7. ASYMPTOTIC BEHAVIOUR OF $\Psi(\alpha,\beta,\gamma;x)$ for $x \to \infty$.

In order to derive the asymptotic behaviour of $\Psi(\alpha,\beta,\gamma;x)$ for large x, we recall for the sake of convenience the following generalization of Poincaré's definition of a asymptotic expansion [21] :

DEFINITION 7.1 " A sequence $\{\phi_s(x)\}$ of functions such that

$$(7.1) \qquad \frac{\phi_{s+1}(x)}{\phi_s(x)} \to 0$$

for $x \to +\infty$ and any $s = 0,1,2, \ldots,$ is called an asymptotic sequence or scale as $x \to +\infty$.

Consider now a scale $\{\phi_s(x)\}$ as $x \to +\infty$, and let $f(x)$, $f_n(x)$ $(n = 0,1,2,\ldots)$ be functions such that for every non-negative integer N, the quantity

$$(7.2) \qquad \left| \frac{f(x) - \sum_{s=0}^{N-1} f_s(x)}{\phi_N(x)} \right|$$

is bounded for $x \to +\infty$.

Then the series $\sum_{s=0}^{\infty} f_s(x)$ is said to be a generalized asymptotic expansion with respect to the scale $\{\phi_s(x)\}$, and one writes

$$(7.3) \qquad f(x) \sim \sum_{s=0}^{\infty} f_s(x); \quad \{\phi_s(x)\} \quad \text{as} \quad x \to +\infty \text{ ."}$$

The following theorem holds:

THEOREM 7.2 "The function

$$(7.4)\qquad \psi(\alpha,\beta,\gamma;x) = \int_x^\infty dt\ t^{\alpha-1}\left[1-\left(1-\frac{e^{-t}}{t^\beta}\right)^\gamma\right],\ (x>0)$$

admits the asymptotic behaviour

$$(7.5)\qquad \psi(\alpha,\beta,\gamma;x) \sim x^{\alpha-\beta-1}\, e^{-x} \sum_{s=0}^{\infty} A_s(x)\ \frac{1}{x^s}\ ,$$

for $x \to +\infty$, in the generalized sense of Poincaré (see Def. 7.1) with respect to the scale $\{\frac{1}{x^s}\}$, where

$$(7.6)\qquad A_s(x) = (-1)^{s+1}\ \frac{1}{\Gamma(-\gamma)} \sum_{m=0}^{\infty} \frac{\Gamma(m+1-\gamma)\Gamma(s-\alpha+(m+1)\beta+1)e^{-mx}}{(m+1)!(m+1)^{s+1}\Gamma(-\alpha+(m+1)\beta+1)x^{m\beta}}$$

the series on the right of (7.6) being uniformly convergent for any $x > \bar{x}$, where $\bar{x}$ is such that the inequality $e^{-x} < x^\beta$ is verified".

The proof of this theorem will be obtained with the help of a few lemmas. More specifically;

Lemma 7.3. " The following inequality holds:

$$(7.7)\qquad \left|(1+y)^{\alpha-1} - \sum_{m=1}^{N-1}(-1)^{m-1}\ \frac{\Gamma(m-\alpha)}{\Gamma(1-\alpha)}\ \frac{y^{m-1}}{(m-1)!}\right| \le$$

$$\le \frac{1}{(N-2)!}\left|\frac{\Gamma(N-\alpha)}{\Gamma(1-\alpha)}\right| y^{N-1}\ e^{|\alpha-N|y}\ ,$$

where $y \ge 0$ ".

Proof. Recall first that, as is known, if f(y) is a function having continuous derivatives up to the (N-1)-th order enclosed, then

$$(7.8)\quad f(y) - \sum_{k=0}^{N-2} \frac{f^{(k)}(0)}{k!} y^k = \frac{1}{(N-2)!} \int_0^y f^{(N-1)}(t)(y-t)^{N-2} dt.$$

If we deal with the case $f(y) = (1+y)^{\alpha-1}$ and put k=m-1, Eq. (7.8) gives rise to the inequality

$$(7.9)\quad \left| (1+y)^{\alpha-1} - \sum_{m=1}^{N-1} (-1)^{m-1} \frac{\Gamma(m-\alpha)}{\Gamma(1-\alpha)} \frac{y^{m-1}}{(m-1)!} \right| \leq$$

$$\leq \frac{1}{(N-2)!} \int_0^y dt \left| f^{(N-1)}(t) \right| |y-t|^{N-2} ,$$

where

$$(7.10)\quad f^{(N-1)}(t) = (-1)^{N-1} \frac{\Gamma(N-\alpha)}{\Gamma(1-\alpha)} (1+t)^{\alpha-N}.$$

Since $1+t \leq e^t$ for any t such that $0 \leq t \leq y$, the assertion comes out immediately from (7.9).

Lemma 7.4. " Let N be a positive integer such that $N \geq 2$. Then

$$(7.11)\quad \left| \Gamma(\alpha,x) x^{-\alpha} e^x - \sum_{m=1}^{N-1} (-1)^{m-1} \frac{\Gamma(m-\alpha)}{\Gamma(1-\alpha)} \frac{1}{x^m} \right| \leq$$

$$\leq (N-1) \left| \frac{\Gamma(N-\alpha)}{\Gamma(1-\alpha)} \right| \frac{1}{\left[x - |\alpha-N|^N\right]} ,$$

for every $x > | \alpha - N |$".

Proof. Consider the Laplace transform

$$(7.12) \qquad \int_0^\infty dz\; e^{-zx} f(z)$$

where

$$(7.13) \quad f(z) = (1+z)^{\alpha-1} - \sum_{m=1}^{N-1} (-1) \frac{\Gamma(m-\alpha)}{\Gamma(1-\alpha)} \frac{1}{(m-1)!} z^{m-1} .$$

In virtue of Lemma 7.3 one has

$$(7.14) \quad \left| \int_0^\infty dz\; e^{-zx} f(z) \right| \leq (N-1) \left| \frac{\Gamma(N-\alpha)}{\Gamma(1-\alpha)} \right| \frac{1}{[x - |\alpha - N|]^N} ,$$

for $x > | \alpha - N |$.

On the other hand, we can write

$$(7.15) \quad \int_0^\infty dz\; e^{-zx} f(z) = x^{-\alpha} e^{x} \Gamma(\alpha,x) - \sum_{m=1}^{N-1} (-1)^m \frac{\Gamma(m-\alpha)}{\Gamma(1-\alpha)} \frac{1}{x^m} ,$$

where

$$(7.16) \qquad (\alpha,x) = x^{\alpha} e^{-x} \int_0^\infty dz\; e^{-zx} (1+z)^{\alpha-1} .$$

The Lemma follows then from (7.15) and (7.14).

Lemma 7.5. "Let N be a positive integer such that $N \geq 2$ and

$$(7.17) \quad A_s(x) = (-1)^{s+1} \frac{1}{\Gamma(-\gamma)} \sum_{m=0}^{\infty} \frac{\Gamma(m+1-\gamma)\Gamma(s-\alpha+(m+1)\beta+1) e^{-mx}}{(m+1)!(m+1)^{s+1} \Gamma(1-\alpha+(m+1)\beta) x^{m\beta}} ,$$

where α,β and γ are fixed parameters.

Then, if ε is any arbitrary positive number,

$$|x^{N-1}\{\psi(\alpha,\beta,\gamma;x)x^{\beta-\alpha+1}e^{x} \sum_{s=0}^{N-2} A_s(x)\frac{1}{x^s}\}| \leq$$

(7.18)

$$\leq \frac{(N-1)(1+\varepsilon)^N}{|\Gamma(-\gamma)|}\sum_{m=0}^{\infty}\frac{|\Gamma(m+1-\gamma)||\Gamma(N-\alpha+(m+1)\beta)}{(m+1)!(m+1)^N|\Gamma(1-\alpha+(m+1)\beta)|}\frac{e^{-mx}}{x^{m\beta}},$$

for $x > [|\beta|+|\alpha-N|]\frac{1+\varepsilon}{\varepsilon}$, where $\psi(\alpha,\beta,\gamma;x)$ is defined by (1.6)".

Proof. Consider the integral representation of $\psi(\alpha,\beta,\gamma;x)$ as given by (1.9).

One has

$$(7.19)\qquad (1-Y)^{\gamma} = 1 + \frac{1}{\Gamma(-\gamma)}\sum_{n=1}^{\infty}\frac{\Gamma(n-\gamma)}{n!}Y^n,$$

where

$$(7.20)\qquad Y \equiv \frac{e^{-x(1+y)}}{x^{\beta}(1+y)^{\beta}} < 1,$$

for each $y > 0$.

Since the series on the right of (7.19) converges uniformly for $|Y| \leq 1-\varepsilon$ (ε being such that $0 < \varepsilon < 1$), from (1.9) one gets integrating term by term

$$\psi(\alpha,\beta,\gamma;x) = -\frac{x^{\alpha}}{\Gamma(-\gamma)}\sum_{n=1}^{\infty}\frac{\Gamma(n-\gamma)}{n!}\frac{e^{-nx}}{x^{n\beta}}\int_0^{\infty}dt(1+y)^{\alpha-1-n\beta}e^{-nxy} =$$

$$= - \frac{1}{\Gamma(-\gamma)} \sum_{n=1}^{\infty} \frac{\Gamma(n-\gamma)}{n!} n^{-\alpha+n\beta} \Gamma(\alpha-n\beta,nx) \quad ,$$

where the representation (7.16) has been used.

Putting m = n-1, Eq. (7.21) can be expressed as

$$\psi(\alpha,\beta,\gamma;x) = - \frac{1}{\Gamma(-\gamma)} x^{\alpha-\beta} e^{-x} \sum_{m=0}^{\infty} \frac{\Gamma(m+1-\gamma)}{(m+1)!} x^{-m\beta} e^{-mx} \quad \cdot$$

(7.22)

$$\cdot \quad \{[(m+1)x]^{-\alpha+(m+1)\beta} e^{(m+1)x} \Gamma(\alpha-(m+1)\beta,(m+1)x)\} .$$

Using now (7.22) and recalling (7.17), we can write

$$\psi(\alpha,\beta,\gamma;x) x^{\beta-\alpha+1} e^{x} - \sum_{s=0}^{N-2} A_s(x) \frac{1}{x^s} =$$

$$= - \frac{1}{\Gamma(-\gamma)} \sum_{m=0}^{\infty} \frac{\Gamma(m+1-\gamma)}{(m+1)!} x^{-m\beta+1} e^{-x} \quad \cdot$$

(7.23)

$$\cdot \quad \{[(m+1)x]^{-\alpha+(m+1)\beta} e^{(m+1)x} \Gamma(\alpha-(m+1)\beta,(m+1)x -$$

$$- \sum_{s=0}^{N-2} (-1)^s \frac{\Gamma(s-\alpha+(m+1)\beta+1)}{\Gamma(1-\alpha+(m+1)\beta)[(m+1)x]^{s+1}}\} .$$

Multiplying both sides of (7.23) by x^{N-1}, with the help of Lemma 7.4 one obtains the inequality (see Remark 7.6):

$$\left| x^{N-1}\{\psi(\alpha,\beta,\gamma;x)x^{\beta-\alpha-1} - \sum_{s=0}^{N-2} A_s(x) \frac{1}{x^s}\} \right| \leq$$

$$\leq \frac{(N-1)}{|\Gamma(-\gamma)|} \sum_{m=0}^{\infty} \frac{|\Gamma(m+1-\gamma)|}{(m+1)!} e^{-m\beta} e^{-mx} .$$

(7.24)

$$\cdot \left| \frac{\Gamma(N-\alpha+(m+1)\beta)}{\Gamma(1-\alpha+(m+1)\beta)} \right| \frac{x^N}{[(m+1)x-|\alpha-(m+1)\beta-N|]^N} ,$$

for $x > |\beta| + |\alpha-N|$.

Now we notice that

$$(7.25) \qquad \frac{x^N}{[(m+1)x-|\alpha-(m+1)\beta-N|]^N} \leq \frac{1}{(m+1)^N} \left[1+ \frac{(m+1)|\beta|+|\alpha-N|}{(m+1)(x-|\beta|)-|\alpha-N|} \right]$$

Furthermore, for any $\varepsilon > 0$ there esists a value of x, say x_ε, such that for any <u>m</u>:

$$(7.26) \qquad \frac{(m+1)|\beta|+|\alpha-N|}{(m+1)(x-|\beta|)-|\alpha-N|} < \varepsilon ,$$

for each $x > x_\varepsilon$.

In fact, the validity of (7.26) is assured for any m whenever $x > x_\varepsilon$, where

$$(7.27) \qquad x_\varepsilon = [|\beta| + |\alpha-N|] \frac{1+\varepsilon}{\varepsilon} .$$

Finally, Lemma (7.5) follows from (7.24) after taking into account (7.25) and (7.26).

Remark 7.6. The use of Lemma 7.4 in deriving the result (7.24) implies the evaluation of the Laplace transform

$$\int_0^\infty dt\, e^{-\lambda t} t^{N-1},$$

where $\lambda = (m+1)x - |\alpha-(m+1)\beta-N|$, which exists if and only if $(m+1)x > |\alpha-(m+1)\beta-N|$ for any m.

Since $|\alpha-(m+1)\beta-N| \leq (m+1)|\beta| + |\alpha-N|$,

we have

$$(m+1)x - |\alpha-(m+1)\beta-N| \geq (m+1)(x-|\beta|) - |\alpha-N|.$$

Thus we need to require that $(m+1)(x-|\beta|)-|\alpha-N| > 0$ for any m, the latter being satisfied when $x > |\beta|+|\alpha-N|$.

Lemma 7.6. "The series

$$(7.28) \qquad \sum_{m=0}^{\infty} \frac{\Gamma(m+1-\gamma)\Gamma(s-\alpha+(m+1)\beta+1)x^{-m\beta}e^{-mx}}{(m+1)!(m+1)^{s+1}\Gamma(1-\alpha+(m+1)\beta)},$$

which defines the function $(-1)^{s+1}\Gamma(-\gamma)A_s(x)$, converges absolutely and uniformly for any x greater than a certain $\bar{x}$ satisfying the inequality $e^{-x} < x^{\beta}$".

Proof. Since

$$(7.29) \qquad \frac{1}{(m+1)^{s+1}} \; \frac{\Gamma(s+1-\alpha+(m+1)\beta)}{\Gamma(1-\alpha+(m+1)\beta)} \sim O(m^{-1})$$

as $m \to +\infty$, from a certain value of m, say m_0, onwards it turns out that

$$(7.30) \qquad \frac{1}{(m+1)^{s+1}} \left| \frac{\Gamma(s+1-\alpha+(m+1)\beta)}{\Gamma(1-\alpha+(m+1)\beta)} \right| < 1.$$

Hence

$$\frac{|\Gamma(m+1-\gamma)|\,|\Gamma(s+1-\alpha+(m+1)\beta)|\,x^{-m\beta}e^{-mx}}{(m+1)!\,(m+1)^{s+1}|\Gamma(1-\alpha+(m+1)\beta)|} \leq$$

(7.31)

$$\leq \frac{|\Gamma(m+1-\gamma)|}{(m+1)!}\, e^{-mx}x^{-m\beta},$$

for $m \geq m_o$.

From (7.31) one deduces that the series (7.28) is majorised by

$$(7.32)\qquad \sum_{m=m_o}^{\infty} \frac{\Gamma(m+1-\gamma)}{(m+1)!}\, e^{-mx}\, x^{-m\beta}.$$

Recall now that (see (3.1))

$$(7.33)\qquad \sum_{m=0}^{\infty} \frac{\Gamma(m+1-\gamma)e^{-mx}x^{-m\beta}}{(m+1)!} = \Gamma(-\gamma)\left[\left(1-\frac{e^{-x}}{x^{\beta}}\right)^{\gamma}-1\right] x^{\beta}e^{x},$$

being the series on the left absolutely and uniformly convergent for any $x>\bar{x}$, where $\bar{x}$ is a certain value verifying the inequality $e^{-x}<x^{\beta}$. The assertion arises therefore from the fact that (7.32) is the m_o-th remainder of the series appearing in (7.33).

Lemma 7.7. "The series

$$(7.34)\quad \sum_{m=o}^{\infty} \frac{\Gamma(m+1-\gamma)}{(m+1)!(m+1)^N} \left| \frac{\Gamma(N-\alpha+(m+1)\beta)}{\Gamma(1-\alpha+(m+1)\beta)} \right| x^{-m\beta} e^{-mx},$$

which appears on the right of (7.18), converges uniformly for any x greater than a certain $\bar{x}$ verifying the inequality $e^{-x} < x$. Furthermore, one has

$$(7.35)\quad \left| x^{N-1}\{\psi(\alpha,\beta,\gamma;x)x^{\beta-\alpha+1}e^{x} - \sum_{s=o}^{N-2} A_s(x) \right| \frac{1}{x^s} \to \text{const},$$

as $x \to +\infty$, $A_s(x)$ being defined by (7.17) and

$$(7.36)\quad \text{const} \le |\gamma|(N-1)(1+\varepsilon)^N \left| \frac{\Gamma(N-\alpha+\beta)}{\Gamma(1-\alpha+\beta)} \right| ,$$

where $N \ge 2$ and ε is any arbitrary positive number".

Proof. The first part of the lemma follows directly from Lemma (7.28).

As a consequence, the results (7.35) and (7.36) arise immediately from (7.18).

In virtue of the series of lemmas from (7.3) to (7.7), the basic Theorem 7.2 is thus completely proved.

8. SOME SPECIAL CASES.

a) "Asymptotic expansion of the incomplete Γ-function".

The expression (7.6) can be written as

$$A_s(x)=(-1)^{s+1}\{(-\gamma)\ \frac{\Gamma(s-\alpha+\beta+1)}{\Gamma(-\alpha+\beta+1)}\quad +$$

(8.1)