c - \mathrm{div} b.$$

Then by the divergence theorem

$$\int_\Omega v\mathcal{A}u\, dx = \int_\Omega u\mathcal{A}^*v\, dx + \int_{\partial\Omega} \big(\langle ADu, \nu\rangle v - \langle ADv, \nu\rangle u + \langle B, \nu\rangle uv\big) d\mathcal{H}^{n-1}$$

for all $u, v \in C^2(\Omega) \cap C^1(\bar{\Omega})$. We let $\nu_A := A\nu$ and $\rho(x) := \frac{\langle \nu_A(x), \nu(x)\rangle}{\langle \beta(x), \nu(x)\rangle}$, and define a vector field by

$$\tau := \nu_A - \rho\beta.$$

We observe that $\langle \tau, \nu\rangle = 0$ and that

$$\langle D, \nu_A\rangle = \rho\langle D, \beta\rangle + \langle D, \tau\rangle. \tag{2.75}$$

Since $\rho(x) \neq 0$ for all $x \in \partial\Omega$, we can define β^* by

$$\rho\beta^* := \nu_A + \tau$$

so that

$$\langle D, \nu_A \rangle = \rho \langle D, \beta^* \rangle - \langle D, \tau \rangle. \tag{2.76}$$

We see that β^* so defined is a non-tangent vector field on $\partial\Omega$, indeed $\rho\langle \beta^*, \nu\rangle = \langle \nu_A, \nu \rangle$. From (2.75) and (2.76) we get

$$\langle ADu, \nu\rangle v - \langle ADv, \nu \rangle u = \rho(v\langle Du, \beta\rangle - u\langle Dv, \beta^*\rangle) + \langle D(uv), \tau\rangle$$

Finally we define γ^* by

$$\rho\gamma^* := \rho\gamma - \langle B, \nu\rangle + \mathrm{div}\tau$$

and the formally adjoint operator $\mathcal{B}^*$ of $\mathcal{B}$ on $\partial\Omega$ by

$$\mathcal{B}^* = \sum_{i=1}^n \beta_i^* D_i + \gamma^*. \tag{2.77}$$

Finally, applying the divergence theorem, we obtain

$$\int_\Omega v\mathcal{A}u\,dx = \int_\Omega u\mathcal{A}^* v\,dx + \int_{\partial\Omega} \rho(v\mathcal{B}u - u\mathcal{B}^* v)\;d\mathcal{H}^{n-1}$$

for all $u, v \in C^2(\Omega) \cap C^1(\bar{\Omega})$.

Henceforth we focus our attention to a particular choice of the boundary operator $\mathcal{B}$. We select the conormal boundary operator

$$\mathcal{B}(x, D) = \sum_{i,j=1}^n a_{ij}(x)\nu_i(x) D_j, \tag{2.78}$$

in this way the formally adjoint operator $\mathcal{B}^*$ is defined as follows

$$\mathcal{B}^* = \langle D, \nu_A \rangle - \langle B, \nu \rangle$$

(since $\rho = 1$, $\tau = 0$, $\beta^* = \nu_A$ and $\gamma^* = -\langle B, \nu\rangle$), and $\mathcal{A}^*$ is defined in (2.74). We suppose that a_{ij}, b_i and c are real valued functions such that

$$a_{ij} = a_{ji}, \quad a_{ij},\, b_i \in W^{2,\infty}(\Omega), \qquad c \in L^\infty(\Omega). \tag{2.79}$$

Assumption (2.79) guarantees that hypotheses in Section 2.1 are satisfied both for the couple of operators $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{A}^*, \mathcal{B}^*)$ and Theorem 2.2.10 can be applied to each of them. We set

$$M_1 = \max_{i,j}\{\|a_{ij}\|_{W^{2,\infty}(\Omega)}, \|b_i\|_{W^{2,\infty}(\Omega)}, \|c\|_{L^\infty(\Omega)}\}. \tag{2.80}$$

Now, we consider the realization of $\mathcal{A}$ with homogeneous boundary condition given by $\mathcal{B}$ as in (2.78) in the Banach space $W_*^{-1,p}$, so defined

$$E_p : D(E_p) = W^{1,p}(\Omega) \subset W_*^{-1,p}(\Omega) \to W_*^{-1,p}(\Omega) \tag{2.81}$$

by

$$\langle E_p u, v\rangle_* = a(u, v) \qquad u \in W^{1,p}(\Omega),\, v \in W^{1,p'}(\Omega) \tag{2.82}$$

where

$$a(u,v) = -\int_\Omega \langle ADu, Dv\rangle\, dx + \int_\Omega \langle B, Du\rangle v\, dx + \int_\Omega cuv\, dx \tag{2.83}$$

in $W^{1,p}(\Omega)\times W^{1,p'}(\Omega)$. Analogously we could define the realization of $(\mathcal{A}^*, \mathcal{B}^*)$ in $W_*^{-1,p'}$ in this way:

$$E_{p'} : D(E_{p'}) = W^{1,p'}(\Omega) \subset W_*^{-1,p'}(\Omega) \to W_*^{-1,p'}(\Omega) \tag{2.84}$$

by

$$\langle E_{p'}u, v\rangle_* = a^*(u,v) \qquad u \in W^{1,p'}(\Omega),\ v \in W^{1,p}(\Omega) \tag{2.85}$$

where

$$a^*(u,v) = -\int_\Omega \langle ADu, Dv\rangle\, dx + \int_\Omega \langle B, Dv\rangle u\, dx + \int_\Omega cuv\, dx \tag{2.86}$$

in $W^{1,p'}(\Omega) \times W^{1,p}(\Omega)$.

We start with two technical results involving L^p estimates that are true for both E_p and $E_{p'}$ and that for simplicity are stated only in one case.

Theorem 2.4.2. *The operator E_p is sectorial in $W_*^{-1,p}(\Omega)$. In particular there is a constant $\omega_p \in \mathbf{R}$ depending on n, p, μ, M_1, Ω such that for each $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda > \omega_p$ and for each $f \in W_*^{-1,p}(\Omega)$ the solution $u \in W^{1,p}(\Omega)$ of the equation $(\lambda - \mathcal{A})u = f$ satisfies*

$$|\lambda|\|u\|_{W_*^{-1,p}(\Omega)} + |\lambda|^{1/2}\|u\|_{L^p(\Omega)} + \|u\|_{W^{1,p}(\Omega)} \le K_1\|f\|_{W_*^{-1,p}(\Omega)} \tag{2.87}$$

where $K_1 > 0$ is a constant independent of λ and f.

Proof. Denote by A_p^B the realization of $\mathcal{A}$ in L^p with homogeneous boundary conditions $\mathcal{B}u = 0$ and analogously ${A^*}_{p'}^{B^*}$ the realization of $\mathcal{A}^*$ in $L^{p'}$ with homogeneous boundary conditions $\mathcal{B}^*u = 0$. We know that $D(A_p^B) = \{u \in W^{2,p}(\Omega) :\ \mathcal{B}u = 0 \text{ in } \partial\Omega\}$. Then for each $u \in D({A^*}_{p'}^{B^*})$ and $v \in L^p(\Omega)$, we have $\langle {A^*}_{p'}^{B^*}u, v\rangle = \langle u, ({A^*}_{p'}^{B^*})^*v\rangle$ where $({A^*}_{p'}^{B^*})^*$ is the adjoint of ${A^*}_{p'}^{B^*}$ and belongs to $\mathcal{L}(L^p(\Omega), (D({A^*}_{p'}^{B^*})')$ where $(D({A^*}_{p'}^{B^*})'$ is the dual space of $D({A^*}_{p'}^{B^*})$. Note that the restriction of $({A^*}_{p'}^{B^*})^*$ to $D(A_p^B)$ coincides with A_p^B. Therefore, from the complex interpolation theory (see Theorem A.3.5), we have that $({A^*}_{p'}^{B^*})^*$ is a bounded linear operator from $[L^p(\Omega), D(A_p^B)]_{1/2}$ to $[(D({A^*}_{p'}^{B^*}))', L^p(\Omega)]_{1/2}$ where $[\cdot,\cdot]_{1/2}$ is the complex interpolation space of order 1/2, (see Section A.3 for the relevant definitions and results). Using [39, Theorem 4.1], which holds for domains with uniformly smooth boundary, we can characterize the complex interpolation spaces in the following way:

$$[L^p(\Omega), D(A_p^B)]_{1/2} = W^{1,p}(\Omega)$$

$$[(D({A^*}_{p'}^{B^*}))', L^p(\Omega)]_{1/2} = [L^{p'}(\Omega), D({A^*}_{p'}^{B^*})]'_{1/2} = (W^{1,p'}(\Omega))' = W_*^{-1,p}(\Omega) \tag{2.88}$$

where in the first equality in (2.88) we have used (A.16). Therefore the restriction of $({A^*}_{p'}^{B^*})^*$ to the space $W^{1,p}(\Omega)$ is a bounded linear operator from $W^{1,p}(\Omega)$ to $W_*^{-1,p}(\Omega)$ and coincides with E_p.

Now, we show that there exists a constant k_1 such that for each λ with $\operatorname{Re}\lambda$ large enough,

$$\|(\lambda - A_p^B)^{-1}\|_{\mathcal{L}(L^p, D(A_p^B))} \le k_1. \tag{2.89}$$

Since A_p^B is a sectorial operator, there exists $\omega_1 \in \mathbf{R}$ such that for each $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda \geq \omega_1$ and for each $f \in L^p(\Omega)$ the equation

$$(\lambda - \mathcal{A})u = f$$

admits a solution $u \in W^{2,p}(\Omega)$ with $\mathcal{B}u = 0$ in $\partial\Omega$ satisfying (2.48). Hence

$$\begin{aligned}\|u\|_{D(A_p^B)} &= \|u\|_{L^p(\Omega)} + \|\mathcal{A}u\|_{L^p(\Omega)} \leq (1+|\lambda|)\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} \\ &\leq (\frac{1+|\lambda|}{|\lambda|} + 1)\|f\|_{L^p(\Omega)} \leq k_1\|f\|_{L^p(\Omega)}\end{aligned}$$

for $\operatorname{Re}\lambda$ large. Analogously, there exists a constant $\omega_2 \in \mathbf{R}$ and $k_2 > 0$, such that

$$\|(\lambda - A^*{}_{p'}^{B^*})^{-1}\|_{\mathcal{L}(L^{p'}, D(A^*{}_{p'}^{B^*}))} \leq k_2 \tag{2.90}$$

for $\operatorname{Re}\lambda > \omega_2$. Using (2.90) we get that

$$[(\lambda - A^*{}_{p'}^{B^*})^{-1}]^* = [(\lambda - A^*{}_{p'}^{B^*})^*]^{-1} \in \mathcal{L}((D(A^*{}_{p'}^{B^*}))', L^p)$$

hence an argument similar to the previous one yields that the operator $[(\lambda - A^*{}_{p'}^{B^*})^{-1}]^*$ belongs to $\mathcal{L}(W_*^{-1,p}(\Omega), W^{1,p}(\Omega))$ and coincides with $(\lambda - E_p)^{-1}$.
Set $K = k_1 + k_2$ and $\omega_p > \omega_1 \vee \omega_2$; then, for every λ with $\operatorname{Re}\lambda > \omega_p$ and for every $f \in W_*^{-1,p}(\Omega)$ we have that $\|u\|_{W^{1,p}(\Omega)} \leq K\|f\|_{W_*^{-1,p}(\Omega)}$ where $u = (\lambda - E_p)^{-1}f$. Then, for every $v \in W^{1,p'}(\Omega)$,

$$\langle f, v\rangle_* = \lambda\langle u, v\rangle_* - \langle E_p u, v\rangle_*$$

Thus

$$\begin{aligned}|\langle u, v\rangle_*| &\leq |\lambda|^{-1}\left(|\langle E_p u, v\rangle_*| + |\langle f, v\rangle_*|\right) \\ &\leq c|\lambda|^{-1}\left(\|u\|_{W^{1,p}(\Omega)}\|v\|_{W^{1,p'}(\Omega)} + \|f\|_{W_*^{-1,p}(\Omega)}\|v\|_{W^{1,p'}(\Omega)}\right) \\ &\leq c|\lambda|^{-1}\left(K\|f\|_{W_*^{-1,p}(\Omega)}\|v\|_{W^{1,p'}(\Omega)} + \|f\|_{W_*^{-1,p}(\Omega)}\|v\|_{W^{1,p'}(\Omega)}\right)\end{aligned}$$

Hence we have proved that

$$|\lambda|\|u\|_{W_*^{-1,p}(\Omega)} + \|u\|_{W^{1,p}(\Omega)} \leq c\|f\|_{W_*^{-1,p}(\Omega)}. \tag{2.91}$$

Therefore, (2.87) is consequence of (2.91) and of the fact that

$$(W_*^{-1,p}(\Omega), W^{1,p}(\Omega))_{1/2,p} = L^p(\Omega)$$

for $1 < p < \infty$ (see [46, Section 2.4.2, Theorem 1; Section 4.2.1, Definition 1]). □

Remark 2.4.3. We observe that if $f \in L^p(\Omega)$, then $u = (\lambda - E_p)^{-1}f \in D(A_p^B)$ and therefore $\mathcal{B}u = 0$ in $\partial\Omega$.

Lemma 2.4.4. *Let $p \geq 2$ and $f \in W_*^{-1,p}(\Omega)$ with $f = f_0 - \sum_{i=1}^n D_i f_i$; then for each $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda > \omega_p$, for each $r < 1$ and for each $x_0 \in \overline{\Omega}$, the solution $u \in D(E_p)$ of the equation $\lambda u - \mathcal{A}u = f$ satisfies the following estimate*

$$\|u\|_{W^{1,p}(\Omega_{x_0,r})} \leq K_2\left\{\sum_{i=1}^n \|f_i\|_{L^p(\Omega_{x_0,2r})} + r\|f_0\|_{L^p(\Omega_{x_0,2r})} + r^{-1}\|u\|_{L^p(\Omega_{x_0,2r})}\right\} \tag{2.92}$$

where $\Omega_{x_0,r}$ is defined in (2.56) and K_2 is a constant independent of λ and f.

Proof. We point out that the space of functions

$$C_\nu^{-1} = \{g = g_0 - \sum_{i=1}^n D_i g_i;\ g_i \in C^1(\overline{\Omega}) \cap L^{p'}(\Omega), \sum_{i=1}^n g_i \nu_i = 0 \ \text{ on } \partial\Omega\}$$

is dense in $W_*^{-1,p}$, because every f_i in the representation of distributions in $W_*^{-1,p}$ as in (2.71) can be approximated in L^p norm. Hence, it is sufficient to prove the claim for functions in C_ν^{-1}. Then, passing to the limit in the estimate we get the claim for every $f \in W_*^{-1,p}(\Omega)$.

Suppose then that $f \in C_\nu^{-1}$; for each $x_0 \in \overline{\Omega}$ and $r < 1$, let $\theta \in C^2(\mathbf{R}^n)$ with $\theta(x) = 1$ for $|x - x_0| \le r$, $\theta(x) = 0$ for $|x - x_0| \ge \sqrt{2}r$, $|D\theta| \le cr^{-1}$ and $\langle AD\theta, \nu\rangle = 0$ in $\partial\Omega$. Such a function can be obtained in the following way: first we consider a cut-off function $\psi \in C^2(\mathbf{R}^n)$, $\psi(x) = 1$ in $B(x_0, r) \cap \Omega$ and $\psi = 0$ in $\Omega \cap (B(x_0, \sqrt{2}r))^c$, then we modify ψ in a neighborhood of the boundary making it constant in the direction $A\nu$ in order that $\langle D\psi, A\nu\rangle = 0$ in $\partial\Omega$. Finally we recover the regularity and preserve the homogeneous boundary condition by convolution with a family of mollifiers whose support is $B(0, \epsilon)$ with ϵ sufficiently small. In this way the function $w := \theta u$ satisfies the equation

$$\lambda w - \mathcal{A} w = E + F + G = g \tag{2.93}$$

where

$$\begin{aligned} E &= -\sum_{i,j=1}^n D_i(a_{ij} u D_j\theta) - \sum_{i=1}^n b_i u D_i\theta \\ F &= -\sum_{i,j=1}^n a_{ij} D_j u D_i\theta \\ G &= -\sum_{i=1}^n D_i(\theta f_i) + \sum_{i=1}^n f_i D_i\theta + \theta f_0 \end{aligned} \tag{2.94}$$

Thus, multiplying (2.93) by w and integrating by parts we get

$$\begin{aligned} \int_\Omega \langle AD(\theta u), D(\theta u)\rangle\, dx &= \int_\Omega \langle B, D(\theta u)\rangle \theta u\, dx - \int_\Omega (\lambda - c)(\theta u)^2\, dx \\ &+ \int_\Omega \langle AD\theta, D(\theta u)\rangle u\, dx - \int_\Omega \langle B, D\theta\rangle \theta u^2\, dx \\ &- \int_\Omega \langle AD\theta, Du\rangle \theta u\, dx + \sum_{i=1}^n \int_\Omega \theta f_i D_i(\theta u)\, dx \\ &+ \sum_{i=1}^n \int_\Omega f_i (D_i\theta)\theta u\, dx + \int_\Omega f_0 \theta^2 u\, dx \end{aligned} \tag{2.95}$$

We point out that in (2.95) all the integrals are on $\Omega \cap B(x_0, \sqrt{2}r)$. Now, using (2.4) and the properties of the function θ we get

$$\begin{aligned}\mu^{-1}\|Du\|^2_{L^2(\Omega\cap B(x_0,r))} \le c\big(r^{-2}\|u\|^2_{L^2(\Omega\cap B(x_0,\sqrt{2}r))} + \sum_{i=1}^n \|f_i\|^2_{L^2(\Omega\cap B(x_0,\sqrt{2}r))}\big)\\ + \int_{\Omega\cap B(x_0,\sqrt{2}r)} \langle B, Du\rangle \theta^2 u\,dx + \int_{\Omega\cap B(x_0,\sqrt{2}r)} \langle AD\theta, Du\rangle \theta u\,dx\\ + \sum_{i=1}^n \int_{\Omega\cap B(x_0,\sqrt{2}r)} \theta^2 f_i D_i u\,dx\end{aligned}$$

Finally, using the inequality $ab \le \varepsilon a^2 + \varepsilon^{-1} b^2$, we prove that there exists a constant c depending on the norm of the coefficients of $\mathcal{A}$ and on the ellipticity constant μ such that

$$\|Du\|_{L^2(\Omega\cap B(x_0,r))} \le c\big(\sum_{i=0}^n \|f_i\|_{L^2(\Omega\cap B(x_0,\sqrt{2}r))} + r^{-1}\|u\|_{L^2(\Omega\cap B(x_0,\sqrt{2}r))}\big) \tag{2.96}$$

which implies the statement for $p = 2$. By Theorem 2.4.2 applied to equation (2.93), we get

$$\begin{aligned}\|\theta u\|_{W^{1,p}(\Omega)} \le K_1\|g\|_{W^{-1,p}_*(\Omega)} \le K_1\Big(\sum_{i=0}^n \|f_i\|_{L^p(\Omega\cap B(x_0,\sqrt{2}r))}\\ + r^{-1}\big(\sum_{i,j=1}^n \|a_{ij}\|_{L^\infty} + \sum_{i=1}^n \|b_i\|_{L^\infty}\big)\|u\|_{L^p(\Omega\cap B(x_0,\sqrt{2}r))}\\ + \sum_{i,j=1}^n \|a_{ij}D_j u D_i\theta\|_{W^{-1,p}_*(\Omega)}\Big)\end{aligned} \tag{2.97}$$

By the Sobolev embedding theorem, every test function $\phi \in W^{1,p'}(\Omega)$ belongs also to $L^{q'}(\Omega)$, with $q' = np/(np - n - p)$, and $\|\phi\|_{L^{q'}(\Omega)} \le k\|\phi\|_{W^{1,p'}(\Omega)}$ with $k = k(p, \Omega)$. Therefore, by (2.96) for $2 < p \le 2n/(n-2)$ if $n > 2$ (for every p if $n \le 2$), we get

$$\begin{aligned}\|a_{ij}D_j u D_i\theta\|_{W^{-1,p}_*(\Omega)} &\le cr^{-1}\|Du\|_{L^{np/(n+p)}(\Omega\cap B(x_0,\sqrt{2}r))}\\ &\le cr^{n(\frac{1}{p}-\frac{1}{2})}\|Du\|_{L^2(\Omega\cap B(x_0,\sqrt{2}r))}\\ &\le cr^{n(\frac{1}{p}-\frac{1}{2})}\big(\sum_{i=0}^n \|f_i\|_{L^2(\Omega\cap B(x_0,2r))} + r^{-1}\|u\|_{L^2(\Omega\cap B(x_0,2r))}\big)\\ &\le c\big(\sum_{i=0}^n \|f_i\|_{L^p(\Omega\cap B(x_0,2r))} + r^{-1}\|u\|_{L^p(\Omega\cap B(x_0,2r))}\big)\end{aligned}$$

where c depends on $n, \|a_{ij}\|_\infty, p, \Omega$ and it may change from a line to the other. Summing up we find

$$\|\theta u\|_{W^{1,p}(\Omega)} \le K_2\Big(\sum_{i=0}^n \|f_i\|_{L^p(\Omega\cap B(x_0,\sqrt{2}r))} + r^{-1}\|u\|_{L^p(\Omega\cap B(x_0,2r))}\Big).$$

Since $\theta u = u$ on $\Omega \cap B(x_0, r)$ we get the statement for every $p \in [2, 2n/(n-2)]$ when $n > 2$ and for all $p \ge 2$ if $n \le 2$. Repeating the same procedure, starting from $p = \frac{2n}{n-2}$

we can prove the statement for every $p \in [2, 2n/(n-4)]$ if $n > 4$, for every p if $n \leq 4$. Thus, after $[n/2]$ steps, the proof is complete. □

The following estimate is proved by using a modification of Stewart's technique. It will be useful in order to obtain the estimate of the gradient of the solution of (2.111) in $L^1(\Omega)$.

Theorem 2.4.5. *Let $p > n$, $f \in W_*^{-1,\infty}(\Omega) \cap W_*^{-1,p}(\Omega)$; then, there exists $\omega_\infty > \omega_p$ such that for each $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda > \omega_\infty$ the solution $u \in D(E_p)$ of $\lambda u - \mathcal{A}u = f$ belongs to $W^{1,p}$ and satisfies*

$$|\lambda|^{1/2}\|u\|_{L^\infty(\Omega)} \leq K_3\|f\|_{W_*^{-1,\infty}(\Omega)}, \tag{2.98}$$

where K_3 is a constant independent of λ, u and f.

Proof. Let $x_0 \in \Omega$ and $r < 1$. Let θ be a cut-off function as the one considered in proof of Lemma 2.4.4: $\theta \in C^2(\mathbf{R}^n)$, $\theta(x) = 1$ on $B(x_0, r)$ $\theta(x) = 0$ outside $B(x_0, 2r)$ and with $\|D^\alpha\theta\|_{L^\infty(\Omega)} \leq cr^{-|\alpha|}$ for each $|\alpha| \leq 2$. As f belongs to $W_*^{-1,\infty}(\Omega)$, it admits a distributional representation $f = f_0 - \sum_{i=1}^n D_i f_i$, where $f_i \in L^\infty(\Omega)$ for each $i = 0, 1, \ldots, n$ and $\sum_{i=0}^n \|f_i\|_{L^\infty(\Omega)} \geq \|f\|_{W^{-1,\infty}(\Omega)}$. Note that $u \in W^{1,p}(\Omega)$ for $p > n$ by Theorem 2.4.2, therefore $\theta u \in W^{1,p}(\Omega)$ and solves

$$(\lambda - \mathcal{A})(\theta u) = g \tag{2.99}$$

where g is defined in (2.94). By (2.97), (2.72) and (2.92), we get

$$\begin{aligned}
\|g\|_{W_*^{-1,p}(\Omega)} &\leq K_4\left\{\|u\|_{W^{1,p}(\Omega_{x_0,2r})} + r^{-1}\|u\|_{L^p(\Omega_{x_0,2r})} + \sum_{i=0}^n \|f_i\|_{L^p(\Omega_{x_0,2r})}\right\} \\
&\leq K_5\left\{\sum_{i=0}^n \|f_i\|_{L^p(\Omega_{x_0,4r})} + r^{-1}\|u\|_{L^p(\Omega_{x_0,4r})}\right\} \\
&\leq K_6 r^{n/p}\left\{\sum_{i=0}^n \|f_i\|_{L^\infty(\Omega)} + r^{-1}\|u\|_{L^\infty(\Omega)}\right\},
\end{aligned} \tag{2.100}$$

where K_4, K_5 and K_6 are constants independent of r, λ, f and u. Since

$$W^{1,p}(\Omega_{x_0,2r}) \hookrightarrow C^0(\overline{\Omega}_{x_0,2r}) \hookrightarrow L^p(\Omega_{x_0,2r})$$

for $p > n$ and the first injection is compact, then for each $\varepsilon > 0$ we get

$$\|\theta u\|_{L^\infty(\Omega_{x_0,2r})} \leq \varepsilon r^{1-n/p}\|\theta u\|_{W^{1,p}(\Omega_{x_0,2r})} + c(\varepsilon)r^{-n/p}\|\theta u\|_{L^p(\Omega_{x_0,2r})}, \tag{2.101}$$

where $c(\varepsilon)$ is independent of r, λ, u and f (see Lemma 5.1 of [30]).

Moreover, (2.73) and the Hölder inequality imply

$$\|\theta u\|_{W_*^{-1,\infty}(\Omega_{x_0,r})} \leq c_2 r^{1-n/p}\|\theta u\|_{L^p(\Omega_{x_0,r})} \leq c_2 r\|\theta u\|_{L^\infty(\Omega)}. \tag{2.102}$$

Therefore, from (2.101) and (2.102) we get

$$r^{-2}\|\theta u\|_{W_*^{-1,\infty}(\Omega)} + r^{-1}\|\theta u\|_{L^\infty(\Omega)} \leq \varepsilon r^{-n/p}\|\theta u\|_{W^{1,p}(\Omega)} + c(\varepsilon)r^{-1-n/p}\|\theta u\|_{L^p(\Omega)}. \tag{2.103}$$

On the other hand, from (2.87)

$$|\lambda|\|\theta u\|_{W_*^{-1,p}(\Omega)} + |\lambda|^{1/2}\|\theta u\|_{L^p(\Omega)} + \|\theta u\|_{W^{1,p}(\Omega)} \le K_1\|g\|_{W_*^{-1,p}(\Omega)}. \tag{2.104}$$

Therefore, by (2.103), (2.104) and (2.100) we deduce

$$\begin{aligned} &r^{-2}\|\theta u\|_{W_*^{-1,\infty}(\Omega)} + r^{-1}\|\theta u\|_{L^\infty(\Omega)} \\ &\quad \le K_1K_6\big(\varepsilon + c(\varepsilon)r^{-1}|\lambda|^{-1/2}\big)\big(r^{-1}\|u\|_{L^\infty(\Omega)} + \sum_{i=0}^n \|f_i\|_{L^\infty(\Omega)}\big). \end{aligned}$$

Set $K_7 = 4K_1K_6$ and choose $\omega_\infty \ge \omega_p$ and $\varepsilon = K_7^{-1}$, $r = K_7c(K_7^{-1})|\lambda|^{-1/2} = K_8|\lambda|^{-1/2}$. Then, if x_0 is a maximum point for the function $|u|$ we obtain

$$K_8^{-2}|\lambda|\|\theta u\|_{W_*^{-1,\infty}(\Omega)} + \frac{1}{2}K_8^{-1}|\lambda|^{1/2}\|u\|_{L^\infty(\Omega)} \le \frac{1}{2}\sum_{i=0}^n \|f_i\|_{L^\infty(\Omega)} \le \|f\|_{W_*^{-1,\infty}(\Omega)}. \tag{2.105}$$

Thus (2.98) is proved. □

2.5 Generation of analytic semigroups in $L^1(\Omega)$

In this section we prove that the realization of uniformly elliptic operators with suitable oblique boundary conditions is sectorial in $L^1(\Omega)$ where Ω is assumed to satisfy (2.2). We consider the operator $\mathcal{A}$ in divergence form with real-valued coefficients

$$\begin{aligned} \mathcal{A}(x,D) &= \sum_{i,j=1}^n D_i(a_{ij}(x)D_j) + \sum_{i=1}^n b_i(x)D_i + c(x) \\ &= \operatorname{div}(A(x)D) + B(x)\cdot D + c(x). \end{aligned} \tag{2.106}$$

We suppose that $\mathcal{A}$ is uniformly μ-elliptic, i.e.,

$$\mu^{-1}|\xi|^2 \le \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \le \mu|\xi|^2, \quad x \in \overline{\Omega}, \xi \in \mathbf{R}^n \tag{2.107}$$

and that

$$a_{ij} = a_{ji}, \quad a_{ij},\, b_i \in W^{2,\infty}(\Omega), \quad c \in L^\infty(\Omega). \tag{2.108}$$

Actually the regularity assumption on the coefficients b_i will be weakened later. Define

$$M_1 = \max_{i,j}\{\|a_{ij}\|_{W^{2,\infty}(\Omega)}, \|b_i\|_{W^{2,\infty}(\Omega)}, \|c\|_{L^\infty(\Omega)}\}. \tag{2.109}$$

We consider the following first order differential operator acting on the boundary

$$\mathcal{B}(x,D) = \langle AD, \nu\rangle = \sum_{i=1}^n a_{ij}(x)\nu_i(x)D_j. \tag{2.110}$$

Since we would like to solve the problem in L^1 by duality from L^∞, we point out that the choice of the coefficients and the assumptions of regularity (2.108) guarantee that hypotheses in Section 2.1 hold also for $(\mathcal{A}^*, \mathcal{B}^*)$; this fact allows us to apply the results of Section 2.3 to the realization of $\mathcal{A}^*$ with homogeneous boundary conditions given by $\mathcal{B}^*$ in $L^\infty(\Omega)$.

In order to deduce a result of generation in $L^1(\Omega)$ we argue as follows. Set

$$D_{\mathcal{A}} = \{u \in L^1(\Omega) \cap C^2(\bar{\Omega});\ \mathcal{A}u \in L^1(\Omega), \mathcal{B}u = 0 \text{ in } \partial\Omega\}.$$

Lemma 2.5.1. *$\mathcal{A} : D_{\mathcal{A}} \subset L^1(\Omega) \to L^1(\Omega)$ is closable in $L^1(\Omega)$.*

Proof. Let (u_j) be a sequence in $D_{\mathcal{A}}$ such that $u_j \to 0$ and $\mathcal{A}u_j \to v$ in $L^1(\Omega)$. Then, integrating by parts,

$$\int_\Omega \varphi\, v\, dx = \lim_{j\to\infty} \int_\Omega \varphi \mathcal{A} u_j\, dx = \lim_{j\to\infty} \int_\Omega u_j \mathcal{A}^* \varphi\, dx = 0$$

for every $\varphi \in C_c^\infty(\Omega)$. Hence $v = 0$, which implies the assertion. □

By Lemma 2.5.1 we can define the realization of $\mathcal{A}$ in L^1 with boundary condition $\mathcal{B}$, (that will be denoted for simplicity by $(A_1, D(A_1))$ to be the closure of $\mathcal{A}_{|D_{\mathcal{A}}}$ in $L^1(\Omega)$, that is, the smallest closed extension of $\mathcal{A}_{|D_{\mathcal{A}}}$ in $L^1(\Omega)$. Then $D(A_1)$ is the closure of $D_{\mathcal{A}}$ with respect to the graph norm in L^1. Now we are in a position to prove the following result.

Theorem 2.5.2. *There exist $C > 0$ and $\omega_1 \in \mathbf{R}$, depending on n, μ, M_1 and Ω, such that for $\operatorname{Re}\lambda \geq \omega_1$ the problem*

$$\begin{cases} \lambda u - \mathcal{A}u = f & \text{in } \Omega \\ \mathcal{B}u = 0 & \text{in } \partial\Omega \end{cases} \tag{2.111}$$

with $f \in L^1(\Omega)$ has a unique solution $u \in L^1(\Omega)$ and

$$|\lambda| \|u\|_{L^1(\Omega)} \leq C \|f\|_{L^1(\Omega)}. \tag{2.112}$$

Proof. First of all we prove that the range of $(\lambda - A_1)$ contains the space of functions $L_c^\infty(\Omega) = \{\psi \in L^\infty(\Omega);\ \operatorname{supp}\psi \subset\subset \Omega\}$ which is dense in $L^1(\Omega)$.
Indeed, let $\pi \in C^2(\Omega)$ be such that

$$\begin{cases} \sum_{i,j=1}^n |D_{ij}\pi| + \sum_{i=1}^n |D_i \pi|^2 \leq c \\ e^{-\pi} \in L^1(\Omega) \\ \langle AD\pi, \nu \rangle = 0 \qquad \text{in } \partial\Omega \end{cases}$$

Moreover, if Ω is unbounded, we also require that $\lim_{|x|\to\infty, x\in\Omega} \pi(x) = +\infty$. Such a π exists. For instance, when $\Omega = \mathbf{R}^n$ one can choose $\pi(x) = \sqrt{1 + |x|^2}$. In the general case one can adapt the previous example modifying π near the boundary in a suitable way.

We define $\Pi(x) = \exp[\pi(x)]$. Then, for every function $\psi \in L_c^\infty(\Omega)$, we get $\Pi\psi \in L_c^\infty(\Omega)$ and

$$\begin{cases} \lambda u - \mathcal{A}u = \psi \in L_c^\infty(\Omega) \\ \mathcal{B}u = 0 \quad \text{in } \partial\Omega \end{cases}$$

if and only if

$$\begin{cases} \lambda \Pi u - \mathcal{A}_\pi(\Pi u) = \Pi\psi \in L_c^\infty(\Omega) \\ \mathcal{B}(\Pi u) = 0 \quad \text{in } \partial\Omega \end{cases} \tag{2.113}$$

where

$$\mathcal{A}_\pi = \mathcal{A} - 2\sum_{i,j=1}^n a_{ij}D_i\pi D_j + \Big(\sum_{i,j=1}^n (D_i(a_{ij}D_j\pi) - a_{ij}D_i\pi D_j\pi) + \sum_{i=1}^n b_i D_i\pi\Big).$$

As it is easily seen, the operator $\mathcal{A}_\pi$ satisfies the assumptions (2.3)-(2.4); moreover, since $\mathcal{A}_\pi^0(x,\xi) = \mathcal{A}^0(x,\xi)$ then $\mathcal{A}_\pi$ satisfies also the root and the complementing conditions. Therefore, by applying Theorem 2.3.3 we get that there exists $\Pi u \in D((A_\pi)_\infty^B) \subseteq L^\infty(\Omega)$ solution of (2.113).
Hence $u \in \{v \in C^1(\bar{\Omega}) \cap L^1(\Omega); \mathcal{A}v \in L^1(\Omega)\}$ and ψ is therefore in the range of $(\lambda - A_1)$.
Now we prove (2.112). Let consider u solution of $\lambda u - \mathcal{A}u = f \in L^1(\Omega)$ and let

$$\mathcal{A}^* = \sum_{i,j=1}^n D_j(a_{ij}D_i) - \sum_{j=1}^n b_j D_j + (c - \operatorname{div} b)$$

Then, from Theorem 2.3.3, it follows that $(A^*)_\infty^{B^*}$ with oblique boundary conditions $\mathcal{B}^*(x,D) = \langle A(x)D, \nu(x)\rangle - \langle B(x), \nu(x)\rangle = 0$ generates an analytic semigroup in $L^\infty(\Omega)$ and so the elliptic problem

$$\begin{cases} \lambda w - \mathcal{A}^* w = \varphi \in L^\infty(\Omega) \\ \mathcal{B}^* w = 0 \quad \text{in } \partial\Omega \end{cases} \tag{2.114}$$

has a unique solution $w \in D((A^*)_\infty^{B^*})$ for $\operatorname{Re}\lambda$ sufficiently large. Moreover, taking $\operatorname{Re}\lambda$ sufficiently large we get

$$|\lambda|\|w\|_{L^\infty(\Omega)} + |\lambda|^{1/2}\|Dw\|_{L^\infty(\Omega)} \le \tilde{K}\|\varphi\|_{L^\infty(\Omega)}.$$

Now, we can apply the method used in Pazy (see [35]) to obtain

$$\begin{aligned} \|u\|_{L^1(\Omega)} &= \sup\Big\{\int u(x)\varphi(x)dx;\ \varphi \in L_c^\infty(\Omega),\ \|\varphi\|_{L^\infty(\Omega)} \le 1\Big\} \\ &\le \sup\Big\{\int u(x)(\lambda - \mathcal{A}^*)w_\varphi dx;\ w_\varphi \in L^\infty(\Omega)\,\text{solution of (2.114)},\ \|\varphi\|_{L^\infty(\Omega)} \le 1\Big\} \\ &\le \sup\Big\{\int w_\varphi(\lambda - \mathcal{A})u dx;\ w_\varphi \in L^\infty(\Omega)\,\text{solution of (2.114)},\ \|\varphi\|_{L^\infty(\Omega)} \le 1\Big\} \end{aligned}$$

in particular,

$$\|u\|_{L^1(\Omega)} \le \tilde{K}|\lambda|^{-1}\|f\|_{L^1(\Omega)}.$$

So, $(\lambda - A_1)$ is an injective operator with closed range in $L^1(\Omega)$ and the proof is complete. □

The following theorem establishes further properties of the resolvent operator.

Theorem 2.5.3. *Under the assumptions of Theorem 2.5.2, there exist $\omega_1' \geq \omega_1$, $K' \geq K$ and $\theta_1' \in (\pi/2, \theta_1)$ depending on n, μ, M_1 and Ω such that for every λ such that $|\arg(\lambda - \omega_1')| < \theta_1'$, the solution of* (2.111) *satisfies*

$$|\lambda|^{1/2}\|Du\|_{L^1(\Omega)} \leq K'\|f\|_{L^1(\Omega)}. \tag{2.115}$$

Proof. Let $\phi = \mathrm{div}\psi$ with ψ any function in $L^\infty(\Omega, \mathbf{R}^n)$. By the estimate (2.98) we know that for λ with $\mathrm{Re}\,\lambda > \omega_\infty$, the solution of the following problem

$$\begin{cases} \lambda v - \mathcal{A}^* v = \mathrm{div}\psi \\ \mathcal{B}^* v = 0 \quad \text{ on } \partial\Omega \end{cases} \tag{2.116}$$

satisfies

$$|\lambda|^{1/2}\|v\|_{L^\infty(\Omega)} \leq K_3\|\mathrm{div}\psi\|_{W_*^{-1,\infty}(\Omega)}. \tag{2.117}$$

We notice that

$$\|\mathrm{div}\psi\|_{W_*^{-1,\infty}} = \sup\{\langle \mathrm{div}\psi, \varphi\rangle :\ \varphi \in W^{1,1}(\Omega),\ \|\varphi\|_{W^{1,1}(\Omega)} \leq 1\} \leq \|\psi\|_{L^\infty}. \tag{2.118}$$

Now, if u is the solution of (2.111), we get

$$\begin{aligned}
\|Du\|_{L^1(\Omega)} &= \sup\left\{\int_\Omega \langle Du(x), \psi(x)\rangle dx :\ \psi \in C_c^\infty(\Omega; \mathbf{R}^n),\ \|\psi\|_{L^\infty(\Omega)} \leq 1\right\} \\
&= \sup\left\{\int_\Omega u(x)\,\mathrm{div}\psi(x) dx :\ \psi \in C_c^\infty(\Omega; \mathbf{R}^n),\ \|\psi\|_{L^\infty(\Omega)} \leq 1\right\} \\
&\leq \sup\left\{\int_\Omega u(x)\,\mathrm{div}\psi(x) dx : \psi \in C_c^\infty(\Omega; \mathbf{R}^n),\ \|\mathrm{div}\psi\|_{W_*^{-1,\infty}(\Omega)} \leq 1\right\} \\
&= \sup\left\{\int_\Omega u\,(\lambda - \mathcal{A}^*)\,v_\psi\,dx :\ v_\psi \text{ solution of (2.116)},\ \|\mathrm{div}\psi\|_{W_*^{-1,\infty}(\Omega)} \leq 1\right\} \\
&= \sup\left\{\int_\Omega [(\lambda - \mathcal{A})u]\,v_\psi\,dx :\ v_\psi \text{ solution of (2.116)},\ \|\mathrm{div}\psi\|_{W_*^{-1,\infty}(\Omega)} \leq 1\right\} \\
&\leq C\sup\left\{\|f\|_{L^1(\Omega)}\|v_\psi\|_{L^\infty(\Omega)} :\ v_\psi \text{ solution of (2.116)},\ \|\mathrm{div}\psi\|_{W_*^{-1,\infty}(\Omega)} \leq 1\right\}.
\end{aligned} \tag{2.119}$$

Now, taking into account (2.117), we get

$$\|Du\|_{L^1(\Omega)} \leq K'|\lambda|^{-1/2}\|f\|_{L^1(\Omega)}.$$

□

As a consequence of Theorem 2.5.2 we have that A_1 is sectorial, that is there exist $K \in \mathbf{R}$ and $\theta_1 \in (\pi/2, \pi)$ such that

$$\Sigma_{\theta_1,\omega_1} = \{\lambda \in \mathbf{C};\ \lambda \neq \omega_1, |\arg(\lambda - \omega_1)| < \theta_1\} \subset \rho(A_1)$$

and

$$\|R(\lambda, A_1)\|_{\mathcal{L}(L^1(\Omega))} \leq \frac{K}{|\lambda - \omega_1|}$$

holds for each $\lambda \in \Sigma_{\theta_1,\omega_1}$.