One can easily verify that every element of $\mathcal{C}$ is a right tail of S preordered with respect to relation defined in (j). On the other hand if $(S,\lesssim)$ is a preordered set and $\mathcal{R}$ is the set of the right tails of $(S,\lesssim)$ then for every $x,y \in S$ the following properties hold

(e) $x \lesssim y$ iff $\mathfrak{x} \lesssim \mathfrak{y}(\mathcal{R})$ according to (j), and $x \lesssim y$ iff $\mathfrak{x} \lesssim \mathfrak{y}(\mathcal{R}_0)$,

where $\mathcal{R}_0$ is the set of the principal filters of $(S,\leq)$ generated by an elment of S. And hence, as a consequence of (jj), the following property holds:

(ee) $x \leq y$ iff $\mathfrak{y} \leq \mathfrak{x}\ (\mathcal{R}'_0)$,

where $\mathcal{R}'_0$ is the set of all set complements in S of the elements of $\mathcal{R}_0$.

Now we can give the following

THEOREM 1. Let $\mathcal{C} \leq \mathcal{P}(S)$ be, $Y \in \mathcal{C}$ and $y \in Y$. Then Y is a point closure in $\mathcal{C}$ with respect to y iff y is minimum in Y with respect to relation defined in (j).

PROOF. In fact: y is minimum in $Y \Longleftrightarrow \forall x \in Y : \mathcal{C}_y \subseteq \mathcal{C}_x \Longleftrightarrow \forall x \in Y$, $\forall Z \in C_y : Z \in \mathcal{C}_x \Longleftrightarrow \forall x \in Y, \forall Z \in \mathcal{C}_y : x \in Z \Longleftrightarrow \forall Z \in \mathcal{C}_y : Y \subseteq Z$.

Q.E.D.

N. 3. A CHARACTERIZATION OF V-PRIME AND STRONGLY V-PRIME ELEMENTS OF A POSET.

Henceforth let $(S,\leq)$ be a poset. Then we can consider the function $g : S \to \mathcal{P}(S)$ mapping an element $x \in S$ into the set $g(x) = \{y \in S : x \not\leq y\} = S - r(x)$. Clearly f is an injective function; moreover $\forall x,y \in S$ $x \leq y$ iff $g(x) \leq$

$\leq g(y)$, in fact $x \leq y$ iff the principal filter $r(x)$ includes the principal filter $r(y)$, and hence $x \leq y$ iff $g(x) = S - r(x) \subseteq S - r(y) = g(y)$; thus g is an isomorphism from $(S,\leq)$ onto $(g(S),\subseteq)$. Now we want to prove the following.

THEOREM 2. The ordered pair $((g(S),\subseteq),g)$ is a U-proper set representation of $(S,\leq)$.

PROOF. Since g is an isomorphism from $(S,\leq)$ onto $(g(S),\subseteq)$ we need only prove that $(g(S),\subseteq)$ is a U-proper set poset. Thus if $Y \subseteq S$ then $\bigcap_{y\in Y} r(y) = r(Y) = \bigcup_{x\in r(Y)} r(x)$ (see (ii) of N. 2). Whence $\bigcup_{y\in Y}(S-r(y)) = S-r(Y) =$

$= \bigcap_{x\in r(Y)}(S-r(x)) = \bigcap_{x\in r(Y)} g(x)$.

In particular if $Y = \{y_1,\ldots,y_n\}$ and $A_i = g(y_i) = S-r(y_i)$ $(i=1,\ldots,n)$

then

$$\bigcup_{i=1}^{n} A_i = \bigcap_{x\in r(Y)} g(x)$$

and hence $\bigcup_{i=1}^{n} A_i$ is equal to the set intersection of all $g(x)$ such that

$g(x) \supseteq \bigcup_{i=1}^{n} A_i$. Q.E.D.

Now we can give the following

THEOREM 3. Let c be an element of S and let c be not minimum in $(S,\leq)$. Then c is strongly v-prime in $(S,\leq)$ iff for ever U-proper set representation $((f(S),\subseteq),f)$ of $(S,\leq)$ $f(c)$ is a point closure in $f(S)$.

PROOF. Let $f(c)$ be a point closure in $f(S)$ for every U-proper set representation $((f(S),\subseteq),f)$. Now then we consider the U-proper set representation of theorem 2; thus, as a consequence of the fact that c is not minimum in $(S,\leq)$, the set $\{y\in S: c \nleq y\}$ is different from $\emptyset$ and it is a point closure in $g(S)$.

Hence, as a consequence of theorem 1 and of (ee) in N. 2, $g(c)$ has a maximum in $(S,\leq)$, therefore c is a strongly v-prime element of $(S,\leq)$.

Conversely let c be a strongly v-prime element in $(S,\leq)$; in such a case the set $\{y\in S: c\nleq y\}$ has a maximum b. Now let $((f(S),\subseteq),f)$ be an arbitrary u-proper set representation of $(S,\leq)$. Since $c\nleq b$ then $f(c)\nsubseteq f(b)$ and hence we can consider an element $x\in f(c)-f(b)$. We want to prove that $f(c)$ is a point closure with regard to x. In fact if Z is an element of $f(S)$ such that $x\in Z$ then there exists $z\in Z$ such that $f(z) = Z$. Now then $c\leq z$; in fact assume, ab absurdo, that $c\nleq z$, then $z\leq b$, thus $Z = f(z) \subseteq f(b)$ and hence $x\in f(b)$. Since $c\leq z$ iff $f(c) \subseteq f(z) = Z$, the theorem follows.

Q.E.D.

THEOREM 4. An element $c \in S$ is v-prime in $(S,\leq)$ iff a $\cup$-proper set representation $((f(S),\leq),f)$ of $(S,\leq)$ exists such that $f(c)$ is a point closure in $f(S)$.

PROOF. Let $((f(S),\leq),f)$ be a $\cup$-proper set representation of $(S,\leq)$ such that $f(c)$ is a point closure in $f(S)$ with regard to x; moreover let $y,z \in S$ such that $c\nleq y$ and $c\nleq z$. Then $(f(c)\nsubseteq f(y)$ and $f(c)\nsubseteq f(z)$, thus $x\notin f(y)$ and $x\notin f(z)$; moreover (since $(f(S),\subseteq)$ is a $\cup$-proper set poset) $f(y)\cup f(z)$ is equal to set intersection of all $f(t)$ including $f(y)$ and $f(z)$. As a consequence an element $t\in S$ exists such that $f(t)\supseteq f(z)\cup f(y)$ and $x\notin f(t)$, thus $f(c)\nsubseteq f(t)$ and hence $c\nleq t$ but $y\leq t$ and $z\leq t$. That means that the subset $\{s\in S : c\nleq s\}$ is v-directed .

Convensely let c be a v-prime element in $(S,\leq)$ and $((f(S),\subseteq),f)$ a $\cup$-proper set representation of $(S,\leq)$. If $f(c)$ is a point closure in $f(S)$ we have nothing to prove. If not let us consider the set $X' = X\cup\{X\}$, ((cfr. [1], p. 62, proof of theorem 31), where $X = \bigcup_{t \in S} f(t)$, and the function $f' : S \to \mathcal{P}(X')$ such that for every $s \in S$ $f'(s)=f(S)$ iff $c \not\leq s$ and $f'(s) = f(s) \cup \{X\}$ iff $c \leq f(s)$. Clearly f' is an injective function and for every, $s_1, s_2 \in S$ $s_1 \leq s_2$ iff $f'(s_1) \subseteq f'(s_2)$, moreover $f'(c)$ is point closure with regard to X in $f'(S)$. Then we must only prove that $((f'(S), \subseteq), f')$ is a $\bigcup$-proper set representation of $(S, \leq)$.

Now if $s_1, \ldots, s_n \in S$ then $\bigcup_{i=1}^{n} f(s_i)$ is equal to the set intersection of all the elements of $f(S)$ that include every $f(s_i)$. Hence let's consider the following two cases:

CASE 1 : for every $i = 1, \ldots, n$: $c \not\leq s_i$. Then for every $i = 1, \ldots, n$ $f'(s_i) = f(s_i)$, moreover (since c is v-prime in $(S, \leq)$) an element $s \in S$ exists such that $c \not\leq s$ and for every $i = 1, \ldots n$ $s_i \leq s$, thus $f'(s) = f(s)$ hence $\bigcup_{i=1}^{n} f'(s_i) (= \bigcup_{i=1}^{n} f(s_i))$ is equal to the set intersection of all the elements of $f'(S)$ that include every $f'(s_i) (= f(s_i))$.

CASE 2 : for some i: $c \leq x_i$. Then $f'(s_i) = f(s_i) \cup \{X\}$; moreover for every $s \in S$ such that $f'(s)$ includes every $f'(s_i)$ one has $f'(s) \supseteq f'(s_i)$, thus $c \leq s_i \leq s$ and hence $f'(s) = f(s) \cup \{X\}$. Then in this case too we can conclude that $\bigcup_{i=1}^{n} f'(s_i) (= \bigcup_{i=1} f(s_i) \cup \{x\})$ is equal to the set intersection of all the element of $f'(S)$ including every $f(s_i)$.

Q.E.D.

REMARK.

We observe that if $(S, \leq)$ has at least a v-prime element then a $\bigcup$-proper set representation $((f(S), \subseteq), f)$ of $(S, \leq)$ exists such that f maps every v-prime element of $(S, \leq)$ in a point closure. In fact let A be the set of all v-prime elements of $(S, \leq)$, $((f(S), \subseteq), f)$ a $\bigcup$-proper set representation of $(S, \leq)$ and B the set of all the elements of A

mapped into a point closure. If $B = A$ we have nothing to prove. Now we suppose that $B \neq A$ and

(m) $$(\bigcup_{s\in S} f(s)) \cap (A-B) = \emptyset \quad (3).$$

Then we consider the function f' that maps every $s \in S$ into the set $f(s) \cup A_s$, where $A_s = \{y \in A - B : y \leq s\}$. Clearly f' is an injective function and for every $s_1, s_2 \in S$ $s_1 \leq s_2$ iff $f(s_1) \cup A_{s_1} \subseteq f(s_2) \cup A_{s_2}$. Moreover if $s_1, \dots, s_n$ are arbitrary elements of S then

$$\bigcup_{i=1}^{n} (f(s_i) \cup A_{s_i}) = (\bigcup_{i=1}^{m} f(s_i)) \cup (\bigcup_{i=1}^{n} A_{s_i}).$$

Now let Z be the set of all upper bounds of $\{s_1, \dots, s_n\}$ in $(S, \leq)$. We want to prove that $\bigcap_{z\in Z} (f(z) \cup A_z) = \bigcup_{i=1}^{n} (f(s_i) \cup A_{s_i}) = \bigcup_{i=1}^{n} (f(s_i)) \cup (\bigcup_{i=1}^{n} A_i)$.

As a consequence of condition (m) $\bigcap_{z\in Z} (f(z) \cup A_z) = (\bigcap_{z\in Z} f(z)) \cup (\bigcap_{z\in Z} A_z)$; moreover we already know that $\bigcap_{z\in Z} f(z) = \bigcup_{i=1}^{n} f(s_i)$ and $\bigcap_{z\in Z} A_z \supseteq \bigcup_{i=1}^{n} A_{s_i}$; then it is sufficient to prove that $\bigcap_{z\in Z} A_z \subseteq \bigcup_{i=1}^{n} A_{s_i}$.

Now if $x \in \bigcap_{z\in Z} A_z$ then x is a v-prime element of $(S, \leq)$ such that $x \leq z$ for every $z \in Z$. Moreover Z is the set of all upper bounds of $\{s_1, \dots, s_n\}$, then as a consequence of the definition of v-prime element $x \leq s_i$ for some $i \in \{1, \dots, n\}$, thus $x \in \bigcup_{i=1}^{n} A_{s_i}$ and hence $\bigcap_{z\in Z} A_z \subseteq \bigcup_{i=1}^{n} A_{s_i}$. From this the enounced assertion follows.

REFERENCE

[1] D.DRAKE and W.J.THRON "On the representations of an abstract lattice as the family of closed sets of a topological space". Trans.of Amer. Math. Soc. 120(1965), 57-71.

(3) The case $(\bigcup_{s\in S} f(s)) \cap (A-B) \neq \emptyset$ can easily be reconducted to condition -)