DEFINITION 4. Let A be an NPCO problem; we say that

i) A is *approximable* if given any $\varepsilon > 0$ there exists an ε-approximate algorithm;

ii) A is *fully approximable* if there exists a polynomial $\lambda x \lambda y[q(x,y)]$ such that for every ε there exists an ε-approximate algorithm A_ε that runs in time bounded by $q(|x|,1/\varepsilon)$

Many results in the recent literature are devoted to establishing whether a given problem is approximable or fully approximable or it cannot be approximated. For example it is known that the MAX-SUBSET-SUM problem is fully approximable while the MIN-CHROMATIC-NUMBER problem has been proven not to be approximable for $\varepsilon < 1$ (if $P \neq NP$). A list of papers dealing with results in this area is provided by Garey and Johnson (1977). At present no result is known that shows that a problem is approximable but not fully approximable neither is known any precise characterization of the class of problems which are approximable or fully approximable. The results given by Paz and Moran (1977) and Garey and Johnson (1978) are nevertheless an important step forward in this direction. For this reason our aim has been to determine conditions for the comparison of these results and at the same time to develop this kind of research and to derive consequences which are useful for a better understanding of the properties of NP-complete optimization problems.

3. TRUNCATED COMBINATORIAL PROBLEMS AND THEIR PROPERTIES

The first approach (Paz and Moran (1977)) to the characterization of NP-complete optimization problems is based on the complexity of the recognition of an infinite sequence of bounded subsets of the associated combinatorial problem. Informally, if we consider the search space that has to be explored in order to find approximate solutions to an optimization problem we may observe the following facts. Clearly, if the size of the search space is polynomial in the size of the input the problem itself is polynomially solvable. In the case of those problems which are in the class NP but which are not known to be polynomial an a priori evaluation of the size of the search space indicates that it grows exponentially. Nevertheless in many cases when we consider the search space that we have to explore in order to find approximate solutions whose measure does not exceed a certain bound, we may notice that it is polynomial. A typical example of this kind of problems is the problem MAX-CLIQUE in which the complete subgraphs of size k in a graph of size n are at most $\binom{n}{k}$, that is polynomial in n. Since this does not happen in all cases it suggests the following definition.

DEFINITION 5. Let A be an NPCO problem; let A^c be the associated combinatorial problem. *A truncated combinatorial problem* of A is a set

$$A^c_{\bar{k}} = \left\{ \langle x,k \rangle \in A^c \,|\, \tilde{m}(x) \leq k \leq \tilde{m}(x) + \bar{k} \right\}$$

where $\bar{k}$ is any nonnegative integer.

Note that the sequence $\{A^c_{\bar{k}}\}^{\infty}_{\bar{k}=0}$ approximates the set A^c in a sense which is analogous to the definition of limit recursion approximation (Gold (1965)).

DEFINITION 6. A is *simple* if, for every $\bar{k}$, $A^c_{\bar{k}}$ is polynomially decidable. A is *rigid* if it is not simple.

Note that if A is rigid there exists an integer $\bar{k}$ such that $A^c_{\bar{k}}$ is p-complete that is $A^c_{\bar{k}}$ is in P if and only if P = NP (see Sahni 75).
Examples of simple NPCO problems, besides MAX-CLIQUE, are MAX-SATISFIABILITY, MIN-CHROMATIC NUMBER etc.

Definitions 5 and 6 are slightly modified with respect to the corresponding definitions in Paz and Moran (1977). In fact we always start from the set A^c_0 in which all pairs $\langle x,\tilde{m}(x)\rangle$ are included and, as long as $\bar{k}$ increases,we go further and further from the worst solution to the optimal solution.

For example the problem MIN-CHROMATIC-NUMBER, which is rigid according to the original definitions, is simple in our case and this is because, given any h, the set of possible colourings of a graph of N nodes with N-h colours has polynomial size in N.

On the other side an example of rigid NPCO problem is provided by MAX-WEIGHTED-SATISFIABILITY because if we allow weights equal to zero even the set A^c_0 is in this case NP-complete because in order to decide whether a formula w has measure 0 we first need to prove that it is satisfiable (Ausiello et al. (1978)). Note that if we instead do not allow weights equal to zero the problem MAX-WEIGHTED-SATISFIABILITY can be proved to be simple.

Note that if a problem is simple, then its worst solution is actually a trivial solution, that is it can be always found in polynomial time.

The concept of simple problem can be strengthened in the following way:

DEFINITION 7. An NPCO problem A is p-*simple* if there is a polynomial Q such that, for every k, A^c_k is recognizable in deterministic time bounded by $Q(|x|,k)$.

Typical examples of p-simple problems are MAX-SUBSET-SUM, JOB-SEQUENCING-WITH-DEADLINES etc., while the above listed simple problems are not p-simple. We will discuss later on this claim.

Beside offering a first classification of NPCO problems, the concepts of simplicity and p-simplicity are relevant because it has been proven by Paz and Moran (1977) that a necessary condition for a problem A to be approximable (fully-approximable) is that A is a simple (p-simple) NPCO problem and clearly these properties still hold under our definitions.

Actually the fact that until now no problem has been shown to be approximable and not fully-approximable, determines a greater attention on the concept of p-simplicity; but in order to prove that a problem is not p-simple it is very hard to show that no algorithm which is polynomial in $|x|$ and k can exist. Much easier is to use the following definitions

DEFINITION 8. An NPCO problem A is *strongly simple* if, given any polynomial q, $A_q^c = \{\langle x,k \rangle \in A^c \mid \tilde{m}(x) \leq k \leq \tilde{m}(x)+q(|x|)\}$ is decidable in polynomial time. A is *weakly rigid* if there exists a polynomial p such that A_p^c is NP-complete.

Since a p-simple problem is strongly simple, to show that a problem is weakly rigid is a very easy method to prove that a problem is not p-simple and therefore not fully approximable. For example weakly rigid problems are MAX-CLIQUE, MAX-SATISFIABILITY, MIN-CHROMATIC-NUMBER and the proof is based on the fact that, for all these problems, for q(n) increasing more rapidly than n, $A_q^c = A^c$.

This fact suggests an even easier condition that is sufficient for a problem not to be fully approximable.

PROPOSITION 1. Let A Be an NPCO problem. If there exists a polynomial p such that for all $x \in INPUT_A$, $m^*(x) - \tilde{m}(x) \leq p(|x|)$ then A is not fully approximable.

PROOF. In fact in order to be fully approximable, A should satisfy the property that A_p^c is recognizable in polynomial time but, by hypothesis, we have that $A_p^c = A^c$ and, hence, A_p^c is NP-complete.

QED

For some problems, like MAX-CLIQUE and MIN-CHROMATIC NUMBER, Proposition 1 can be immediately applied. In fact in these cases $p(|x|) = |x|$.
In some other case, in order to apply Proposition 1, we may prove a stronger result that is useful for showing that a problem is weakly rigid.

THEOREM 1. Let A and B be two NPCO problems; if there exists a reduction $f = \langle f_1, f_2 \rangle$ from A to B such that f satisfies the following property: $f_2(x,k) \leq p(|f_1(x)|)$ for some polynomial p and all $x \in INPUT_A$, $k \in Q_A$, then B is not fully approximable.

PROOF. If B was fully approximable then for every polynomial q we should have B_q^c recognizable in polynomial time. If we now consider the set B_p^c if we could decide within polynomial time whether, given any pair $\langle y,h \rangle$ with $\tilde{m}(y) \leq h \leq \tilde{m}(y) + p(|y|)$, h is the measure of an approximate solution of y, then within polynomial time we could decide A^c. In fact in order to decide A^c in polynomial time, given a pair $\langle x,k \rangle$ we could compute in polynomial time $f_1(x)$ and $f_2(x,k)$ and since $f_2(x,k) \leq p(|f_1(x)|)$ we could use the decision procedure B_p^c to check whether $f_2(x,k)$ is the measure of an approximate solution of $f_1(x)$.

QED

Note that in theorem 1 the condition on f_2 may regard only a subset of B while in Proposition 1 all inputs must satisfy the hypothesis that $(m^*(x)-\tilde{m}(x)) \leq p(|x|)$.

Furthermore Theorem 1 partially characterizes the reductions between an arbitrary problem and a weakly rigid one. For example if we consider the trivial reduction (inclusion) from SIMPLE-MAX-CUT to MAX-CUT, we see that the image of SIMPLE-MAX-CUT is a subset of MAX-CUT where the measure is bounded by the number of nodes of the graph and this fact is sufficient to deduct that MAX-CUT is not fully approximable.

In the following we will continue the study of the characterization of reductions between problems belonging to different classes, and we will show how some of the considered properties can be inherited by polynomial reduction, under some natural hypothesis.

THEOREM 2. Let A and B be two NPCO problems such that $A \leq B$ via the reduction $f=\langle f_1,f_2\rangle$; if A is rigid and if there exists a monotonous function g such that for every $x \in \mathrm{INPUT}_A$, $k \in Q_A$ $f_2(x,k) \leq g(k)$ then B is rigid.

PROOF. If A is rigid there must be an integer $\bar{k}$ such that

$$A^c_{\bar{k}} = \left\{\langle x,k\rangle \mid \langle x,k\rangle \in A^c \text{ and } \tilde{m}(x) \leq k \leq \tilde{m}(x)+\bar{k}\right\}$$

is P-complete. By hypothesis, if we take $\bar{\bar{k}}=g(\bar{k})$ then

$$B^c_{\bar{\bar{k}}} = \left\{\langle y,h\rangle \mid \langle y,h\rangle \in B^c \text{ and } \tilde{m}(y) \leq h \leq \tilde{m}(y)+g(\bar{k})\right\}$$

contains $f(A^c_{\bar{k}})$ and, hence if there was a polynomial algorithm for $B^c_{\bar{\bar{k}}}$ it could be used to decide $A^c_{\bar{k}}$ in polynomial time. In fact in order to decide whether $\langle x,k \rangle$ belongs to $A^c_{\bar{k}}$ in the case $k \leq \bar{k}$ (otherwise we trivially know that $\langle x,k \rangle$ does not belong to $A^c_{\bar{k}}$), we may consider $\langle f_1(k), f_2(x,k) \rangle$ and decide whether it belongs to $B^c_{\bar{\bar{k}}}$. QED

REMARK. Note that under the same conditions if $A \leq B$ and B is simple A must be simple. This result shows that no polynomial reduction from a rigid problem to a simple problem is possible unless the function f_2 is such that for no computable function g it is true that for every x and every k $f_2(x,k) \leq g(k)$. In other words $f_2(x,k)$ cannot be dependent only on k but must eventually increase with respect to x.

Notice that theorem 2 strengthens another result given in Paz and Moran (1977) where g is not an arbitrary monotonous function but just a polynomial and the only considered case is when $f_2(x,k)$ is *equal* to g(k).

When we pass from simple problems to strongly simple problems we obtain the following result.

THEOREM 3. Let A and B be two NPCO problems and $A \leq B$ via the reduction $f = \langle f_1, f_2 \rangle$. If there exists a polynomial t such that for all $x \in \mathrm{INPUT}_A$ and $k \in Q_A$ $f_2(x,k) - \tilde{m}(f_1(x)) \leq$ $\leq t(|x|, k - \tilde{m}(x))$ then B strongly simple implies A strongly simple.

PROOF. If B is strongly simple then for all polynomials p we know that the set B^c_p must be polynomially recognizable. Now, let us consider any polynomial r and the set

$$A^c_r = \left\{ \langle x,k \rangle \mid \langle x,k \rangle \in A^c \text{ and } \tilde{m}(x) \leq k \leq \tilde{m}(x) + r(|x|) \right\}$$

we shall show that A^c_r is polynomially decidable. In fact, given $\langle x,k \rangle$, if $k > \tilde{m}(x)+r(|x|)$ or if $k \leq \tilde{m}(x)$ we immediately know that $\langle x,k \rangle$ does not belong to A^c_r. On the other side, if $\tilde{m}(x) \leq k \leq \tilde{m}(x)+r(|x|)$ let us consider the following set:

$$f(A^c_r) = \left\{ \langle f_1(x), f_2(x,k'(x)+\tilde{m}(x)) \rangle \mid \langle x,k'(x)+\tilde{m}(x) \rangle \in A^c \text{ and } 0 \leq k'(x) \leq r(|x|) \right\} ;$$

where $k'(x) = k - \tilde{m}(x)$

by hypothesis $f(A^c_r)$ is included in the set

$$S = \left\{ \langle y,h \rangle \mid \langle y,h \rangle \in B^c \text{ and } \tilde{m}(y) \leq h \leq \tilde{m}(y)+t(|x|,r(|x|)) \right\}$$

Since we know that if A^c and B^c are NP-complete sets and $A^c \leq B^c$ via $\langle f_1, f_2 \rangle$ then we must have $|x| \leq q(|f_1(x)|)$ for every x and a polynomial q, then there must exist a polynomial r' such that

$$B^c_{r'} = \left\{ \langle y,h \rangle \mid \langle y,h \rangle \in B^c \text{ and } \tilde{m}(y) \leq h \leq \tilde{m}(y) + r'(|y|) \right\} \supseteq S.$$

So in order to decide whether $\langle x,k \rangle \in A^c_r$ we may use the reduction f and the polynomial algorithm that decides whether $\langle f_1(x), f_2(x,k) \rangle$ belongs to $B^c_{r'}$. Hence A^c_r is also polynomially decidable.

QED

An interesting consequence of this fact is that, given a problem *A* which is not strongly simple and a problem *B* which is strongly simple any reduction from *A* to *B* must violate the hypothesis.

This means that in a reduction between *A* and *B* the measure must increase exponentially. If we consider similar reductions given by Karp (1972) (e.g. EXACT-COVER≤KNAPSACK) we notice that this is the case and by theorem 3 we may argue that no "easier" reduction may be found.

An analogous result holds in the case of p-simple problems. First of all we prove the following lemma:

LEMMA. Let A be an NPCO problem If A is p-simple, then, for every polynomial p, $A_p^c=\{\langle x,k\rangle \mid \langle x,k\rangle \in A^c \wedge \tilde{m}(x) \leq k \leq \tilde{m}(x) + p(|x|)\}$ is recognizable in $Q(|x|,p(|x|))$ where Q is a polynomial.

PROOF. Let A be p-simple. Given a polynomial p, we can decide $\langle x,k\rangle \in Q_p^c$ in $Q(|x|,p(|x|)$.
In fact if $k > \tilde{m}(x) + p(|x|)$ or $k < \tilde{m}(x)$, it is obvious that $\langle x,k\rangle$ does not belong to A_p^c. Differently, we can use the following algorithmic procedure

1) compute $\bar{k} = p(|x|)$

2) decide if $\langle x,k\rangle \in Q_{\bar{k}}^c$ in $Q(|x|,\bar{k})$

QED

The following theorem holds:

THEOREM 4. Under the same hypotheses of Theorem 3, B p-simple implies A p-simple

PROOF. For every $\bar{k}$ we show that we can decide $A_{\bar{k}}^c$ in time polynomial in $|x|$ and $\bar{k}$. In fact, given $\langle x,k\rangle$, if $\tilde{m}(x) \leq k \leq \tilde{m}(x)+\bar{k}$ (the other cases are trivial), we consider $f\ (A_{\bar{k}}^c)$ which is included in the set $\bar{S} = \{\langle y,h\rangle \mid \langle y,h\rangle \in B^c \wedge \tilde{m}(y) \leq h \leq \tilde{m}(y)+t(|x|,\bar{k})\}$.
Furthermore if we consider the polynomial $r(u,\bar{k})=t(q(u),\bar{k})$ where t and q are as in theorem 3, B_r^c contains $\bar{S}$ and, by the lemma, B_r^c is decidable in time $Q(|y|,r(|y|,\bar{k}))$. Using the reduction f and the property of B_r^c we may decide whether $\langle x,k\rangle \in A_{\bar{k}}^c$ within time

$$Q(|f_1(x)|,t(q(|f_1(x)|),\bar{k}))=Q(p(|x|),t(q(p(|x|)),\bar{k}))$$

(due to the polynomiality of the reduction f) what means that the decision time is bounded by a polynomial in $|x|$ and $\bar{k}$.

QED

Since no example is known of a problem which is strongly simple and not p-simple no application of theorem 4 can be provided which is different from the application given at the end of theorem 3.

As a conclusion of this paragraph we may observe that the results provided insofar have a twofold implication. On one side they can be used in order to characterize the computational complexity of one problem with respect to the given definitions, on the other side they establish conditions on the type of reductions that can be found among problems belonging to different classes, such as those discussed at the end of theorem 2 and theorem 3. As a further example we may observe that in the case of the reduction from PARTITION to MAX-CUT the existence of a much more succint reduction than the one given by Karp is ensured by noting that the first problem is strongly simple while the second is weakly rigid.

4. STRONG NP-COMPLETENESS AND ITS RELATION TO RIGIDITY

In the preceding paragraph we have seen that in some cases the characterization of a problem B that is not fully approximable comes out of the fact that we can reduce an NP-complete combinatorial problem A^c into a subset of B^c in which the measure is bounded by a polynomial.Garey and Johnson give another way of considering subsets of the set INPUT of a problem to study the different characteristics of NPCO problems. Their paper (1978) is an attempt to understand the different roles that numbers play in NPCO problems. Let