on X and D is any divisor, then

$$\ell(D) = \deg D + 1 - g + \ell(W-D).$$

8. THE OSCULATING HYPERPLANE OF A CURVE

Let X be an irreducible, non-singular, projective, algebraic curve of genus g defined over K but viewed as the set of points defined over $\bar{K}$, and let $f : X \to \mathscr{C} \subset PG(n,\bar{K})$ be a suitable rational map. Then $\mathscr{C}$ is viewed as the set of branches of X.

Assume that $\mathscr{C}$ is not contained in a hyperplane. The <u>degree</u> d of $\mathscr{C}$ is the number of points of intersection of $\mathscr{C}$ with a generic hyperplane. For any hyperplane H, if n_p is the intersection multiplicity of H and $\mathscr{C}$ at P, then

$$H \cdot \mathscr{C} = \sum_{P \in \mathscr{C}} n_P \, P$$

is a <u>divisor</u> of degree $d = \Sigma n_P$. Also

$$\mathscr{D} = \{H.\mathscr{C} \mid H \text{ a hyperplane}\}$$

is a <u>linear system</u>. In this case, $D \sim D'$ for any D, D' in $\mathscr{D}$. Hence $\mathscr{D}$ is contained in the <u>complete</u> linear system $|D| = \{D' \mid D' \sim D\}$, where D is some element of $\mathscr{D}$.

A complete linear system defines an embedding $f : X \to \mathscr{C}$ given by

$$f(Q) = P(f_o(Q), \ldots, f_n(Q))$$

where $\{f_o, \ldots, f_n\}$ is a basis of

$$L(D) = \{g \in \bar{K}(X) \mid \mathrm{div}(g) + D \geq 0\}.$$

Given a linear system $\mathcal{D}$, the complete system containing $\mathcal{D}$ has the same degree as $\mathcal{D}$ and possibly larger dimension. Hence, although not necessary, it is simpler to consider complete linear systems, and this we do.

Let $\mathcal{C}$ of degree d have associated complete linear system $\mathcal{D}$ and let P be a fixed point of $\mathcal{C}$. Let $\mathcal{D}_i$ be the set of hyperplanes passing through P with multiplicity at least i. Then

$$\mathcal{D} = \mathcal{D}_o \supset \mathcal{D}_1 \supset \ldots \supset \mathcal{D}_d \supset \mathcal{D}_{d+1} = \emptyset.$$

Each $\mathcal{D}_i$ is a projective space. If $\mathcal{D}_i \neq \mathcal{D}_{i+1}$, then $\mathcal{D}_{i+1}$ has codimension one in $\mathcal{D}_i$. Such an i is a $(\mathcal{D},P)$-<u>order</u>. So the $(\mathcal{D},P)$-orders are $j_o, \ldots, j_n$, where

$$0 = j_o < j_1 < j_2 < \ldots < j_n \leq d.$$

Note that $j_1 = 1$ if and only if P is non singular.

For example, let $\mathcal{C}$ be a plane cubic.
Then

$$(j_o, j_1, j_2) = \begin{cases} (0,1,2) & \text{if P is neither singular nor an inflexion,} \\ (0,1,3) & \text{if P is an inflexion,} \\ (0,2,3) & \text{if P is singular.} \end{cases}$$

Note that, as the points of $\mathcal{C}$ are viewed as branches, each branch has a unique tangent.

The <u>Hasse derivative</u>, satisfies the following properties:

(i) $D_t^{(i)}(\Sigma a_j t^j) = \Sigma a_j \binom{j}{i} t^{j-i}$;

(ii) $D_t^{(i)}(fg) = \sum_{j=o}^{i} D_t^{(j)} f \cdot D_t^{(i-j)} g$;

(iii) $D_t^{(i)} D_t^{(j)} = \binom{i+j}{i} D_t^{(i+j)}$.

The unique hyperplane with intersection multiplicity j_n at P is the <u>osculating hyperplane</u> H_P and has equation

$$\det \begin{bmatrix} x_o & \cdots & x_n \\ D^{(j_o)} f_o & & D^{(j_o)} f_n \\ \vdots & & \vdots \\ D^{(j_{n-1})} f_o & & D^{(j_{n-1})} f_n \end{bmatrix} = 0$$

For example, if $\mathscr{C}$ is the twisted cubic in PG(3,K),

$$(f_o, f_1, f_2, f_3) = (1, t, t^2, t^3),$$

$$(j_o, j_1, j_2, j_3) = (0, 1, 2, 3).$$

The osculating hyperplane at $P(1,t,t^2,t^3)$ is

$$\det \begin{bmatrix} x_o & x_1 & x_2 & x_3 \\ 1 & t & t^2 & t^3 \\ 0 & 1 & 2t & 3t^2 \\ 0 & 0 & 1 & 3t \end{bmatrix} = 0 ;$$

that is,

$$t^3 x_o - 3t^2 x_1 + 3t x_2 - x_3 = 0.$$

The point P on $\mathscr{C}$ is a <u>Weierstrass point</u>, W-point for short, if $(j_o, j_1, \ldots, j_n) \neq (0, 1, \ldots, n)$.

Since $\mathcal{D}$ is complete, the Riemann-Roch theorem gives that, if $d>2g-2$, then

(i) $n = d-g$;

(ii) $\dim \mathcal{D}_i = d-g-i$ for $i \leq d - 2g + 1$;

(iii) $j_i = i$ for $i \leq d - 2g$.

Let $L_i = \cap$ hyperplanes meeting $\mathcal{C}$ at P with $n_P \geq j_i+1$. Then L_i is dual to $\mathcal{D}_i$ and

$$L_o \subset L_1 \subset L_2 \subset \ldots \subset L_{n-1}.$$

Also $L_o = \{P\}$, the set L_1 is the tangent line at P, and L_{n-1} is the osculating hyperplane at P.

The point P is a $\mathcal{D}$-<u>osculation point</u> if $j_n>n$, that is, there exists a hyperplane H such that $n_P > n$.

The integers j_i are characterized by the following result.

THEOREM 8.1 : (i) If $j_o,\ldots,j_{i-1}$ are known, then j_i is the smallest integer r such that $D^{(r)}f(Q)$ is linearly independent of $\{D^{(j_o)}f(Q),\ldots,D^{(j_{i-1})}f(Q)\}$; the latter set spans L_{i-1}.

(ii) If $0 \leq r_o < .. < r_s$ are integers such that $D^{(r_o)}f(Q),\ldots,D^{(r_s)}f(Q)$ are linearly independent, then $j_i \leq r_i$.