$$\begin{array}{ccc} J'(R,N) & \longrightarrow & R \times T\,N \\ {\scriptstyle j^1c}\nwarrow & & \nearrow{\scriptstyle (id_R,dc)} \\ & R & \end{array}$$

b) We get $J'(M,R) \cong R \times T^*M$.

This isomorphism is the unique map $J'(M,R) \to R \times T^*M$ that makes commutative the following diagram, for each function $f : M \to R$,

$$\begin{array}{ccc} J'(M,R) & \longrightarrow & R \times T^*M \\ {\scriptstyle j^1f}\nwarrow & & \nearrow{\scriptstyle (f,df)} \\ & M & \end{array}$$

c) There is a unique map (which is an isomorphism)

$$J^2(R,N) \to R \times s\,T^2N$$

such that the following diagram is commutative, for each curve $c : R \to N$,

$$\begin{array}{ccc} J^2(R,N) & \longrightarrow & R \times s\,T^2\,N \\ {\scriptstyle j^2c}\nwarrow & & \nearrow{\scriptstyle (id_R,d^2c)} \\ & R & \end{array}$$

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2 - JETS OF SECTIONS.

Let $\eta \equiv (E,p,M)$ be a bundle.

1 DEFINITION.

The JET SPACE, of order i, OF SECTIONS $M \to E$, is the set

where $$J^iE \equiv \bigsqcup_{p\in M} \mathcal{S}_{p/\rho^i_p} ,$$

a) $\mathcal{S}_p$ is the se of C^∞ sections $M \to E$ defined in a neighbourhood of p;

b) ρ^i_p is the restriction of the equivalence relation defined in (1,1) .

Let us remark that we get

$$J^i(M,N) = J^i(M \times N),$$

considering $M \times N$ as a trivial bundle on M.

2 PROPOSITION.

a) J^iE is a submanifold of $J^i(M,E)$.

b) The following diagram is commutative, for each $i > j$

$$\begin{array}{ccc} J^iE & \hookrightarrow & J^i(M,E) \\ {\scriptstyle \sigma^{ji}} \downarrow & & \downarrow {\scriptstyle \sigma^{ji}} \\ J^jE & \hookrightarrow & J^j(M,E) \\ & {\scriptstyle \sigma^j} \searrow \quad \swarrow {\scriptstyle \Pi' \circ \sigma^\circ j} & \\ & M & \end{array}$$

c) The triple $\eta^{ji} \equiv (J^iE, \sigma^{ji}, J^iE)$ is a bundle.

d) The triple $\eta^i \equiv (J^iE, \sigma^i, M)$ is a bundle.

e) If $f : M \to E$ is a section,

then $j^if : M \to J^iE$ is a section $\underline{\cdot}$

3 PROPOSITION.

We get a natural isomorphism

$$J^\circ E \cong E \quad \underline{\cdot}$$

4 THEOREM.

$\eta^{01} \equiv (J'E, \sigma^{01}, J^\circ E)$ is an affine subbundle of $(J'(M,E), \sigma^{01}, J^\circ(M,E))$

over the inclusion $J^\circ E \to J^\circ(M,E)$,

whose vector bundle is $\bar{\eta}^{01} \equiv (T^*M \otimes_E \nu TE, \Pi_E, E)$.

PROOF.

We have
$$J'E = \bigsqcup_{e \in E} \{\Phi_e\}$$

where $\Phi_e : T_{p(e)}M \to T_e E$

is any linear map such that

a) $Tp \circ \Phi_e = id_{T_{p(e)}M}$ $\doteq$

5 THEOREM.

$\eta^{12} \equiv (J^2E, \sigma^{12}, J'E)$ is an affine subbundle of $(J^2(M,E), \sigma^{12}, J'(M,E))$

over the inclusion $J'E \to J'(M,E)$, whose vector bundle is

$$\bar{\eta}^{12} \equiv (J'E \times_E (T^*M \vee_M T^*M \otimes_E \nu TE), \bar{\sigma}^{12}, J'(M,E))$$

PROOF.

We have $J^2E = \bigsqcup_{\varphi_e \in J'E} \{\bar{\Phi}_{\Phi_e}\}$

where $\bar{\Phi}_{\Phi_e} : T^2_{p(e)}M \to T^2_e E$

is any linear map as in (1.7), that satisfies the further condition

a) $T^2p \circ \bar{\Phi}_{\Phi_e} = id_{T^2_{p(e)}M}$ $\doteq$

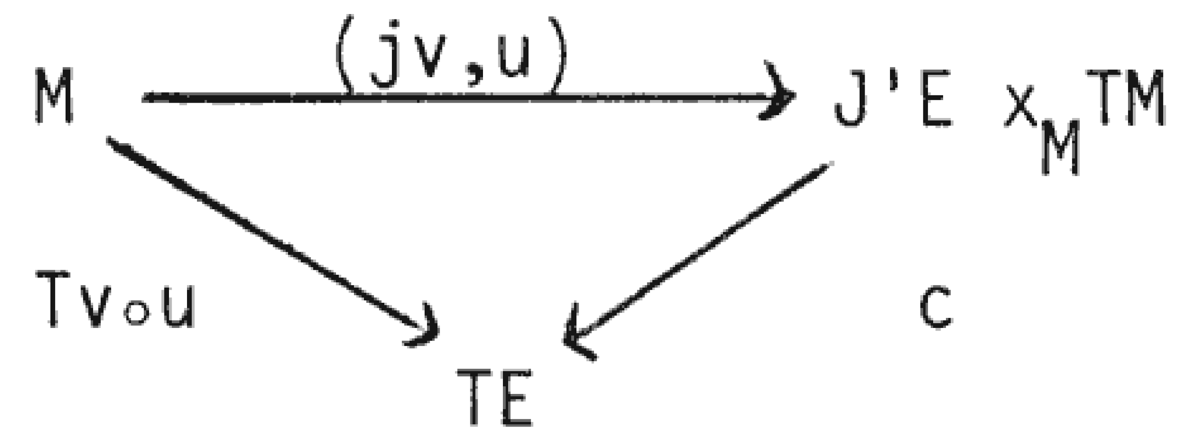
This theorem can be generalized to higher orders.

6 PROPOSITION.

There is a unique map

$$c : J'E \times_M TM \to TE$$

such that, for each section $u : M \to TM$, $v: M \to E$, the following diagram is commutative



Such a map is given by

$$c : (\Phi_e, u_{p(e)}) \to \Phi_e(u_{p(e)}) \in T_e E .$$

c is an affine morphism on hTE and a linear morphism on $J'E \to E$.
Hence the following diagram is commutative

$$\begin{array}{ccccc}
J'E & \xrightarrow{\quad \pi^{01} \quad} & & & E \\
\uparrow{\scriptstyle \pi'} & & & & \uparrow{\scriptstyle \pi_E} \\
J'E \times_M TM & \xrightarrow{\quad c \quad} & & & TE \\
& {\scriptstyle \pi^{01}\times id_{TM}} \searrow & hTE & \swarrow {\scriptstyle h} &
\end{array}$$

7 PROPOSITION.

There is a unique map

$$c^* : T^*M \times_M J'\,T\,M \to T\,T^*M$$

such that the following diagram is commutative

$$\begin{array}{ccc}
T^*M \times_M J'\,T\,M \times_M T\,M & \xrightarrow{\quad (soc^*, c) \quad} & T^*T\,M \times_{TM} T\,T\,M \\
\downarrow & & \downarrow {\scriptstyle <,>} \\
0 & \hookrightarrow & R
\end{array}$$

8 PROPOSITION.

Let $\eta \equiv (E,p,M)$ be an affine (vector) bundle, whose vector bundle is $\bar{\eta} \equiv (\bar{E},\bar{p},M)$.

Then $\eta^i \equiv (J^i E, \sigma^i, M)$ is an affine (vector) bundle and

$$\overline{J^i E} = J^i \bar{E} \quad .$$

PROOF.

The affine (vector) operations on E are compatible with respect to the equivalence relations ρ^i_p $\dot{-}$

9 COROLLARY.

Let $k > h > 0$.

Let $\eta \equiv (E,p,M)$ be a vector bundle.

Then $\eta^{hk} \equiv (J^kE, \sigma^{hk}, J^kE)$ is an affine bundle, whose vector bundle $\bar{\eta}^{hk}$ is the pull back bundle of $(\mathrm{Ker}\ \sigma^{hk}, \sigma^k, M)$ with respect to the map σ^h.

Namely the following diagram is commutative

$$\begin{array}{ccc} \bar{J}^{hk}E & \longrightarrow & \mathrm{Ker}\ \sigma^{hk} \\ \downarrow & & \downarrow \sigma^k \\ J^h & \xrightarrow{\sigma^h} & M \end{array} .$$

Moreover, if $h \equiv k-1$, we get

$$\bar{J}^{hk}E = J^hE \times_M (\vee_{kM} T^*M \otimes_M E) ,$$

where $\vee_{kM} T^*M$ is the K-symmetrized tensor product of T^*M over M.

Let us remark that, if $E \equiv M \times F$ (i.e. η is a trivial bundle), then we get

$$J^kE = E \oplus_M \mathrm{Ker}\ \sigma^{0k} = F \times \mathrm{Ker}\ \sigma^{0k} .$$

In such a case, we put

$$j'^k u \equiv \pi^2 \circ j^k u .$$

In particular, for $F \equiv R$, we get

$$j'^1 f = d f$$

10 PROPOSITION.

Let $\eta' \equiv (E',p',M)$ and $\eta'' = (E'',p'',M)$ be vector bundles.

There is a unique linear map $t : J^kE' \otimes_M J^kE'' \to J^k(E' \otimes_M E'')$

such that the following diagram is commutative

$$\begin{array}{ccc} J^kE' \otimes_M J^kE'' & \xrightarrow{t} & J^k(E' \otimes_M E'') \\ & \nwarrow \ (j^ku', j^ku'') \qquad j^k(u' \otimes u'') \ \nearrow & \\ & M & \end{array}$$

for each section $u' : M \to E'$, $u'' : M \to E''$.

11 COROLLARY.

Let $\eta \equiv (E,p,M)$ be a vector bundle.

There is a unique linear map on M

$$t : \mathbb{K} \otimes_M J^{k-1}E \to J^kE,$$

where $\mathbb{K}$ is the Kernel of the linear morphism $J^k(M\times R) \to J^\circ(M\times R)$ on M, such that

$$\begin{array}{ccc} \mathbb{K} \otimes_M J^{k-1}E & \xrightarrow{\;t'\;} & J^kE \\ {\scriptstyle j'^k f \otimes j^{k-1}u}\ \nwarrow & & \nearrow\ {\scriptstyle t(j^kf \otimes j^ku) - f\, j^ku\ .} \\ & M & \end{array}$$

As a particular case, we get

$$j^1(fu) = f\, j^1u + d\,f \otimes u,$$

being

$$t'(d\,f \otimes u) = d\,f \otimes u$$

12 PROPOSITION.

Let $\eta \equiv (E,p,M)$ be an affine bundle, whose vector bundle is $\bar{\eta} \equiv (\bar{E},\bar{p},M)$. There is a unique map

$$J'E \times_E J'E \xrightarrow{\;dif\;} T^*M \otimes_M \bar{E}$$

such that, for each vertical curve $c,c' : R \to E$, the following diagram is commutative

$$\begin{array}{ccc} J'E \times_E J'E & \longrightarrow & T^*M \otimes_M \bar{E} \\ {\scriptstyle (j'c,\, j^1c^1)}\ \nwarrow & & \nearrow\ {\scriptstyle j^1(c-c')} \\ & R & \end{array}$$

13 PROPOSITION.

Let $\eta \equiv (E,p,M)$ be a vector bundle and let $\eta^* \equiv (E^*,p',M)$ be the dual one. Ther is a unique map

$$J'E \times_M J'E^* \xrightarrow{\langle\,,\,\rangle} T^*M$$

such that, for each section $u : M \to E$ and $v : M \to E^*$, the following diagram is commutative

$$\begin{array}{ccc} J'E \times_M J'E^* & \xrightarrow{\langle\,,\,\rangle} & T^*M \\ (j^1u, j^1v) \nwarrow & & \nearrow \Pi^2(j^1\langle u,v\rangle) \\ & M & \end{array}$$

Such a map is bilinear .

14 PROPOSITION.

Let $\eta \equiv (E,p,M)$ be a vector bundle and let $g : E \times_M E \to R$ be a pseudo-Riemannian structure.

There is a unique map

$$\check{g} : J'E \times_M J'E \to T^*M$$

such that the following diagram is commutative, for each section $u,v : M \to E$

$$\begin{array}{ccc} J'E \times_M J'E & \xrightarrow{\check{g}} & T^*M \\ (j^1u, j'v) \nwarrow & & \nearrow \Pi^2 \circ j^1(g(u,v)) \\ & M & \end{array} .$$

3 LIE DERIVATIVES.

1 DEFINITION.

A K-LIE-DERIVABLE bundle is a 4-plet

$$\eta \equiv (E,p,M;B),$$

where (E,p,M) is a bundle and

$$B : E \times_M J^kTM \to TE$$

is a bundle morphism on hTE and a linear morphism on E .