2. THE MAXIMUM NUMBER OF POINTS ON AN ALGEBRAIC CURVE

Let $\mathscr{C}$ be an algebraic curve defined over GF(q) of genus g, and let N_1 be the number of points, rational over GF(q), on a non-singular model of $\mathscr{C}$. Define $N_q(g) = \max N_1$, where $\mathscr{C}$ varies over all curves of genus g. We recall the following bounds.

(i) Hasse-Weil: $N_q(q) \leq q+1+2gq^{1/2}$

(ii) Serre: $N_q(g) \leq q+1+g[2q^{1/2}]$

(iii) Ihara: $N_q(g) \leq q+1 - \frac{1}{2}g + \{2(q+1/8)g^2+(q^2-q)g\}^{1/2}$

(iv) Manin: $N_2(q) \leq 2g - \sigma(g)$ as $g \to \infty$

$N_3(g) \leq 3g + \sigma(g)$ as $g \to \infty$

(v) Drinfeld-Vladut: $N_q(g) \leq g(q^{1/2}-1)+\sigma(g)$ as $g \to \infty$.

For a summary of results on $N_q(g)$ and references, see [9] Appendix IV.

The estimates (i) and (ii) are good for $g \leq \frac{1}{2}(q-q^{1/2})$, but not for $g > \frac{1}{2}(q-q^{1/2})$.

One of the aims of these notes is to describe improvements to (i), (ii), (iii). First, it is elementary that (ii) is sometimes better than (i) and never worse.

Let $m = [2q^{1/2}]$. Then $2q^{1/2} = m+\varepsilon$, where $0 \leq \varepsilon < 1$. So

$$[2gq^{1/2}] = [g(m+\varepsilon)] = [gm+g\varepsilon] = gm+[g\varepsilon].$$

3. THE DEDUCTION OF SERRE'S AND IHARA'S RESULTS FROM THE RIEMANN HYPOTHESIS.

(a) Serre's result