0) *Background*(*)

Let G be a *finite directed graph.*

If v,w are two vertices of G, we use the symbol $v \to w$ (resp. $v \not\to w$) to denote that vw is(resp. is not) a directed edge of G. if $v \to w$, we call v a *predecessor* of w and w a *successor* of v.

If, for all $v,w \in G$, we have $(v \to w) \iff (w \to v)$, the graph is called *undirected.*

Let S be a *topological space.*

Given a function $f : S \to G$ from S to G, we denote by capital letter V the set of the f-counterimages of $v \in G$, and if we must display the function f, we put $V^f = f^{-1}(v)$.

A function $f : S \to G$ is called *o-regular* (resp. *o*-regular*) if, for all $v,w \in G$ such that $v \neq w$ and $v \not\to w$, it is $V \cap \bar{W} = \phi$ (resp. $\bar{V} \cap W = \phi$). (See Definition 3).

That is equivalent to saying:

$(v \neq w, V \cap \bar{W} \neq \phi$ and f o-regular$) \Longrightarrow v \to w$

$(v \neq w, V \cap \bar{W} \neq \phi$ and f o^*-regular$) \Rightarrow w \to v$.

A function $f : S \to G$ is called strongly *o-regular* (resp. *strongly o*-regular*) if:

i) f is o-regular (resp. o^*-regular);

ii) for all $v,w \in G$ such that $v \neq w$, $v \not\to w$ and $w \not\to v$ it follows $\bar{V} \cap \bar{W} = \phi$. (See Definition 4).

Let $I = [0,1]$ be the unit interval in R^1. Two o-regular (resp. o^*-regular) functions $f,g : S \to G$ are called *o-homotopic* (resp. *o*-homotopic*) if there exists an o-regular (resp. o^*-regular) function $F : S \times I \to G$ such that, for all $x \in S$, $F(x,0) = f(x)$ and $F(x,1) = g(x)$. The function F is called

(*) The references are ralative to [2]

an *o-homotopy* (resp. *o^*-homotopy*) between f and g. (See Definition 5).

The previous o-homotopy (resp. o^*-homotopy) relation is an equivalence relation in the set of o-regular (resp. o^*-regular) functions from S to G. We denote by $Q(S,G)$ (resp. $Q^*(S,G)$ *the set of the o-homotopy* (resp. *o^*-homotopy*) *classes* of o-regular (resp. o^*-regular) functions.

The graph G^* with the same vertices of G and such that ($u \to v$ in G) $\Longleftrightarrow$ $\Longleftrightarrow$ ($v \to u$ in G^*) is called the *dually directed graph* as regards G. (See Definition 6). Hence, we have:

DUALITY PRINCIPLE. - *Every true proposition in which appear the concepts of o-regularity, o^*-regularity, strongly o-regularity, strongly o^*-regularity, o-homotpoty, o^*-homotopy,* Q(S,G), $Q^*(S,G)$, *remains true if the concepts of o-regularity and o^*-regularity (strongly o-regularity and strongly o^*-regularity, o-homotopy and o^*-homotopy,* Q(S,G) *and* $Q^*(S,G)$ *are interchanged throught the statement of the proposition.*

Moreover, let S' be *a subspace* of S and G' *a subgraph* of G. A function f from the pair S,S' to the pair G,G' is called *o-regular* (resp. *o^*-regular, strongly o-regular, strongly o^*regular*) if both the function $f : S \to G$ and its restriction $f' = f_{/S'} : S' \to G'$ are o-regular (resp. o^*-regular, etc.) functions. (See Definition 7).

Two o-regular (resp. o^*-regular) functions $f,g : S,S' \to G,G'$ are called *o-homotopic* (resp. *o^*-homotopic*), if there exists an o-regular (resp. o^*-regular) function $F : S \times I, S' \times I \to G,G'$, such that for all $x \in S$, $F(x,0)=f(x)$ and $F(x,1) = g(x)$. The function F is called an *o-homotopy* (resp. *o^*-homotopy*) between f and g. (See Definition 8).

In [2] we proved the following results:

R.1.- Let f be an o-regular function from a normal topological space S to a finite directed graph G and Y a closed subset of S. Then, if for $a \in G$ we have $A^f \cap Y = \phi$ and $\overline{A^f} \cap Y \neq \phi$ there exists an o-regular function $g : S \to G$, which is o-homotopic to f and such that $A^g \cap Y = \phi$ (See Lemma 6).

R.2 - In the construction of R.1, if there exist n vertices $p_1, \dots, p_n \in G$, such that $\overline{P_1^f} \cap \dots \cap \overline{P_n^f} = \phi$ then also it follows $\overline{P_1^g} \cap \dots \cap \overline{P_n^g} = \phi$. (See Corollary 7).

R.3.- Let f be an o-regular function from S,S' to G,G', where S is a normal topological space, S' a closed subspace of S, Y a closed subset of S', G a finite directed graph and G' a subgraph of G. Then, if for $a \in G$ we have $A^f \cap Y = \phi$ and $\overline{A^f} \cap Y = \phi$, there exists an o-regular function $g : S,S' \to G,G'$, which is o-homotopic to f and such that $\overline{A^g} \cap Y = \phi$. (See Lemma 11).

R.4.- In the construction of R. 3, if there exist n vertices $p_1, \dots, p_n \in G$ and m vertices $q_1, \dots, q_m \in G'$, such that $\overline{P_1^f} \cap \dots \cap \overline{P_n^f} \cap \overline{Q_1^{f'}} \cap \dots \cap \overline{Q_m^{f'}} = \phi$, then also it follows that $\overline{P_1^g} \cap \dots \cap \overline{P_n^g} \cap \overline{Q_1^{g'}} \cap \dots \cap \overline{Q_m^{g'}} = \phi$. While, from $\overline{P_1^f} \cap \dots \cap \overline{P_n^f} \cap \dots = \phi$, it results $\overline{P_1^g} \cap \dots \cap \overline{P_n^g} \cap S' = \phi$. (See Corollary 12).

By Duality Principle, the results dual to the previous ones are also true for o^*-regular functions.

1) Headed and totally headed subsets of a graph

DEFINITION 1.- Let G be a directed graph and X a non-empty subset of G. A vertex of X is called a **head** (resp. a **tail**) of X in G, if it is a