2 - FURTHER SPACES AND MAPS.

Now we introduce some further notions concerning applied vector spaces and maps.

Vertical and unitary spaces.

1 We introduce the spaces of applied vectors relative to $\bar{\mathbb{S}}$, and $\mathbb{U}$.

DEFINITION.

The VERTICAL SPACE WITH RESPECT TO $(\mathbb{E},t,\mathbb{T})$, or the PHASE SPACE, or the ACCELERATION SPACE, is

$$\mathbb{A} \equiv \check{T}\,\mathbb{E} \equiv \mathrm{Ker}\ \ Tt \ = \mathbb{E} \times \bar{\mathbb{S}} \hookrightarrow T\,\mathbb{E} .$$

The HORIZONTAL SPACE WITH, RESPECT TO $(\mathbb{E},t,\mathbb{T})$ is

$$\mathring{T}\,\mathbb{E} \equiv T\mathbb{E}_{/\check{T}\mathbb{E}} = \mathbb{E} \times \bar{\mathbb{T}} \quad .$$

The UNITARY SPACE, or the VELOCITY SPACE, is

$$\mathbb{V} \equiv \overset{|}{T}\mathbb{E} \equiv (Tt)^{-1}(\mathbb{T} \times 1) = \mathbb{E} \times \mathbb{U} \hookrightarrow \mathbb{E} \quad .$$

2 Let us remember that $T\mathbb{E}$ has two bundle structures, namely

$$(T\mathbb{E}, Tt, T\,\mathbb{T}) \qquad \text{and} \qquad (T\mathbb{E}, \pi_{\mathbb{E}}, \mathbb{E}) .$$

PROPOSITION.

a) $\check{T}\mathbb{E}$ is the submanifold of $T\mathbb{E}$ characterized by $\dot{x}^{\circ} = 0$

$\overset{|}{T}\mathbb{E}$ is the submanifold of $T\mathbb{E}$ characterized by $\dot{x}^{\circ} = 1$.

b) $\check{T}\mathbb{E}$ and $\overset{|}{T}\mathbb{E}$ have two natural bundle structures, namely

$$(\check{T}\mathbb{E}, \check{t}, T) \qquad \text{and} \qquad (\check{T}\mathbb{E}, \check{\pi}_{\mathbb{E}}, \mathbb{E}) .$$

$$(\overset{\prime}{T}\mathbb{E}, t, \mathbb{T}) \quad \text{and} \quad (T\mathbb{E}, \pi_{\mathbb{E}}, \mathbb{E}).$$

c) The sequence $0 \to \acute{T}\mathbb{E} \hookrightarrow T\mathbb{E} \to T\mathbb{E} \to 0$ is exact .

We have not a canonical splitting of $T\mathbb{E}$, as we have not a canonical projection $T\mathbb{E} \to T\mathbb{E}$, or a canonical inclusion $T\mathbb{E} \hookrightarrow T\mathbb{E}$.

In the same way we have not a canonical isomorphism $\acute{T}\mathbb{E} \to \overset{\prime}{T}\mathbb{E}$.

3 We can extend the vertical derivative in terms of applied vectors.

DEFINITION.

Let F be a C^∞ manifold and $f : \mathbb{E} \to \mathbf{F}$ a C^∞ map.

The VERTICAL TANGENT MAP of f, WITH RESPECT TO $(\mathbb{E}, t, \mathbb{T})$, is the map

$$\acute{T}f \equiv Tf_{\acute{T}\mathbb{E}} : \acute{T}\mathbb{E} \to T\mathbb{E} .$$

4 We can view the metric as a function on $\check{T}\mathbb{E}$, which will become the kinetic energy in dynamics.

DEFINITION.

The METRIC FUNCTION is the funtion

$$\check{\mathbf{g}} : \acute{T}\mathbb{E} \to \mathbb{R},$$

given by

$$(e,u) \to \frac{1}{2} u^2 .$$

5 PROPOSITION.

We have

$$\check{\mathbf{g}} = \frac{1}{2} \check{g}_{ij} \dot{x}^i \dot{x}^j .$$

<u>Second order spaces, affine connection and canonical projection.</u>

6 We consider now the second order tangent spaces.

DEFINITION.

The VERTICAL SPACE, WITH RESPECT TO $(\check{T}\mathbb{E},\check{t},\mathbb{T})$, is

$$\check{T}^2\mathbb{E} \equiv \mathrm{Ker}\ \ T\check{t} = \mathbb{E}\times\bar{\mathbb{S}}\times\bar{\mathbb{S}}\times\bar{\mathbb{S}} \hookrightarrow T^2\mathbb{E}\ .$$

The VERTICAL SPACE, WITH RESPECT TO $(\check{T}\mathbb{E},\check{t},\mathbb{T})$ and $(\check{T}\mathbb{E},\check{\pi}_{\mathbb{E}},\mathbb{E})$, is

$$\nu\check{T}^2\mathbb{E} \equiv \mathrm{Ker}\ \ T\check{t}\quad \mathrm{Ker}\ \ T\check{\pi}_{\mathbb{E}} = \mathbb{E}\times\bar{\mathbb{S}}\times 0\times\bar{\mathbb{S}} \hookrightarrow T^2\mathbb{E}.$$

The BIUNITARY SPACE or BIVELOCITY SPACE, is

$$\mathbb{V}^2 \equiv \acute{T}\mathbb{E} \equiv sT\acute{T}\mathbb{E} \equiv \mathbb{E}\times\mathbb{U}\times\bar{\mathbb{S}} \xhookrightarrow{\text{diagonal}} \mathbb{E}\times\mathbb{U}\times\mathbb{U}\times\bar{\mathbb{S}} \hookrightarrow T^2\mathbb{E}\ .$$

The VERTICAL BIUNITARY SPACE, WITH RESPECT TO $(\acute{T}\mathbb{E},\acute{\pi}_{\mathbb{E}},\mathbb{E})$, is

$$\nu\mathbb{V}^2 \equiv \nu\acute{T}^2\mathbb{E} \equiv \nu T\acute{T}\mathbb{E} = \mathbb{E}\times\mathbb{U}\times 0\times\bar{\mathbb{S}} \hookrightarrow T^2\mathbb{E}\ .$$

7 PROPOSITION.

$\check{T}^2\mathbb{E}$ is the submanifold of $T^2\mathbb{E}$ characterized by $\check{x}^\circ = \dot{\check{x}}^\circ = \ddot{x}^\circ = 0$.

$\nu\check{T}^2\mathbb{E}$ " " " " " " " $\check{x}^\circ = \dot{\check{x}}^\alpha = \ddot{x}^\circ = 0$.

$\acute{T}^2\mathbb{E}$ " " " " " " " $\check{x}^\circ = \dot{\check{x}}^\circ = 1, \bar{\check{x}}^i = \dot{\check{x}}^i, \ddot{x}^\circ = 0$.

$\nu\acute{T}^2\mathbb{E}$ " " " " " " " $\check{x}^\circ = 1, \dot{\bar{x}}^\alpha = 0, \ddot{x}^\circ = 0$.

8 Let us consider some important canonical maps, which are used to define the covariant derivatives.

DEFINITION.

a) The AFFINE CONNECTION MAP

$$\Gamma : T^2\mathbb{E} \to \nu T^2\mathbb{E}\ ,$$

given by $(e,u,v,w) \mapsto (e,u,o,w)$,

induces naturally the maps

$$\check{\Gamma} : \check{T}^2\mathbb{E} \to \nu\check{T}^2\mathbb{E}$$

and $$\acute{\Gamma} : \acute{T}^2\mathbb{E} \to \nu\acute{T}^2\mathbb{E} .$$

b) The CANONICAL PROJECTION (which is an isomorphism on fibers).

$$\sqcup : \nu T^2\mathbb{E} \to T\mathbb{E} ,$$

given by $$(e,u,o,w) \mapsto (e,w),$$

induces naturally the maps

$$\check{\sqcup} : \nu\check{T}^2\mathbb{E} \to \check{T}\mathbb{E}$$

and $$\acute{\sqcup} : \nu\acute{T}^2\mathbb{E} \to \check{T}\mathbb{E} \ \underline{.}$$

9 PROPOSITION.

We have
$$\begin{cases} \check{\dot{x}}^{\alpha} \circ \Gamma = \dot{\dot{x}}^{\alpha} \\ \check{\dot{x}}^{\alpha} \circ \Gamma = \dot{\dot{x}}^{\alpha} \\ \dot{\check{x}}^{\alpha} \circ \Gamma = 0 \\ \ddot{x}^{\circ} \circ \Gamma = \ddot{x}^{\circ} \ ; \ \ddot{x}^{k} \circ \Gamma = \ddot{x}^{k} + \check{\Gamma}^{k}_{\alpha\beta}\, \dot{x}^{\alpha}\, \dot{x}^{\beta} . \end{cases}$$

We have
$$\begin{cases} \check{x}^{\alpha} \circ \sqcup = \dot{x}^{\alpha} \\ \dot{x}^{\alpha} \circ \sqcup = \ddot{x}^{\alpha} \end{cases} \qquad \underline{.}$$

10 Then we can introduce the covariant derivative in a way that, not making an essential use of free vectors, can be extended to manifolds.

DEFINITION.

Let $u \equiv (id_{\mathbb{E}}, \check{u}) : \mathbb{E} \to T\mathbb{E}$ and $v \equiv (id_{\mathbb{E}}, \check{v}) : \mathbb{E} \to T\mathbb{E}$ be C^{∞} vector fields.

The COVARIANT DERIVATIVE of v with respect to u is

$$\nabla_u v \equiv \coprod \circ \Gamma \circ T v \circ u = (id_{\mathbb{E}}, D\tilde{u}(\tilde{v})) : E \to T\mathbb{E} \quad .$$

3 - ASSOLUTE KINEMATICS.

Here we introduce the basic elements of one-body kinematics independent of any frame of reference.

Absolute world-line and motion.

1 The basic definition of kinematics is the following. Here we consider a C^∞ world-line extending along the whole T. We leave to the reader the easy generalization to the case when it is C^2 almost every where, or when it extends along an interval of T.

DEFINITION.

A WORLD-LINE is a connected C^∞ submanifold

$$\mathbb{M} \hookrightarrow \mathbb{E}$$

such that $\mathbb{S}_\tau \cap \mathbb{M}$ is a singleton, $\forall \tau \in \mathbb{T}$.

The MOTION, RELATIVE TO THE WORLD LINE $\mathbb{M}$, is the map

$$M : \mathbb{T} \to \mathbb{E}$$

given by $\tau \mapsto$ the unique element $\in \mathbb{S}_\tau \cap \mathbb{M}$.

Henceforth in this section we suppose a world-line $\mathbb{M}$, or its motion M, to be given.