PART ONE. DUALITY THEOREM FOR REGULAR FUNCTIONS.

For brevity, we omit the statements of dual propositions, but if we must refer to them, we denote them by *.

1) Properties of regular and completely regular functions.

DEFINITION 1.- *Let* S *be a topological space,* x *a point of* S, G *a finite directed graph and* f: S → G *a function from* S *to* G. *We call* image-envelope *of* x *by* f, *and we denote by* <f(x)>, *the set of vertices, such that the closures of their* f*-counterimages include the point, i.e.* $v \in \langle f(x) \rangle \Leftrightarrow x \in \overline{V^f}$.

PROPOSITION 1. - *Let* S *be a topological space,* x *a point of* S, G *a finite directed graph and* f: S → G *a function from* S *to* G. *Then the image-envelope of* x *coincides with the intersection of the images of the neighbourhoods of* x, *i.e.* $\langle f(x) \rangle = \cap\{f(U_x) \,/\, U_x$ *is a neighbourhhod of* x$\}$.

Proof.- $v \in \langle f(x) \rangle \Leftrightarrow x \in \overline{V^f} \Leftrightarrow (\forall\, U_x,\ U_x \cap V^f \neq \phi) \Leftrightarrow (\forall\, U_x,\ v \in f(U_x)) \Leftrightarrow v \in \cap f(U_x)$. ■

PROPOSITION 2. - *Let* S *be a topological space,* G *a finite directed graph and* f: S → G *a function from* S *to* G. *Then* f *is an o-regular function, iff, for all* x ∈ S, f(x) *is a head of* <f(x)>, *i.e.* f(x) ∈ H(<f(x)>).

Proof. - i) Let f be an o-regular function, x a point of S, and $v = f(x)$. Then, for all $w \in \langle f(x) \rangle$, i.e. $x \in \overline{W^f}$, we have $V^f \cap \overline{W^f} \neq \phi$. Hence $v \to w$, i.e. $v \in H(\langle f(x) \rangle)$.

ii) For all $x \in S$, let $f(x) \in H(\langle f(x) \rangle)$ be. We have to prove that, for all $v, w \in G$, such that $v \neq w$ and $v \not\to w$, it results that $V^f \cap \overline{W^f} = \phi$. If we assume $x \in V^f \cap \overline{W^f}$, it follows $f(x) = v$, $v \in H(\langle (f(x) \rangle)$ and $w \in \langle f(x) \rangle$, hence $v \to w$. Contradiction.■

PROPOSITION 3. - *Let* S *be a topological space,* G *a finite directed graph and* f: S → G *a function from* S *to* G. *Then* f *is a c.o-regular function, iff, for all* x ∈ S, *it is:*

i) f(x) *is a head of* <f(x)>, *i.e.* f(x) ∈ H(<f(x)>);

ii) <f(x)> *is a totally headed subset of* G.

Proof. - By Proposition 2, we have only to prove that an o-regular function is c.o-regular iff ii) is true.

Then let f be a c.o-regular function. Since each subset $X = \{v_1,\ldots,v_n\}$ of $<f(x)>$ such that $\overline{V_1^f} \cap \ldots \cap \overline{V_n^f} \neq \phi$ can not be a singularity of f, X must be headed.

Conversely, let $<f(x)>$ be totally headed for all $x \in S$. Then if we assume that $X = \{v_1,\ldots,v_n\}$ is a singularity of f, there exists a point $x \in \overline{V_1^f} \cap \ldots \cap \overline{V_n^f}$. Hence the non-headed subset X is included in $<f(x)>$. Contradiction. ∎

REMARK. - Consequently, *if* G *is an undirected graph, a function* f:S → G *is strongly regular i.e. c.regular iff, for all* x ∈ S, <f(x)> *is a totally headed subset of* G. In this case, indeed, we have that "<f(x)> *totally headed*" is equivalent to $H(<f(x)>) = <f(x)>$.

2) Patterns of a function.

DEFINITION 2. - *Let* f: S → G *be a function from a topological space* S *to a finite directed graph* G. *A function* g: S → G *is called an* o-pattern *(resp.* o*-pattern*) of* f, *if, for all* x ∈ S, *it holds* g(x) ∈ H(<(f(x)>) *(resp.* g(x) ∈ T(<f(x)>)*)*.

REMARK. - In general there is no pattern of a given function, because the sets $<f(x)>$ may be non-headed for some $x \in S$.

DEFINITION 3. - *A function* f: S → G *from a topological space* S *to a finite directed graph* G *is called* quasi o-regular *(resp.* quasi o*-regular*), or simply* q.o-regular *(resp.* q.o*-regular*) if the image-envelope* <f(x)> *is headed (resp. tailed) for all* x ∈ S.
Moreover, the function f *is called* completely quasi regular, *or simply* c.q. regular, *if* <f(x)> *is totally headed.*

REMARK 1. - Consequently, if G is an undirected graph, a q. regular function is also regular and a c.q.regular function is also completely regular, i.e. strongly regular.

REMARK 2. - We consider only c.q.regular functions, since by R_a each c.q.o-regular function is also c.q.o*-regular.

PROPOSITION 4. - *An o-regular function is q.o-regular. A c.o-regular function is c.q.regular.*

Proof. - It follows from Propositions 2, 3. ■

PROPOSITION 5. - *A function* f: S → G *is q.o-regular iff there exists an o-pattern of* f.

Proof. - i) Let g be an o-pattern of f. Since, for all $x \in S$, $g(x) \in H(<f(x)>)$, $<f(x)>$ is headed.
ii) Let $<f(x)>$ be headed. In order to construct an o-pattern g of f, we number the vertices of the finite graph G by $v_1,\dots,v_n$. Then, for all $x \in S$, we choose as $g(x)$ the vertex with the lowest index among the vertices of $H(<f(x)>)$. ■

REMARK. - We note that a c.q.regular function is q.o-regular and q.o*-regular. Hence, there exist both o-patterns and o*-patterns for a c.q-regular function.

PROPOSITION 6. - *Let* f: S → G *be a q.o-regular function. Then:*

i) all its o-patterns are o-regular functions;

ii) two o-patterns of f *are o-homotopic to each other.*

Proof. - i) Let $g: S \to G$ be an o-pattern of f. At first, we prove that $\overline{V^g} \subseteq \overline{V^f}$, for each $v \in G$. We have, indeed, $x \in V^g \Rightarrow g(x) = v \Rightarrow v \in \langle f(x)\rangle \Rightarrow x \in \overline{V^f}$. Hence it results $V^g \subseteq \overline{V^f}$ and also $\overline{V^g} \subseteq \overline{V^f}$. Consequently, $\langle g(x)\rangle \subseteq \langle f(x)\rangle$, for all $x \in S$. Now, since $g(x)$ is a head of $\langle f(x)\rangle$, it is also a head of $\langle g(x)\rangle$. Then, by Proposition 2, g is an o-regular function.

ii) Let g,h be two o-patterns of f. The function $F: S\times I \to G$, given by:

$$F(x,t) = \begin{cases} g(x) & \text{for } t = 0 \\ h(x) & \forall t \in]0,1], \end{cases}$$

is a homotopy between g and h. Besides, for all $(x,t) \in S\times I$, it is:

$$\langle F(x,t)\rangle = \begin{cases} \langle g(x)\rangle \cup \langle h(x)\rangle \subseteq \langle f(x)\rangle & \text{for } t = 0, \\ \langle h(x)\rangle & \forall t \in]0,1]. \end{cases}$$

Then, since $g(x)$ and $h(x)$ are heads of $\langle f(x)\rangle$, they are also, respectively, a head of $\langle g(x)\rangle \cup \langle h(x)\rangle$ and a head of $\langle h(x)\rangle$. Consequently, F is an o-regular function.

DEFINITION 4. - *Let* S *be a topological space and* G *a finite directed graph. Two c.o-regular (resp. c.o***-regular) functions* f,g: S → G *are called* completely o-homotopic *(resp.* completely o*-homotopic*) or simply* c.o-homotopic *(resp.* c.o*-homotopic*) if there exists a homotopy* F *between* f *and* g*, which is a c.o-regular (resp. c.o***-regular) function.* F *is called a* complete o-homotopy *(resp.* complete o*-homotopy*), or simply a* c.o-homotopy *(resp.* c.o*-homotopy*).*

PROPOSITION 7. - *Let* f: S → G *be a c.q.regular function. Then:*

i) all its o-patterns are c.o-regular functions;

ii) any two o-patterns of f *are c.o-homotopic to each other.*

Proof. - i) Like in Proposition 6, we prove that $\langle g(x)\rangle \subseteq \langle f(x)\rangle$, for all $x \in S$. Consequently, since $\langle f(x)\rangle$ is totally headed, also $\langle g(x)\rangle$ is totally headed.

Hence, by i) of Proposition 6 and by Proposition 3, g is c.o-regular.

ii) We define the homotopy like in Proposition 6. Since, $\forall\, x \in S$, $f(x)$ is totally headed, the subsets $\langle g(x)\rangle \cup \langle h(x)\rangle$ and $\langle h(x)\rangle$ are also totally headed. Hence, $\forall\, (x,t) \in S\times I$, $F(x,t)$ is totally headed and so is a c.o-homotopy between g and h, by Proposition 3. ■

3) Duality Theorem for complete homotopy classes.

We see it is possible to construct homotopy classes, by considering only c. regular functions and c.regular homotopies.

PROPOSITION 8. - *The c.o-homotopy is an equivalence relation in the set of c.o-regular functions from* S *to* G.

Proof. - The relation obviously satisfies the reflexive and symmetric properties. (See [2], Remark to Definition 5). Also the transitive property is true. In fact, let F (resp. J) be a c. o-homotopy between the c. o-regular functions f and g (resp. g and k). Then the function $K\colon S\times I \to G$, given by:

$$K(x,t) = \begin{cases} F(x,3t) & \forall\, x \in S, \quad \forall\, t \in [0,\frac{1}{3}] \\ g(x) & \forall\, x \in S, \quad \forall\, t \in [\frac{1}{3},\frac{2}{3}] \\ J(x,3t-2) & \forall\, x \in S, \quad \forall\, t \in [\frac{2}{3},1] \end{cases},$$

is an o-homotopy between f and k.

We have to prove that k is a c.o-regular function. Let us assume that the image-envelope of the point (x,t) is non-totally headed. Then, if $t \leqslant \frac{1}{3}$, also the image-envelope of $(x,3t)$ is non-totally headed for the function F. If $t \geqslant \frac{2}{3}$, also the image-envelope of $(x,3t-2)$ is non-totally headed for the function J. If $\frac{1}{3} < t < \frac{2}{3}$, also the image-envelope of the point x is non-totally headed for the function g. Anyhow, we obtain a non-totally headed image-envelope for a c.o-regular function. This contradicts to Proposition 3. ■

REMARK. - By considering as homotopy between f and g that given by the sum (see [2], Remark to Defintion 5), we obtain only an o-regular function, in general.

DEFINITION 5. - *Let* S *be a topological space and* G *a finite directed graph. We denote by* $Q_c(S,G)$ *(resp.* $Q^*_c(S,G)$*) the set of c.o-homotopy (resp. c.o**-*homotopy) classes.*

REMARK. - We note that $Q^*_c(S,G)$ coincides with $Q_c(S,G^*)$, and $Q^*_c(S,G^*)$ with $Q_c(S,G)$.

THEOREM 9. - *Let* S *be a topological space and* G *a finite directed graph. Then there exists a natural bijection from the set of complete o-homotopy classes* $Q_c(S,G)$ *to the one of complete o**-*homotopy classes* $Q^*_c(S,G)$.

Proof. - We denote by $F_c(S,G)$ (resp. $F^*_c(S,G)$) the set of all the c.o-regular (resp. c.o*-regular) functions. We define a relation $\phi\colon F_c(S,G) \to F^*_c(S,G)$ which sends each $f \in F_c(S,G)$ in any its o*-pattern $\phi(f)$ and similarly a relation $\psi\colon F^*_c(S,G) \to F_c(S,G)$ which sends each $h \in F^*_c(S,G)$ in any its o-pattern $\psi(h)$.

i) ϕ induces a function Φ from $Q_c(S,G)$ to $Q^*_c(S,G)$.

By the Remark to Proposition 5 and by i) of Proposition 7 the relation ϕ is defined on all the set $F_c(S,G)$ and by ii) of Proposition 7 every o*-pattern of f is o*-homotopic to $\phi(f)$. Then we define a function $\overline{\phi}\colon F^*_c(S,G) \to Q_c(S,G)$ by putting:

$$\forall f \in F_c(S,G), \quad \overline{\phi}(f) = \{\phi(f)\}.$$

Now let g be a function c.o-homotopic to f by the homotopy H, and let $\phi(g)$ be an o*-pattern of g. We construct the c.o-homotopy:

$$F(x,t) = \begin{cases} f(x) & 0 \leq t \leq \frac{1}{3}, \\ H(x,3t-1) & \frac{1}{3} \leq t \leq \frac{2}{3}, \\ g(x) & \frac{2}{3} \leq t \leq 1. \end{cases}$$

Let $\hat{F}$ be an o*-pattern of F, it follows from Proposition 7* that $\hat{F}$ is a c.o*-homotopy between the restrictions $\hat{f} = \hat{F}/_{S\times\{0\}}$ and $\hat{g} = \hat{F}/_{S\times\{1\}}$. Since $f = F/_{S\times\{0\}}$ and H does not interfere in the construction of $\hat{f}$, $\hat{f}$ is an o*-pattern of f. Similarly, $\hat{g}$ is an o*-pattern of g. Then by Proposition 7*, $\phi(f)$ and $\hat{f}$ are c.o*-homotopic, and the same happens for $\phi(f)$ and $\hat{g}$. For the relation is transitive, $\phi(f)$ is c.o*-homotopic to $\phi(g)$.

Since the function $\overline{\phi}$ is compatible with the c.o-homotopy relation in $F_c(S,G)$, ϕ induces a function Φ from $Q_c(S,G)$ to $Q^*_c(S,G)$ given by:

$\forall\ \alpha \in Q_c(S,G),\quad \Phi(\alpha) = \{\phi(f)\}$, where f is a representative of α.

ii) ψ induces a function Ψ from $Q^*_c(S,G)$ to $Q_c(S,G)$.

By dual arguments we can prove that the required function Ψ is individualized by putting:

$\forall\ \beta \in Q^*_c(S,G),\quad \Psi(\beta) = \{\psi(h)\}$, where h is a representative of β.

iii) Φ and Ψ are bijective functions.

We have only to prove that $\Psi\Phi$ is the identity in $Q_c(S,G)$ and $\Phi\Psi$ the one in $Q^*_c(S,G)$. Let α be a class of $Q_c(S,G)$ and $f \in \alpha$ a c.o-regular function. We have $\Phi(\alpha) = \{\phi(f)\}$, and, successively, $\Psi\Phi(\alpha) = \{\psi\phi(f)\}$. We observe that the function $\psi\phi(f)$ is c.o-regular by Propositions 7, 7*. Following i) of the proof of Proposition 6, it results, $\forall\ v \in G$, $\overline{V^{\psi\phi(f)}} \subseteq \overline{V^{\phi(f)}} \subseteq \overline{V^f}$, then like ii) of the same proof, we can construct a c.o-homotopy between f and $\psi\phi(f)$. Consequently, $\Psi\Phi(\alpha) = \{\psi\phi(f)\} = \{f\} = \alpha$. Similarly, it results, $\forall\ \beta \in Q^*_c(S,G)$, $\Phi\Psi(\beta) = \beta$. ■

4) Duality theorem for homotopy classes.

By the two Normalization Theorems R_h, R_e, the duality can be extended to the homotopy classes $Q(S,G)$ and $Q^*(S,G)$.

PROPOSITION 10. - *Let* S×I *be a normal topological space and* G *a finite directed graph. Then there exists a natural bijection from the set of c.o-homotopy classes* $Q_c(S,G)$ *to the one of o-homotopy classes* $Q(S,G)$.

Proof. - Let $F(S,G)$ and $F_c(S,G)$ be the sets of o-regular and c.o-regular functions from S to G and $j\colon F_c(S,G) \to F(S,G)$ the identical embedding. Obviously, j is compatible with the c.o-homotopy relation in $F_c(S,G)$ and with the o-homotopy relation in $F(S,G)$, hence j induces a function J from $Q_c(S,G)$ to $Q(S,G)$. Moreover, J is onto by R_h, and it is one to one by R_e. ■

Finally, by Propositions 10, 10^* and Theorem 9 we obtain:

THEOREM 11. - *Let S be a countably paracompact normal space and G a finite directed graph. Then there exists a natural bijection from the set of o-homotopy classes O(S,G) to the one of o^*-homotopy classes $O^*(S,G)$.*

Proof. In fact the assumption on *S* is equivalent to suppose that *S* and $S\times I$ are normal spaces. (See Introduction). ■

REMARK 1. - In general the previous result does not hold for any topological space. (See Example 13.5).

REMARK 2. - In the foregoing conditions it follows that the sets *Q(S,G)*, $Q(S,G^*)$, $Q^*(S,G)$, $Q^*(S,G^*)$ can be identified.

PART TWO. DUALITY THEOREM FOR REGULAR FUNCTIONS BETWEEN PAIRS.

5) *Balanced functions.*

We can characterize the regular functions between pairs, similarly to Propositions 2, 3, by the following:

PROPOSITION 12. - *Let f: S,S' → G,G' be a function from a pair of topological spaces S,S' to a pair of finite directed graphs G,G' and f': S' → G' the restriction of f: S → G to S'. Then f is an o-regular function, iff f(x) is a head of <f(x)> in G, for all x ∈ S; while f'(x) is a head of <f'(x)> in G', for all x ∈ S'. Moreover, f is c.o-regular, iff also the subsets <f(x)> are totally headed in G and all the subsets <f'(x)> are totally headed in G'.* ■

REMARK. - Consequently, *if G is an undirected graph, a function* f: S,S' → G,G'