case, analysis just carries over. It is like the situation in which you have an inverse function theorem valid for functions in three variables. It is then trivial to obtain (by holding one variable constant) a similar theorem for functions of two variables.

In any case, I tried to see if one could use a result like Theorem 2 for Fréchet spaces very different from $C^\infty(T)$. As we will see, the restriction to essentially this one space was no accident and it is necessary to change things quite a bit if we want to find an implicit function theorem valid in different kinds of Fréchet space.

Before we can get very far with such a program , it is necessary to say something about these spaces.

STRUCTURE OF FRÉCHET SPACES.

Recall that a Fréchet space is a complete, metrizable, locally convex space. Equivalently, it is a vector space E which is complete under a certain translation invariant metric and on which is defined an increasing sequence of seminorms (sub-additive , positive scalar homogeneous real-valued functions) $(\|\cdot\|_k)_{k\geq 0}$ such that a sequence (x_n) in E converges to x in E iff $\lim_n \|x_n - x\|_k = 0$ $(k=0,1,2,\ldots)$.

In all of our applications we will take the seminorms $\|\cdot\|_k$ to be norms (that is, $\|x\|_k = 0$ iff $x = 0$). Somewhat more complicated is the fact that the Fréchet spaces we consider will all be nuclear. It is best to defer the definition of nuclear until we are in a more concrete situation.

The basic references for all of our discussion of the structure of nuclear Fréchet spaces will be [1], [5] and [10]. For us, the best starting point is to list some examples:

$C^\infty(T)$ - *The space of infinitely differentiable real-valued functions on the unit circle equipped with the topology of uniform convergence of each derivative.*

$H(\mathbb{C})$ - *The space of complex-valued functions of one complex variable, analytic in the complex plane, equipped with the compact-open topology.*

$H(\mathbb{D})$ - *The space of complex-valued functions of one complex variable, analytic in the open unit disk, equipped with the compact-open topology.*

For each of these spaces we give one possible choice of the norms which define the topology.

$C^\infty(T)$: $\|x\|_k = \sup\{|x^p(t)| : p = 0, \ldots, k, \quad t \in T\}$

$H(\mathbb{C})$: $\|x\|_k = \sup\{|x(t)| : |t| \leq k\}$

$H(\mathbb{D})$: $\|x\|_k = \sup\{|x(t)| : |t| \leq \frac{k}{k+1}$

Each of these examples has what is called a coordinate representation or basis. For example, in $C^\infty(T)$ we can represent functions by their Fourier series and so write,

$$C^\infty(T) = \{\xi = (\xi_n) : (\xi_n) \text{ is the sequence of Fourier coefficients of an element of } C^\infty(T)\} =$$

$$= \{\xi = (\xi_n) : \sup_n |\xi_n| e^{k \log n} < \infty , \qquad k = 0, 1, 2, \ldots\} ,$$

where the second equality represents a standard fact about asymptotic behavior of Fourier coefficients of differentiable functions. More is true. If we set $\|x\|_k = \sup_n |\xi_n| e^{k \log n}$, where $x \in C^\infty(T)$ and (ξ_n) is its sequence of Fourier coefficients, then this definition of $\|\cdot\|_k$ works just as well to define the topology of $C^\infty(T)$.

The same thing can be done for $H(\mathbb{C})$ and $H(\mathbb{D})$ using power series expansions.

We obtain,

$$H(\mathbb{C}) = \{\xi = (\xi_n) : \|\xi\|_k = \sup_n |\xi_n| \, e^{kn} < \infty, \qquad k=0,1,2,\ldots\}$$

$$H(\mathbb{D}) = \{\xi = (\xi_n) : \|\xi\|_k = \sup_n |\xi_n| \, e^{-\frac{n}{k}} < \infty, \qquad k=0,1,2,\ldots\}$$

Clearly we can abstract all this by writing down an infinite matrix of positive numbers $a = (a_{nk})$ instead of $e^{k \log n}$, e^{kn} or $e^{-\frac{n}{k}}$.

Then the Fréchet space we obtain is given by

$$K(a) = \{\xi = (\xi_n) : \|\xi\|_k = \sup_n |\xi_n| \, a_{nk} < \infty, \qquad k=0,1,2,\ldots\}$$

and we only need assume that $a_{nk} \leq a_{n,k+1}$. The condition that $K(a)$ is nuclear can then be expressed as follows :

$$\forall k \quad \exists j \quad \text{such that} \quad \sum_n \frac{a_{nk}}{a_{nj}} < \infty$$

If e_n is the sequence which is 0 except in the n^{th} coordinate where it is 1, then each element of $K(a)$ can be expanded as an infinte series in (e_n) and considering the coefficient sequences gives $K(a)$ back.

It is a simple but informative exercise to verify the details of this general formulation for the three examples mentioned above.

The notion of coordinate representation leads to another notion that permits us to construct other examples of nuclear Fréchet spaces. Let $\alpha = (\alpha_n)$ be a subsequence of the sequence $\mathbb{N}$ of positive integers. If E is a nuclear Fréchet space given via a coordinate representation, then $(E)_\alpha$ denotes all of those elements of E whose corresponding sequence ξ has the property that $\xi_n = 0$ unless n is one of the terms of α.

Thus $(K(a))_\alpha$ is $K(b)$ where $b_{nk} = a_{\alpha_n,k}$.

It is clear that $(E)_\alpha$ is again a nuclear Frechét space. It is complemented in the sense that, if $\beta = N \sim \alpha$, then E is isomorphic to the product $(E)_\alpha x(E)_\beta$. We call $(E)_\alpha$ a coordinate subspace of E.

This completes the preliminaries for the structure theory and now I would like to say something about the content of this theory. Generally speaking the questions considered are of the following type. Given a nuclear Fréchet space E and another one F, find quantitative conditions which determine whether F is isomorphic to a subspace of E. Usually E is a space which can be given in a coordinate representation, say $E = K(a)$. This may or may not be the case for F. If it is, say F is isomorphic to K(b), then the condition is given in terms of the matrices a,b. If not, then the condition is in terms of the norms.

The structure theory involves much more than I have mentioned. It is possible to replace subspace by quotient, or even complemented subspace. There are investigations which try to determine what effect this has in guaranteeing that F has a coordinate representation. Other approximation properties have been studied and there is a lot of work in determining when concrete function spaces fall under this theory. I have not tried to give a serious bibliography here. Many results are quite recent and just now beginning to find their way into print.

It will be useful for us to go a bit beyond these generalities and to give at least one definite example of how the structure theory works. We consider the possibility that a space K(b) is isomorphic to a subspace of $C^\infty(T)$. We will use the fact (mentioned above) that $C^\infty(T)$ is isomorphic to K(a) where $a_{nk} = e^{k \log n} = n^k$.

A necessary condition can be derived quite easily as follow. We have an isomorphism $A : K(b) \to K(a)$ so (after passing to a subsequence on k if necessary) we can write

$$\|x\|_k \le C_k^1 \|Ax\|_{k+1} \le C_k^2 \|x\|_{k+2} \qquad (x \in K(b),\ k=1,2,\dots).$$

Applying this with $x = e_n$ we can derive

$$\frac{1}{C_k^3} \frac{\|Ae_n\|_{k+1}}{\|Ae_n\|_k} \le \frac{\|e_n\|_{k+2}}{\|e_n\|_{k-1}} = \frac{b_{n,k+2}}{b_{n,k-1}} \le C_k^4 \frac{\|Ae_n\|_{k+3}}{\|Ae_n\|_{k-2}} \qquad (n,k=1,2,3,\dots)$$

Now if Ae_n is the sequence $(\xi_\nu^n)_\nu$ in $K(a)$ we have

$$\|Ae_n\|_k = \sup_\nu |\xi_\nu^n| \nu^k = |\xi_{g_n^k}| (g_n^k)^k ,$$

where g_n^k is the largest value of ν at which the sup occurs. (We use here the fact that $\sup_\nu |\xi_\nu^n| \nu^k < \infty$ for every k implies $\lim_\nu |\xi_\nu^n| \nu^k = 0$).

Now using the property of sup we have, for any k,j,

$$\frac{|\xi_{g_n^k}|(g_n^k)^j}{|\xi_{g_n^k}|(g_n^k)^k} \le \frac{|\xi_{g_n^j}|(g_n^j)^j}{|\xi_{g_n^k}|(g_n^k)^k} \le \frac{|\xi_{g_n^j}|(g_n^j)^j}{|\xi_{g_n^j}|(g_n^j)^k}$$

from which we conclude that

$$(g_n^k)^{j-k} \le \frac{\|Ae_n\|_j}{\|Ae_n\|_k} \le (g_n^j)^{j-k} .$$

Together with our first inequality, and writing only C for any positive constant independent of n, we obtain,

$$\frac{1}{C} \frac{b_{n,k+2}}{b_{n,k-1}} \le (g^{k+3})^5 \le \frac{\|Ae_n\|_{k+8}}{\|Ae_n\|_{k+3}} \le C \frac{b_{n,k+9}}{b_{n,k+2}}$$

which, passing to a subsequence, gives

$$\frac{b_{nk}}{b_{n,k-1}} \leq C \frac{b_{n,k+1}}{b_{nk}} .$$

It is very interesting that this simple condition which we derived is also sufficient and we have,

THEOREM 3.

A nuclear Fréchet space E *with a coordinate representation* K(b) *is isomorphic to a subspace of* $C^\infty(T)$ *iff, after passing to an appropriate subsequence on* k *, we have, for every* k ,

$$\sup_n \frac{(b_{nk})^2}{b_{n,k-1}\, b_{n,k+1}} < \infty .$$

A proof of this and many similar results can be found in [1]. What is even more striking is that results of this kind with equally simple conditions can be obtained without the assumption that E has a coordinate representation (see [9], [11] and [12]).

One important consequence of this characterization is that neither $H(\mathbb{D})$ nor any of its coordinate subspaces is isomorphic to a subspace of $C^\infty(T)$. To see this we use the fact that any coordinate subspace of $H(\mathbb{D})$ is isomorphic to a space K(b) where $b_{nk} = e^{-\frac{\alpha_n}{j(k)}}$ and $(j(k))$ is any subsequence of $\mathbb{N}$. This gives,

$$\frac{(b_{n,j(k)})^2}{b_{n,j(k-1)}\, b_{n,j(k+1)}} = e^{\alpha_n \left(\frac{1}{j(k-1)} + \frac{1}{j(k+1)} - \frac{2}{j(k)}\right)} \geq e^{\alpha_n \left(\frac{1}{j(k-1)} - \frac{2}{j(k)}\right)} .$$

Thus, if $j(k) > 2j(k-1)$ this quantity is unbounded and passing to a subsequence on k will not help. Therefore the condition of Theorem 3 is violated.

CONNECTIONS. -

Now we turn to the main topic of these lectures which is the description of certain connections between the structure theory and inverse function theorems. Our first remark is an observation that shows how special is the relation (3) which the smoothing operators are assumed to satisfy.

If we have (3), then any $x \in E$, $\theta > 0$ and $k < j < \ell$ we would have,

$$\|x\|_j \leq \|S_\theta x\|_j + \|x-S_\theta x\|_j \leq C(\theta^{j-k}\|x\|_k + \theta^{j-\ell}\|x\|_\ell).$$

We can use calculus to show that the right hand side achieves its minimum value when

$$\theta = \left(\frac{\|x\|_\ell\,(\ell-j)}{\|x\|_k\,(j-k)}\right)^{\frac{1}{\ell-k}},$$

and substituting this value for θ with $k=j-1$, $\ell=j+1$ yields

$$(\|x\|_j)^2 \leq C\,\|x\|_{j-1}\,\|x\|_{j+1} .$$

This immediately implies the condition of Theorem 3 so, with the above discussion, we may conclude that Theorem 2 does not hold for $H(\mathbb{D})$. Actually we have the following much stronger result of D. Vogt.

THEOREM 4. -

A nuclear Fréchet space E *has a family of smoothing operators satisfyng* (3) *if and only if* E *is isomorphic to a coordinate subspace of* $C^\infty(T)$.