where

$$c'(\bar{t}) = \begin{cases} \dfrac{||u_\circ||_\infty}{r} + \left(\dfrac{||u_\circ||_\infty}{r} b + d\ r \right) \dfrac{e^{b\bar{t}}-1}{b} & \text{if} \quad b \neq 0 \\ \dfrac{||u_\circ||_\infty}{r} + d\ r\ \bar{t} & \text{if} \quad b = 0 \end{cases}$$

In each case we can have $c'(\bar{t}) < 1$ provided $r > ||u_\circ||_\infty$. $\blacksquare$

Hence we can conclude with the following

THEOREM (2). If $u_\circ \in X_\circ^+ \cap X_\infty$ and $r > ||u_\circ||_\infty$ then the equation (13) has a unique local solution $u \in S^+(r)$.

Remark (3) For fixed $\bar{t}$ and $r = ||u_\circ||_\infty$ we can always choose a so as P_1 maps $S(r)$ into itself. In fact if $g \in S(r)$ we have

$$||P_1 g)(t)||_\infty \leq r + r(b+dr) \ \frac{e^{b\bar{t}}-1}{b} \qquad \text{for} \quad t \in [0,\bar{t}]$$

and so

$$||(P_1 g)\ (t)||_\infty \leq r \qquad \text{for} \quad t \in [0,\bar{t}]$$

when

$$b+dr \leq 0 \quad , \text{ i.e.} \qquad a \geq \frac{1}{T} + dr \quad .$$

Nevertheless this result does not enable us to improve theorem (2) by removing the condition $r > ||u_\circ||_\infty$, because it was used in theorem (1), from which the theorem (2) comes.

In other words given $[0,\bar{t}]$ and $r = ||u_\circ||_\infty$, P_1 maps $S(r)$ into itself but P_1 can be noncontractive.

5. Global mild solution.

As in [1] we introduce the functional

$$J\ f = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dv \int_{V_1}^{V_2} f(x,v,w)dw \qquad \text{for} \quad f \in \mathcal{X}.$$

We have:

Lemma (9). (a) $|Jf| \leq ||f||$ and so $J \in X^* = B(X;R)$

(b) $J\, Z_o(t)f = J\, f$ for $f \in X_o$ and $t \geq 0$

(c) $J\, F(f) = 0$ for $f \in X_o \cap X_\infty$.

Applying the functional J to (10) we have

$Ju(t) = J\, u_o$ for $t \in [0,\bar{t}]$

and because the solution is positive if $u_o \in X_o^+ \cap X_\infty$

we have

(16) $||u(t)|| = ||u_o||$ for $t \in [0,\bar{t}]$.

The physical meaning of this result is the invariability of the total number of the vehicles on the motorway with respect to the time. This is natural because the motorway is supposed to have no entrances or exits.
This fact allowed us to obtain the global solution in [1], but here we need information about $||u(t)||_\infty$ and not about $||u(t)||$.
We can prove that P is strictly contractive over S(r) also with respect to the norm $||\cdot||_\infty$, using the inequalities

$||F(f) - F(g)||_\infty \leq \frac{c^3+d}{2}(||f||_\infty+||g||_\infty)||f-g||_\infty$ for $f,g \in X_o \cap X_\infty$ and for $u,w \in S(r)$

$||P(u)-P(w)||_\infty \leq ||u-w||_\infty(c^3+d)r\, T(e^{t/T}-1)$. Then by well-known techniques(see e.g. [8] pag. 48) we can obtain that

$$||u(t)||_\infty \leq \frac{||u_o||_\infty\, e^{\bar{t}/T}}{1 - d\, T\, ||u_o||_\infty\, (e^{\bar{t}/T} -1)} \qquad \text{for} \qquad t \in [0,\bar{t}]$$

provided that

$$e^{\bar{t}/T} < \frac{1}{d\, T||u_o||_\infty} + 1 \qquad \text{and} \qquad ||u_o||_\infty \neq 0 \, .$$

From this last inequality it is clear that the time interval $[0,\bar{t}]$ will increase when $||u_o||_\infty$ decreases.

This result is also justified from the physical point of view,because once the are more than a suitable number of vehicles at each point of the motorway there will be a traffic jam.

In particular for $||u_o||_\infty = 0$ (i.e. $u_o(x,v,w) = 0$ a.e.) we have the global solution (i.e. for $t \in [0,\infty)$)

6. Connexion with the mollified problem.

As we said in the introduction, the mollified version of the problem (1) has a unique strict global solution $u_\varepsilon(t)$ and if $u_\circ \in X_\circ^+ \cap X_\infty$ we have

$$u_\varepsilon(t) = Z_\circ(t)u_\circ + \int_0^t Z_\circ(t-s)\, F_\varepsilon(u_\varepsilon(s))ds \qquad \text{for } t \geq 0$$

If $[0,\bar{t}]$ is the existence interval for the solution of the problem (13), we have for $t \in [0,\bar{t}]$:

$$(17) \quad ||u_\varepsilon(t) - u(t)|| \leq \int_0^t ||F_\varepsilon(u_\varepsilon(s)) - F(u(s))||ds.$$

The aim of this section is to prove the following

THEOREM (3). If $u_\circ \in X_\circ^+ \cap X_\infty$, $u(t)$ is the mild solution of the problem (1) in the interval $[0,\bar{t}]$ and $u_\varepsilon(t)$ is the strict global solution of the mollified version of the problem (1), then we have

$$(18) \quad \lim_{\varepsilon \to 0+} ||u_\varepsilon(t)-u(t)|| = 0 \qquad \text{uniformly in } t \in [0,\bar{t}] .$$

PROOF

If $f,g \in X_\circ \cap X_\infty$ then

$$(19) \quad ||F_\varepsilon(f) - F(g)|| \leq ||F_\varepsilon(f) - F_\varepsilon(g)||+||F_\varepsilon(g)-F(g)|| \leq 2\delta(||f|| + ||g||)||f-g|| + ||F_\varepsilon(g)-F(g)||$$

where $\delta = (v_2-v_1)\, ||k_\varepsilon||_\infty$ (see [1]).

Since we proved that the norm of the solution is invariable both in [1] and in this paper (see (16)), we have

$$||u_\varepsilon(t)|| = ||u_\circ|| = ||u(t)|| \qquad \text{for } t \in [0,\bar{t}]$$

and then, from (17) and (19)