STRUCTURE, CONGRUENCES AND VARIETIES OF COMPLETELY REGULAR SEMIGROUPS

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1. LOCAL AND GLOBAL STRUCTURE

One area of research in the field of Semigroup Theory in which there have been significant successes in recent years has been the subject of completely regular semigroups. The aim of these lectures is to give a brief review of some of the achievements in the theory of completely regular semigroups. We will start with some familiar and well known results and concepts.

An element a of a semigroup S is _regular_ if there exists an element x in S such that $a = axa$ and a semigroup S is _regular_ if every element of S is regular.

If $a, x \in S$, a semigroup, are such that $a = axa$ and $y = xax$, then a simple calculation will verify that $a = aya$ and $y = yay$. Such an element y is called an _inverse_ of a.

An element a of a semigroup S is _completely regular_ if there exists an element $x \in S$ such that $a = axa$ and $ax = xa$. In particular, x must be an inverse of a.

LEMMA 1.1. For any element a in a semigroup S, the following statements are equivalent.

(i) a is completely regular.

(ii) a has an inverse with which it commutes.

(iii) H_a is a subgroup.

We say that a semigroup S is _completely regular_ if every element of S is completely regular.

LEMMA 1.2. For any semigroup S the following statements are equivalent.

(i) S is completely regular.

(ii) S is a union of (disjoint) groups.

(iii) Every $\mathcal{H}$ - class of S is a group.

NOTATION Let $\mathcal{CR}$ denote the class of all completely regular semigroups and for any $a \in S \in \mathcal{CR}$, let a^{-1} denote the inverse of a in the (group) $\mathcal{H}$ - class H_a and let a^0 denote the element $aa^{-1} = a^{-1}a$, the identity of the group H_a.

It is not hard to see that the class $\mathcal{CR}$ is closed with respect to products and homomorphic images. However, the additive group of integers is completely regular but has the infinite cyclic semigroup of positive integers, which is not completely regular, as a subsemigroup. Thus the class $\mathcal{CR}$ is not closed under subsemigroups. On the other hand, any subsemigroup of a completely regular semigroup which is closed under inverses ($a \longrightarrow a^{-1}$) is also completely regular.

These observations suggest considering completely regular semigroups not simply as semigroups but as semigroups endowed with a unary operation ($a \longrightarrow a^{-1}$). This has now become the accepted viewpoint from which to study the class $\mathcal{CR}$. When we do this the class $\mathcal{CR}$ becomes a variety of algebras endowed with a binary and a unary operation satisfying the following identities:

$$x(yz) = (xy)z, \quad x = xx^{-1}x, \quad (x^{-1})^{-1} = x, \quad xx^{-1} = x^{-1}x.$$

In this context, consistent with earlier notation, we shall write $x^0 = xx^{-1} = x^{-1}x$.

The manipulation of inverses in completely regular semigroups can present quite a problem. One observation that is sometimes helpful is the following.

LEMMA 1.3. (Petrich and Reilly [19], Lemma 2.8) The variety $\mathcal{CR}$ satisfies the identity

$$(xy)^{-1} = (xy)^0y^{-1}(yx)^0x^{-1}(xy)^0.$$

Recall that a simple semigroup is one without proper ideals. A completely simple semigroup is one which is both completely regular and simple.

Let S be the disjoint union of the semigroups S_α $(\alpha \in Y)$, where Y is a semilattice and $S_\alpha S_\beta \subseteq S_{\alpha\beta}$. Then S is said to be a semilattice of the semigroups S_α, $\alpha \in Y$, and we write $S = (Y;S_\alpha)$. The importance of this concept in the theory of completely regular semigroups was revealed by the following theorem.

THEOREM 1.4. (Clifford [2] and [4], Theorem 4.6) Let S be a completely regular semigroup. Then $\mathcal{D} = \mathcal{J}$ is a congruence, each $\mathcal{J}$-class is a completely simple semigroup and $S/\mathcal{J}$ is a semilattice. Thus S is a semilattice of its $\mathcal{J}$-classes.

This theorem focusses the attention on the class of completely simple semigroups, not just as an interesting special class of completely regular semigroups but as an essential component of the structure of all completely regular semigroups. That the class of completely simple semigroups is an interesting class is also attested to by the fact that it can be characterized in so many different ways, as illustrated in the next theorem.

We adopt the notation $E(S)$ for the set of idempotents of a semigroup S.

THEOREM 1.5. The following conditions on a semigroup S are equivalent.

(i) S is completely simple.

(ii) S is completely regular and satisfies the identity $(axb)^0 = (ab)^0$.

(iii) S is completely regular and satisfies the identity $(axa)^0 = a^0$.

(iv) S is completely regular and, for all $a,b,x \in S$, $ab \,\mathcal{H}\, axb$.

(v) S is completely regular and, for all $a,x \in S$, $a \,\mathcal{R}\, ax$.

(vi) S is regular and, for all $a,b \in S$, aSb is a maximal subgroup of S.

(vii) S is regular and weakly cancellative (that is, $ax = bx$ and $xa = xb$ implies that $a = b$).

(viii) S is regular and $a = axa$ implies that $x = xax$.

(ix) S is regular and every idempotent is primitive in $E(S)$ ($e \in E(S)$ is primitive if $f \in E(S)$ and $ef = fe = e$ implies that $e = f$).

(x) S is simple and $E(S)$ contains a primitive element.

It follows immediately from Theorem 1.5(ii) and (iii) that $\mathcal{CS}$ is a subvariety of $\mathcal{CR}$.

For any 4-tuple $(I,G,\Lambda;P)$ where G is a group, I and Λ are non-empty sets and $P\colon (\lambda,i) \longrightarrow p_{\lambda i}$ is a function from $\Lambda\times I$ to G, let

$\mathcal{M}(I,G,\Lambda;P) = I\times G\times\Lambda$ together with the multiplication

$$(i,g,\lambda)(j,h,\mu) = (i,gp_{\lambda j}h,\mu).$$

It is a straightforward exercise to show that $\mathcal{M}(I,G,\Lambda;P)$ is a completely simple semigroup. This construction is due to Rees and such semigroups are therefore called <u>Rees matrix semigroups</u>. However, Rees matrix semigroups are much more than examples of completely simple semigroups.

<u>THEOREM</u> 1.6. (Rees [31] and [4], Theorem 3.5) Every completely simple semigroup is isomorphic to a Rees matrix semigroup.

The Rees Theorem is tremendously important in the study of completely regular semigroups in general and completely simple semigroups in particular. Congruences and homomorphisms can be effectively studied in terms of the Rees matrix representations following from Theorem 1.6. Indeed, the construction of Rees matrix semigroups is so simple, it would almost seem as if any problem concerning completely simple semigroups could be resolved by the simple expedient of representing all completely simple semigroups as Rees matrix semigroups and then performing the appropriate arithmetic. While many problems are indeed amenable to such an approach it is not universally true as we shall see later.

We can view Clifford's Theorem as giving a global structure to any completely regular semigroup while Rees's Theorem provides a local structure. However, much of the complexity in the study of completely regular semigroups arises in going from the local to the global picture. This is perhaps best illustrated by the following general structure theorem for completely regular semigroups where the "simple" local components interact by means of factors and mappings.

<u>THEOREM</u> 1.7. (Petrich, [17]) For every $\alpha \in Y$ a semilattice, let $S_\alpha = \mathcal{M}(I_\alpha,G_\alpha,\Lambda_\alpha;P_\alpha)$ be normalized at $\alpha \in I_\alpha\cap\Lambda_\alpha$. For $\alpha \geq \beta$, let

(1) $\qquad \langle\ ,\ \rangle: S_\alpha\times I_\beta \longrightarrow I_\beta,$

(2) $\qquad S_\alpha \longrightarrow G_\beta,$ denoted by $a \longrightarrow a_\beta,$

(3) $\qquad [\ ,\ ]:\Lambda_\beta\times S_\alpha \longrightarrow \Lambda_\beta$

be functions such that, for $a \in S_\alpha$, $b \in S_\beta$

(i) if $i \in I_\beta$ and $\lambda \in \Lambda_\beta$, then

$$p_{\lambda\langle a,i\rangle} a_\beta p_{[\beta,a]i} = p_{\lambda\langle a,\beta\rangle} a_\beta p_{[\lambda,a]i}$$

(ii) if $i \in I_\alpha$ and $\lambda \in \Lambda_\alpha$, then

$$a = (\langle a,i\rangle,\ a_\alpha,\ [\lambda,a]).$$

On $S = \bigcup_{\alpha \in Y} S_\alpha$ define a multiplication by

$$(4) \qquad a \circ b = (\langle a,\langle b,\alpha\beta\rangle\rangle, a_{\alpha\beta} p_{[\alpha\beta,a]\langle b,\alpha\beta\rangle} b_{\alpha\beta}, [[\alpha\beta,a],b]).$$

Suppose that

(iii) for $\gamma \le \alpha\beta$, $i \in I_\gamma, \lambda \in \Lambda_\gamma$,

$$(\langle a,\langle b,i\rangle\rangle,\ a_\gamma p_{[\gamma,a]\langle b,\gamma\rangle} b_\gamma, [[\lambda,a],b]) = (\langle a \circ b,i\rangle, (a \circ b)_\gamma, [\lambda, a \circ b]).$$

Then S is a completely regular semigroup whose multiplication restricted to each S_α coincides with the given multiplication. Conversely, every completely regular semigroup is isomorphic to one so constructed.

This result is remarkable for its complete generality. A special case of particular importance arises as follows.

Let $S = (Y;S_\alpha)$ and, for all $\alpha, \beta \in Y$ with $\alpha \ge \beta$, let $\varphi_{\alpha,\beta}: S_\alpha \longrightarrow S_\beta$ be a homomorphism such that

(1) $\varphi_{\alpha,\alpha} = 1_\alpha$,

(2) for $\alpha \ge \beta \ge \gamma$, $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$.

If, in addition, for any $a \in S_\alpha$, $b \in S_\beta$, we have $ab = a\varphi_{\alpha,\alpha\beta} b\varphi_{\beta,\alpha\beta}$, then we say that S is a <u>strong semilattice of the semigroups</u> S_α and write $S = [Y;S_\alpha, \varphi_{\alpha,\beta}]$. Clearly, any strong semilattice of completely simple semigroups is completely regular. There are various nice characterizations of the semigroups that arise in this way. We require a few preliminary concepts.

Recall that a <u>normal band</u> is a band which satisfies the identity $axyb = ayxb$ and that a semigroup is a <u>normal cryptogroup</u> if $\mathcal{H}$ is a congruence on S and $S/\mathcal{H}$ is a normal band.

For any completely regular semigroup S, let the relation $\le$ be defined in S by: for $a,b \in S$

$$a \le b \iff a = eb = bf, \quad \text{for some } e,f \in E(S).$$

Let S be a completely regular semigroup with completely simple components S_α, $\alpha \in Y$. If S is such that, for $\alpha, \beta \in Y$ with $\alpha \ge \beta$, and any idempotent e in S_α there exists a unique idempotent f in S_β

with $e \geq f$, then S is said to <u>satisfy $\mathcal{D}$-majorization</u>.

Let

$\mathcal{CS}$ = the variety of completely simple semigroups

$\mathcal{S}$ = the variety of semilattices.

We can now provide a number of different characterizations of normal cryptogroups.

<u>THEOREM</u> 1.8. For any semigroup S the following statements are equivalent.

(i) S is a normal cryptogroup.

(ii) S is completely regular and, for all $e \in E(S)$, eSe is an inverse semigroup.

(iii) S is completely regular and, for all $e \in E(S)$, $E(eSe)$ is a semilattice.

(iv) S is completely regular and satisfies $\mathcal{D}$-majorization.

(v) $S = (Y;S_\alpha)$ is completely regular with completely simple components S_α and for all $\alpha, \beta \in Y$ with $\alpha \geq \beta$ and for all $a \in S_\alpha$, there exists a unique element $a^* \in S_\beta$ with $a^* \leq a$.

(vi) $S = (Y;S_\alpha)$ is a strong semilattice of the completely simple semigroups S_α, $\alpha \in Y$.

(vii) S is regular and a subdirect product of completely simple semigroups with, possibly, a zero adjoined.

(viii) $S \in \mathcal{CS} \vee \mathcal{S}$.

2. <u>CONGRUENCES</u>

We begin our treatment of congruences with congruences on completely simple semigroups. With the aid of the Rees Theorem, congruences on completely simple semigroups can be described fairly completely. The details of the following treatment can be found in Howie [10].

Let $S = \mathcal{M}(I,G,\Lambda;P)$. A triple $(\mathcal{S},N,\mathcal{T})$, where $\mathcal{S}$ is an equivalence relation on I, $\mathcal{T}$ is an equivalence relation on Λ and N is a normal subgroup of G, is said to be <u>admissible</u> if

$$(i,j) \in \mathcal{S} \text{ or } (\lambda,\mu) \in \mathcal{T} \implies p_{\lambda i}p_{\mu i}^{-1}p_{\mu j}p_{\lambda j}^{-1} \in N.$$

For any admissible triple $(\mathcal{S},N,\mathcal{T})$, define the relation $\rho_{(\mathcal{S},N,\mathcal{T})}$ on S by

$$(i,a,\lambda)\ \rho_{(\mathcal{S},N,\mathcal{T})}\ (j,b,\mu) \iff (i,j) \in \mathcal{S},\ (\lambda,\mu) \in \mathcal{T} \text{ and } p_{\zeta i}ap_{\lambda x}p_{\mu x}^{-1}b^{-1}p_{\zeta j}^{-1} \in N \text{ for some (all) } x \in I,\ \zeta \in \Lambda.$$

THEOREM 2.1. For any admissible triple $(\mathcal{S},N,\mathcal{T})$, $\rho_{(\mathcal{S},N,\mathcal{T})}$ is a congruence on $S = \mathcal{M}(I,G,\Lambda;P)$ and all congruences on S are of this form.

Given the stucture theorems of Clifford (Theorem 1.4), Rees (Theorem 1.6) Petrich (Theorem 1.7), it would be natural to investigate the properties of congruences on a completely regular semigroups by considering their restrictions to the completely simple components and how they can be reconstituted from these components. This approach has been succesfully explored by Petrich [18]. However, here I wish to explore an approach to the study of congruences which is less direct but which has provided a rich harvest of insights into not only the behaviour of congruences but also the lattice of varieties of completely regular semigroups.

DEFINITION Let ρ be a congruence on a completely regular semigroup S. Then the kernel of ρ is

$$\ker \rho = \{ a \in S: a\ \rho\ a^0\}$$

and the trace of ρ is

$$\operatorname{tr} \rho = \rho|_{E(S)}.$$

The key observation about the kernel and trace of a congruence is that in combination they completely determine the congruence.

LEMMA 2.2. (Pastijn and Petrich [14], Lemma 2.10) Let ρ be a congruence on a completely regular semigroup S. Then, for any elements $a,b \in S$,

$$a\ \rho\ b \iff a^0\ \operatorname{tr}\rho\ b^0 \quad \text{and} \quad ab^{-1} \in \ker\rho.$$

Proof. Let $a,b \in S$ and $a\ \rho\ b$. Then $a^0\ \rho\ b^0$ and $ab^{-1}\ \rho\ b^0$. Hence $a^0\ \operatorname{tr}\rho\ b^0$ and $ab^{-1} \in \ker\rho$. Conversely, suppose that $a^0\ \operatorname{tr}\rho\ b^0$ and $ab^{-1} \in \ker\rho$. Then

$$\begin{aligned} b &= b(b^{-1}b)b^{-1}b \\ &\rho\ b(a^{-1}a)b^{-1}b \\ &= ba^{-1}(ab^{-1})b \\ &\rho\ ba^{-1}(ab^{-1})(ab^{-1})b \\ &= b(a^{-1}a)b^{-1}a(b^{-1}b) \end{aligned}$$

$$
\begin{aligned}
&\rho \quad b(b^{-1}b)b^{-1}a(a^{-1}a)\\
&= \quad bb^{-1}a\\
&\rho \quad aa^{-1}a\\
&= \quad a.
\end{aligned}
$$

<u>COROLLARY</u> 2.3. (Feigenbaum [5], Theorem 4.1) Let λ, ρ be congruences on a completely regular semigroup S. Then

$$\lambda = \rho \quad \Longleftrightarrow \quad \ker \lambda = \ker \rho \quad \text{and} \quad \operatorname{tr} \lambda = \operatorname{tr} \rho.$$

This leads to natural questions concerning the nature of those subsets of a completely regular semigroup which are kernels for congruences and those equivalence relations on the set of idempotents which are the traces of congruences. The treatment presented here is essentially that of Pastijn and Petrich [14], specialized to completely regular semigroups as in (Petrich and Reilly [24]).

<u>DEFINITION</u> A subset K of a completely regular semigroup S is said to be a <u>normal subset</u> of S if it satisfies the following conditions:

(K1) $E(S) \subseteq K$,

(K2) $k \in K \Rightarrow k^{-1} \in K$,

(K3) $xy \in K \Rightarrow yx \in K$, $\quad (x,y \in S)$,

(K4) $x, x^0 y \in K \Rightarrow xy \in K$ $\quad (x,y \in S)$.

For any subset K of a semigroup S, we denote by π_K the largest congruence on S for which K is a union of π_K-classes. Then

$$a \, \pi_K \, b \quad \Longleftrightarrow \quad [xay \in K \Longleftrightarrow xby \in K \quad (x,y \in S^1)]$$

If γ is a relation on a semigroup S, then we denote by γ^* the congruence on S generated by γ, and if γ is an equivalence relation then we denote by γ^0 the largest congruence on S contained in γ.

<u>THEOREM</u> 2.4. (Pastijn and Petrich [14], Lemmas 2.4, 2.9 and Petrich and Reilly [24]) Let K be a subset of a completely regular semigroup S. Then the following statements are equivalent.

(1) K is a normal subset of S.

(2) K is the kernel of some congruence on S.

(3) K is the kernel of π_K.

When (1) - (3) hold, $\{(k,k^0): k \in K\}^*$ is the smallest congruence and π_K is the largest congruence on S with kernel K.

Next we consider the relations on the set of idempotents that arise from congruences.

<u>DEFINITION</u> Let S be a completely regular semigroup and τ be an equivalence relation on E(S). Then τ is a <u>normal equivalence</u> if it satisfies the following condition:

$$e \ \tau \ f \ \Rightarrow \ (xey)^0 \ \tau \ (xfy)^0 \qquad (x,y \in S^1).$$

<u>THEOREM</u> 2.5. (Pastijn and Petrich [14], Lemma 1.3 and Petrich and Reilly [24]) Let S be a completely regular semigroup and τ be an equivalence relation on E(S). Then the following conditions are equivalent.

(1) τ is a normal equivalence.

(2) τ is the trace of some congruence on S.

(3) $\tau = \operatorname{tr} \tau^*$.

When (1) - (3) hold, then τ^* is the smallest congruence and $(\mathcal{H}\tau\mathcal{H})^0$ is the largest congruence on S with trace τ.

Having successfully characterized those subsets of S that can be kernels and those equivalences on E(S) that can be traces, it is natural to consider when a normal subset and a normal equivalence can be combined to be the kernel and trace of a single congruence.

<u>DEFINITION</u> Let S be a completely regular semigroup, K be a normal subset of S and τ be a normal equivalence relation on E(S). Then (K,τ) is a <u>congruence pair</u> for S if K is a normal subset, τ is a normal equivalence and the following conditions are satisfied:

$$\text{(CP1)} \quad e \ \tau \ f \ \Longrightarrow \ [xey \in K \ \Longleftrightarrow \ xfy \in K, \text{ for all } x,y \in S^1]$$

$$\text{(CP2)} \quad k \in K \ \Longrightarrow \ (xky)^0 \ \ \tau \ \ (xk^0y)^0, \text{ for } x,y \in S^1.$$

From the definition of π_K it follows that (CP1) could be replaced by the equivalent condition

$$\text{(CP1)}^* \quad e \ \tau \ f \ \Longrightarrow \ e \ \pi_K \ f, \quad (\text{equivalently, } \tau \subseteq \operatorname{tr} \pi_K)$$

or, alternatively, invoking (K3) we could replace (CP1) by

$$\text{(CP1)}^{**} \quad e \ \tau \ f \ \Longrightarrow \ [ex \in K \ \Longleftrightarrow \ fx \in K].$$

In the same spirit, (CP2) can be replaced by the equivalent condition

$(CP2)^*$ $K \subseteq \ker (\mathcal{H}\tau\mathcal{H})^0$.

For any congruence pair (K,τ) for S, define the relation $\rho_{(K,\tau)}$ on S by

$$a \, \rho_{(K,\tau)} \, b \quad \Longleftrightarrow \quad a^0 \, \tau \, b^0 \, , \; ab^{-1} \in K \qquad (a,b \in S).$$

THEOREM 2.6. (Pastijn and Petrich [14], Theorem 2.13 and Petrich and Reilly [24]) Let S be a completely regular semigroup, K be a normal subset of S and τ be a normal equivalence relation on $E(S)$. Then the following statements are equivalent.

(1) (K,τ) is a congruence pair for S.

(2) $\pi_K \cap (\mathcal{H}\tau\mathcal{H})^0$ has kernel K and trace τ.

(3) There exists a congruence ρ on S with kernel K and trace τ.

(4) There is a unique congruence ρ on S with kernel K and trace τ.

Whenever (1) - (4) hold, the unique congruence on S with kernel K and trace τ is

$$\rho_{(K,\tau)} = \pi_K \cap (\mathcal{H}\tau\mathcal{H})^0.$$

3. KERNEL AND TRACE RELATIONS

Throughout this section, let S denote a completely regular semigroup and $\mathcal{C}(S)$ its lattice of congruences. Let the kernel relation K and the trace relation T be defined on $\mathcal{C}(S)$ as follows.

$$\lambda \, K \, \rho \iff \ker \lambda = \ker \rho \qquad (\lambda,\rho \in \mathcal{C}(S))$$

$$\lambda \, T \, \rho \iff \operatorname{tr} \lambda = \operatorname{tr} \rho \qquad (\lambda,\rho \in \mathcal{C}(S))$$

Clearly K and T are both equivalence relations. As an immediate consequence of Corollary 2.2, we have

LEMMA 3.1. $K \cap T = \epsilon$, the identical relation.

We consider K first. As a related characterization of the kernel relation we have the following interesting obvservation.

LEMMA 3.2. (Pastijn and Petrich [14], Lemma 3.9) Let $\lambda,\rho \in \mathcal{C}(S)$. Then

$$\lambda \, K \, \rho \iff \lambda \cap \mathcal{H} = \rho \cap \mathcal{H}.$$

Proof. First suppose that $\ker\lambda = \ker\rho$. Then

$$\begin{aligned} a\ \lambda\cap\mathcal{H}\ b &\iff a\ \mathcal{H}\ b,\ a\ \lambda\ b \\ &\iff a\ \mathcal{H}\ b,\ ab^{-1}\ \lambda\ b^0 \qquad (\text{since } a^0 = b^0) \\ &\iff a\ \mathcal{H}\ b,\ ab^{-1} \in \ker\lambda = \ker\rho \\ &\iff \ldots\ldots \\ &\iff a\ \rho\cap\mathcal{H}\ b. \end{aligned}$$

Thus $\lambda\cap\mathcal{H} = \rho\cap\mathcal{H}$. Conversely, let $\lambda\cap\mathcal{H} = \rho\cap\mathcal{H}$. Then

$$\begin{aligned} a \in \ker\lambda &\implies a\ \lambda\cap\mathcal{H}\ a^0 \\ &\implies a\ \rho\cap\mathcal{H}\ a^0 \\ &\implies a \in \ker\rho \end{aligned}$$

so that $\ker\lambda \subseteq \ker\rho$ and, by symmetry, equality follows.

NOTATION Let $\mathcal{K}(S)$ denote the set of normal subsets of S ordered by set theoretic inclusion.

For any family $\{K_i : i \in I\}$ of normal subsets of S, it is clear that $\bigcap_{i\in I} K_i$ is again a normal subset of S. From this it follows that $\mathcal{K}(S)$ is a complete lattice with respect to the operations

$$K_1 \wedge K_2 = K_1 \cap K_2 \qquad \text{and} \qquad K_1 \vee K_2 = \cap\{K \in \mathcal{K}(S) : K_1 \cup K_2 \subseteq K\}.$$

THEOREM 3.3. (Pastijn and Petrich [14], Lemma 2.9 and Petrich and Reilly [24]) The mapping

$$\ker : \rho \longrightarrow \ker\rho \qquad (\rho \in \mathcal{C}(S))$$

is a complete $\cap$ - homomorphism of $\mathcal{C}(S)$ onto $\mathcal{K}(S)$ which induces the relation K on $\mathcal{C}(S)$. For all $\rho \in \mathcal{C}(S)$ the K - class of ρ is an interval $[\rho_K, \rho^K]$ where

$$\rho_K = (\rho\cap\mathcal{H})^* \qquad \text{and} \qquad \rho^K = \pi_{\ker\rho}.$$

Unfortunately, K is not always a congruence. Let G be any non-trivial group, $Y = \{0,1\}$ be the two element semilattice and $S = G \times Y$. Let ϵ denote the identical relation, ω the universal relation, σ the minimum group congruence and ρ the Rees congruence determined by the ideal $G \times \{0\}$. Then $\epsilon\ K\ \sigma$ but

$$\epsilon \vee \rho = \rho \qquad \text{and} \qquad \sigma \vee \rho = \omega$$

where ρ and ω do not have the same kernels.

However, there are circumstances under which K is a congruence. The method of proof used by Pastijn to establish the fact (Theorem 4.4 below) that K is a congruence on the lattice of fully invariant congruences on the free completely regular semigroup suggests the following discussion. We begin with completely simple semigroups. Let $(\mathscr{S},N,\mathscr{T})$, $(\mathscr{S}',N',\mathscr{T}')$ and $(\mathscr{P},M,Q)$ be admissible triples for $S = \mathcal{M}(I,G,\Lambda;P)$ and let

$$\rho = \rho_{(\mathscr{S},N,\mathscr{T})}, \quad \rho' = \rho_{(\mathscr{S}',N',\mathscr{T}')} \quad \text{and} \quad \sigma = \sigma_{(\mathscr{P},M,Q)}.$$

A straightforward calculation will show that

$$\ker \rho = \{(i,a,\lambda) : ap_{\lambda i} \in N\}$$

with similar expressions for $\ker \rho'$ and $\ker \sigma$. Consequently,

$$\ker \rho = \ker \rho' \iff N = N'.$$

Now it is also the case that

$$\rho \vee \sigma = \rho_{(\mathscr{S}\vee\mathscr{P},MN,\mathscr{T}\vee Q)}$$

so that

$$\ker \rho \vee \sigma = \{(i,a,\lambda) : ap_{\lambda i} \in MN\}$$

with a similar expression for $\ker \rho' \vee \sigma$. Therefore, it is clear that

$$\ker \rho = \ker \rho' \implies \ker \rho \vee \sigma = \ker \rho' \vee \sigma$$

whence K is a congruence on $\mathcal{C}(S)$ and the mapping ker is a homomorphism on $\mathcal{C}(S)$ for any completely simple semigroup S.

This observation has consequences for any completely regular semigroup. To see this, let $S = \bigcup_{\alpha\in Y} S_\alpha$ be a completely regular semigroup with completely simple components S_α and let $\rho, \rho', \sigma \in (\mathcal{D}]$, the sublattice of $\mathcal{C}(S)$ consisting of those congruences contained in $\mathcal{D}$, be such that $\ker \rho = \ker \rho'$. Let

$$\rho_\alpha = \rho_{|S_\alpha}, \quad \rho'_\alpha = \rho'_{|S_\alpha} \quad \text{and} \quad \sigma_\alpha = \sigma_{|S_\alpha} \qquad (\alpha \in Y).$$

Then $\ker \rho_\alpha = \ker \rho'_\alpha$. Also

$$\rho \vee \sigma = \cup\, \rho\circ\sigma\circ\rho\circ\ldots\circ\rho \qquad \text{where the union runs over compositions of arbitrary length}$$

$$= \cup \bigcup_{\alpha\in Y} \rho_\alpha\circ\sigma_\alpha\circ\ldots\circ\rho_\alpha \qquad \text{since } \rho, \sigma \in (\mathcal{D}]$$

$$= \bigcup_{\alpha\in Y} \cup\rho_\alpha\circ\sigma_\alpha\circ\ldots\circ\rho_\alpha$$

$$= \bigcup_{\alpha\in Y} \rho_\alpha\vee\sigma_\alpha.$$

Hence

$$(\rho \vee \sigma)_\alpha = \rho_\alpha \vee \sigma_\alpha$$

and

$$\ker \rho \vee \sigma = \cup \ker(\rho \vee \sigma)_\alpha$$

$$= \cup \ker (\rho_\alpha \vee \sigma_\alpha)$$

$$= \cup (\ker \rho_\alpha \vee \ker \sigma_\alpha) \qquad \text{since ker is a homomorphism when applied to the lattice of congruences on a completely simple semigroup}$$

$$= \cup (\ker \rho'_\alpha \vee \ker \sigma_\alpha)$$
$$= \ldots\ldots$$
$$= \ker \rho' \vee \sigma.$$

Thus we have established the following theorem:

THEOREM 3.4. For any completely regular semigroup, the mapping $\ker$ is a homomorphism on $(\mathcal{D}]$.

Parallelling Lemma 3.1, we have the following result characterizing the trace relation.

LEMMA 3.5. (Pastijn and Petrich [14], Lemma 6.5) Let $\lambda, \rho \in \mathcal{C}(S)$. Then

$$\lambda \; T \; \rho \quad \Longleftrightarrow \quad \lambda \vee \mathcal{H} = \rho \vee \mathcal{H}.$$

Combining Lemmas 3.1 and 3.5, we obtain a rather curious test for the equality of congruences.

LEMMA 3.6. Let $\lambda, \rho \in \mathcal{C}(S)$. Then

$$\lambda = \rho \quad \Longleftrightarrow \quad \lambda \cap \mathcal{H} = \rho \cap \mathcal{H} \quad \text{and} \quad \lambda \vee \mathcal{H} = \rho \vee \mathcal{H}.$$

In dealing with expressions of the form $\rho \vee \mathcal{H}$, it is sometimes useful to know the following simpler descriptions.

LEMMA 3.7. For any $\rho \in \mathcal{C}(S)$,

$$\rho \vee \mathcal{H} = \rho\mathcal{H}\rho = \mathcal{H}\rho\mathcal{H}.$$

NOTATION Let $\mathcal{T}(S)$ denote the set of all normal equivalence relations on $E(S)$.

Clearly the intersection of any family of normal equivalences is again a normal equivalence. From this it follows that the set $\mathcal{T}(S)$ is a complete lattice with respect to the operations

$$\sigma \wedge \tau = \sigma \cap \tau \quad \text{and} \quad \sigma \vee \tau = \cap \{\rho \in \mathcal{T}(S) : \sigma \cup \tau \subsetneq \rho\}.$$

THEOREM 3.8. (Pastijn and Petrich [14], Theorem 4.20) The mapping

$$\mathrm{tr} : \rho \longrightarrow \mathrm{tr}\, \rho \qquad (\rho \in \mathcal{C}(S))$$

is a complete homomorphism of $\mathcal{C}(S)$ onto $\mathcal{T}(S)$ inducing the relation T on $\mathcal{C}(S)$. Moreover, for each $\rho \in \mathcal{C}(S)$, the T-class of ρ is an interval $[\rho_T, \rho^T]$ where

$$\rho_T = (\text{tr}\ \rho)^* \qquad \text{and} \qquad \rho^T = (\rho \vee \mathcal{H})^0.$$

In contrast to the fact that K need not always be a congruence on $\mathcal{C}(S)$, we have the following immediate consequence of Theorem 3.8.

<u>COROLLARY</u> 3.9. T is a complete congruence on $\mathcal{C}(S)$.

From Theorems 3.3 and 3.8, we see that the equivalence relations K and T are such that every class is an interval in the lattice $\mathcal{C}(S)$. These facts, together with Lemma 3.1 enable us to give a purely lattice theoretic proof of the next observation.

<u>PROPOSITION</u> 3.10. (Pastijn and Petrich [14], Theorem 3.5) Let $\rho \in \mathcal{C}(S)$. Then

$$\rho_K \vee \rho_T = \rho = \rho^K \wedge \rho^T.$$

<u>Proof</u>. We have

$$\rho_K \leq \rho_K \vee \rho_T \leq \rho$$

and, by the convexity of the class ρK, it follows that $\rho_K \vee \rho_T \ K \ \rho$. Similarly, $\rho_K \vee \rho_T \ T \ \rho$ which, by Lemma 3.1, implies that $\rho_K \vee \rho_T = \rho$. The second equality in the statement of the proposition follows by duality.

There are two additional relations on $\mathcal{C}(S)$ that are closely related to T. In order to recognize that these relations are natural relatives of K and T, it is helpful to consider slightly different characterizations of K and T.

Let $\rho \in \mathcal{C}(S)$. Then

ρ is <u>idempotent pure</u> if $\ker \rho = E(S)$,

ρ is <u>idempotent separating</u> if $\text{tr}\ \rho = \epsilon$ or, equivalently, $\rho \subseteq \mathcal{H}$.

Clearly,

$$\begin{aligned} \lambda \ K \ \rho &\iff \ker \lambda = \ker \rho = \ker \lambda\cap\rho \\ &\iff \ker \lambda/(\lambda\cap\rho) = E(S/(\lambda\cap\rho)) = \ker \rho/(\lambda\cap\rho) \\ &\iff \lambda/(\lambda\cap\rho) \text{ and } \rho/(\lambda\cap\rho) \text{ are both idempotent pure.} \end{aligned}$$

Similarly,

$$\begin{aligned} \lambda \ T \ \rho &\iff \text{tr}\ \lambda = \text{tr}\ \rho = \text{tr}\ \lambda\cap\rho \\ &\iff \text{tr}\ \lambda/(\lambda\cap\rho) = \epsilon = \text{tr}\ \rho/(\lambda\cap\rho) \end{aligned}$$

$\Longleftrightarrow \quad \lambda/(\lambda\cap\rho)$ and $\rho/(\lambda\cap\rho)$ are both idempotent separating

$\Longleftrightarrow \quad \lambda/(\lambda\cap\rho),\ \rho/(\lambda\cap\rho) \subsetneq \mathcal{H}$.

It is this very last characterization of T that leads to two additional relations on $\mathcal{C}(S)$: for $\lambda,\rho \in \mathcal{C}(S)$,

$$\lambda\ T_\ell\ \rho \quad\Longleftrightarrow\quad \lambda/(\lambda\cap\rho),\ \rho/(\lambda\cap\rho) \subsetneq \mathcal{L}$$

$$\lambda\ T_r\ \rho \quad\Longleftrightarrow\quad \lambda/(\lambda\cap\rho),\ \rho/(\lambda\cap\rho) \subsetneq \mathcal{R}.$$

We refer to T_ℓ as the <u>left trace relation</u> and to T_r as the <u>right trace relation</u> on $\mathcal{C}(S)$.

For any congruence $\rho \in \mathcal{C}(S)$, the <u>left trace</u> and <u>right trace</u> of ρ are defined to be

$$\operatorname{ltr}\rho = (\rho \vee \mathcal{L})^0 \qquad\text{and}\qquad \operatorname{rtr}\rho = (\rho \vee \mathcal{R})^0.$$

Then an equivalent characterization of the relations T_ℓ and T_r is given by the following: for $\lambda,\rho \in \mathcal{C}(S)$,

$$\lambda\ T_\ell\ \rho \iff \operatorname{ltr}\lambda = \operatorname{ltr}\rho \qquad\text{and}\qquad \lambda\ T_r\ \rho \iff \operatorname{rtr}\lambda = \operatorname{rtr}\rho.$$

The parallelism between the relations T, T_ℓ and T_r is brought out strongly in the next result.

<u>THEOREM</u> 3.11. (Pastijn and Petrich [14], Lemma 6.5) The mappings

$$\rho \longrightarrow \rho \vee \mathcal{H}, \qquad \rho \longrightarrow \rho \vee \mathcal{L}, \qquad \rho \longrightarrow \rho \vee \mathcal{R}$$

are complete homomorphisms of the lattice $\mathcal{C}(S)$ into the lattice $\mathcal{E}(S)$ of equivalence relations on S inducing the relations T, T_ℓ and T_r, respectively. Consequently, the relations T, T_ℓ and T_r are complete congruences on $\mathcal{C}(S)$.

As an immediate consequence, to match Lemma 3.5, we have

<u>COROLLARY</u> 3.12. (i) $\lambda\ T_\ell\ \rho \iff \lambda \vee \mathcal{L} = \rho \vee \mathcal{L}$.

(ii) $\lambda\ T_r\ \rho \iff \lambda \vee \mathcal{R} = \rho \vee \mathcal{R}$.

Since T_ℓ and T_r are complete congruences, it follows that all the T_ℓ-classes and T_r-classes are intervals. For any $\rho \in \mathcal{C}(S)$, we define ρ_{T_ℓ}, ρ_{T_r}, ρ^{T_ℓ} and ρ^{T_r} by setting

$$\rho T_\ell = [\rho_{T_\ell}, \rho^{T_\ell}] \qquad\text{and}\qquad \rho T_r = [\rho_{T_r}, \rho^{T_r}].$$

The next result sets out some important basic connections between the relations T, T_ℓ and T_r.

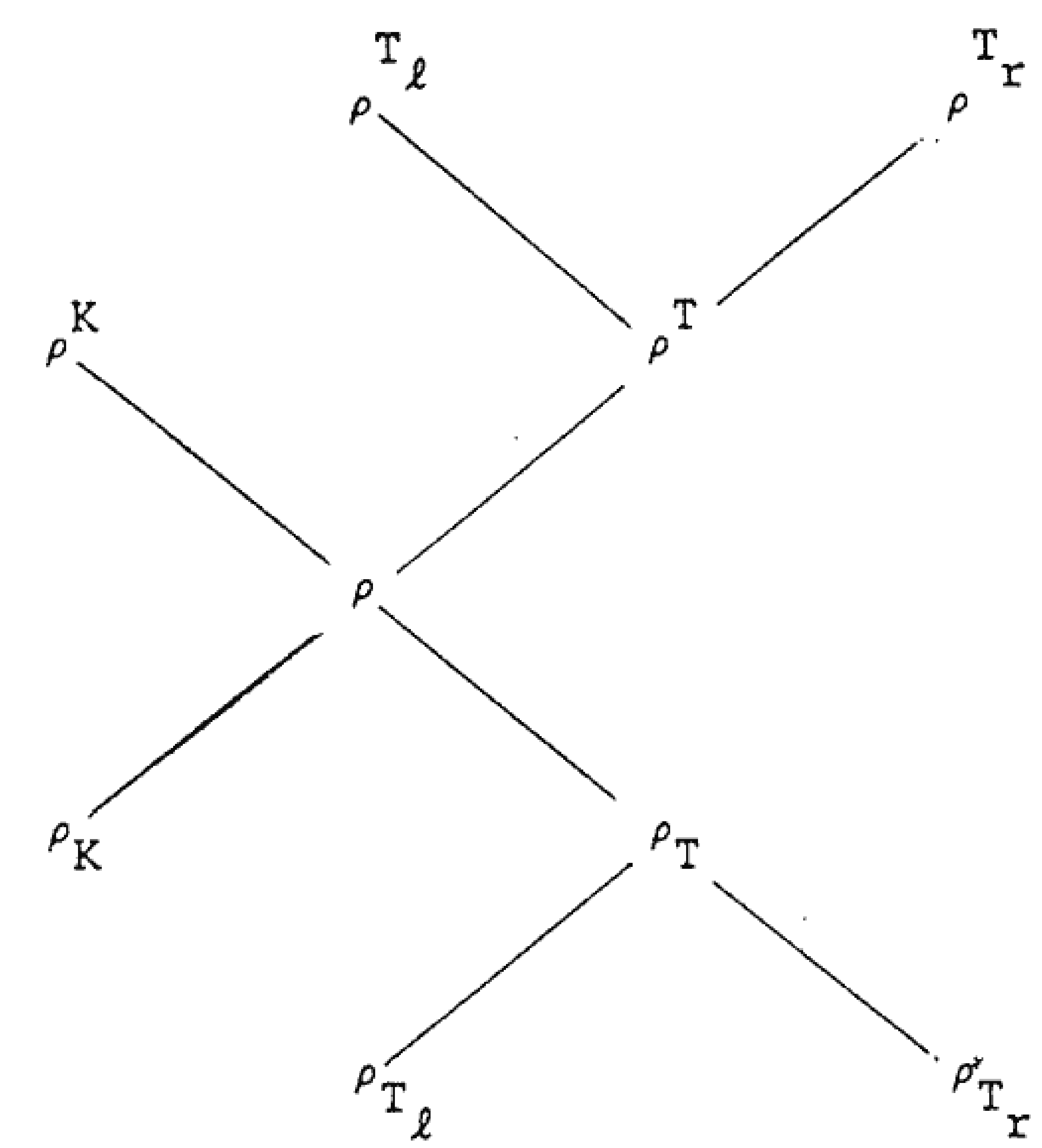
THEOREM 3.13. (Pastijn and Petrich [14], Corollary 4.8 and Theorem 4.14)

(i) $T_\ell \cap T_r = T$.

(ii) For any $\rho \in \mathcal{C}(S)$,

$$\rho_{T_\ell} \vee \rho_{T_r} = \rho_T \qquad \text{and} \qquad \rho^{T_\ell} \wedge \rho^{T_r} = \rho^T.$$

This leads to the following diagram from [14].



In order to give more explicit descriptions of the endpoints of T_ℓ- and T_r-classes, it is convenient to introduce the following relations. Define

$$e \leq_\ell f \quad \Longleftrightarrow \quad e = ef \qquad (e, f \in E(S))$$

and define the relation $\leq_r$ dually.

PROPOSITION 3.14. (Pastijn and Petrich [14], Theorem 4.12)

Let $\rho \in \mathcal{C}(S)$.

(i) $\quad \rho_{T_r} = (\rho \cap \leq_\ell)^* \quad$ and $\quad \rho^{T_r} = (\rho \vee \mathcal{R})^0$.

(ii) $\qquad \rho_{T_\ell} = (\rho \cap \leq_r)^*$ and $\rho^{T_\ell} = (\rho \vee \mathcal{L})^0$.

4. THE LATTICE OF VARIETIES

We shall require some notation. For any subvariety $\mathcal{V}$ of $\mathcal{CR}$, we shall write

$\mathcal{L}(\mathcal{V})$ = the lattice of subvarieties of $\mathcal{V}$

$F\mathcal{V}$ = the relatively free completely regular semigroup in $\mathcal{V}$ on a countably infinite set X

Γ = the lattice of fully invariant congruences on $F\mathcal{CR}$.

Fundamental to the discussion of varieties is the standard correspondence between varieties and fully invariant congruences.

For $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$, let $\rho_{\mathcal{V}}$ be defined on $F\mathcal{CR}$ by

$$\rho_{\mathcal{V}} = \{(u,v) \in F\mathcal{CR}\times F\mathcal{CR}: u\theta = v\theta, \text{ for all homomorphisms } \theta: F\mathcal{CR} \longrightarrow S \in \mathcal{V}\}.$$

Then the mapping

$$\pi: \mathcal{V} \longrightarrow \rho_{\mathcal{V}} \qquad (\mathcal{V} \in \mathcal{L}(\mathcal{CR}))$$

is an antiisomorphism of $\mathcal{L}(\mathcal{CR})$ onto Γ.

The study of $\mathcal{L}(\mathcal{CR})$ involves many special varieties as reference points.

$\mathcal{T}$	- trivial semigroups	$[x = y]$
$\mathcal{LZ}$	- left zero semigroups	$[xy = x]$
$\mathcal{RZ}$	- right zero semigroups	$[xy = y]$
$\mathcal{RB}$	- rectangular bands	$[xyz = xz]$
$\mathcal{ReG}$	- rectangular groups	$[x^0y^0z^0 = (xz)^0]$
$\mathcal{S}$	- semilattices	$[x^2 = x,\ xy = yx]$
$\mathcal{B}$	- bands	$[x^2 = x]$
$\mathcal{NB}$	- normal bands	$[x^2 = x,\ axya = ayxa]$
$\mathcal{G}$	- groups	$[x^0 = y^0]$
$\mathcal{AG}$	- abelian groups	$[x^0 = y^0,\ xy = yx]$
$\mathcal{A}_n$	- abelian groups of exponent n	$[x^0 = y^0,\ xy = yx,\ x^n = x^0]$
$\mathcal{LG}$	- left groups	$[x^0y^0 = x^0]$
$\mathcal{RG}$	- right groups	$[x^0y^0 = y^0]$
$\mathcal{SG}$	- semilattices of groups	$[x^0y^0 = y^0x^0]$
$\mathcal{CS}$	- completely simple semigroups	$[(xyz)^0 = (xz)^0]$
$\mathcal{OG}$	- orthogroups	$[x^0y^0 = (x^0y^0)^0]$
$\mathcal{CG}$	- cryptogroups	$[(x^0y^0)^0 = (xy)^0]$

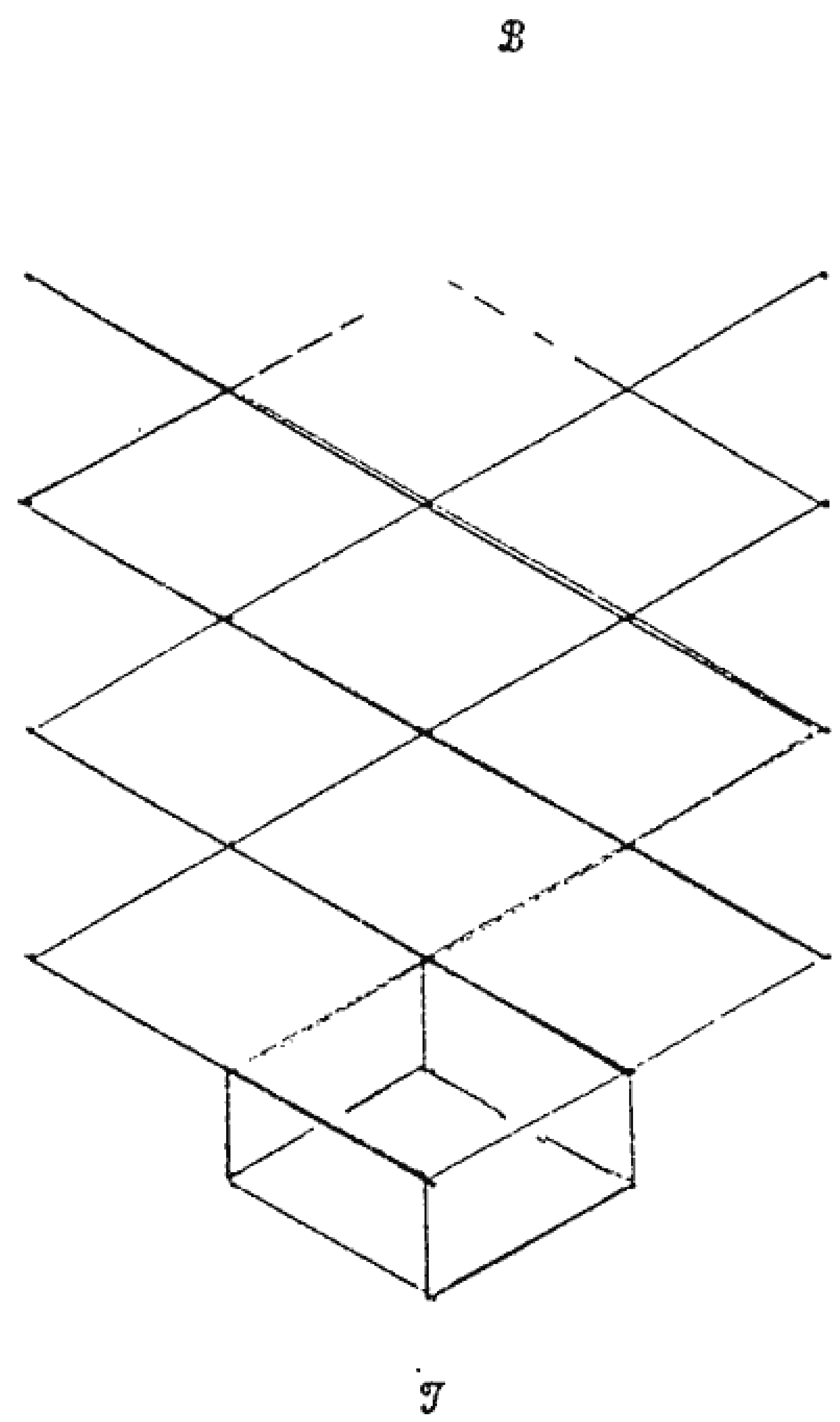
$\mathcal{OCG}$ - orthocryptogroups $[x^0y^0 = (x^0y^0)^0,\ (x^0y^0)^0 = (xy)^0]$

$\mathcal{NCG}$ - normal cryptogroups (completely regular semigroups for which $\mathcal{H}$ is a congruence and $S/\mathcal{H}$ is a normal band).

$\mathcal{LOCG}$ - locally orthodox cryptogroups (that is, all $S \in \mathcal{CR}$ such that $eSe \in \mathcal{OCG}$ for all $e \in E(S)$).

$\mathcal{CLOCG}$ - completely regular semigroups for which the core (that is, the subsemigroup generated by the idempotents) lies in $\mathcal{LOCG}$

The first part of $\mathcal{L}(\mathcal{CR})$ to be studied in any depth was the lattice $\mathcal{L}(\mathcal{B})$ of subvarieties of the variety $\mathcal{B}$ of bands. Here is the familiar diagram for $[\mathcal{S},\mathcal{B}]$ due to Birjukov [1], Fennemore [6] and Gerhard [8]:



The next part of the lattice $\mathcal{L}(\mathcal{CR})$ to be studied in depth (excluding the lattice of varieties of groups, which has been studied for many years, of course) was the lattice $\mathcal{L}(\mathcal{CS})$ of subvarieties of the variety of completely simple semigroups. Most of the work on $\mathcal{L}(\mathcal{CS})$ to date has taken advantage of the description of the free completely simple semigroup described by Clifford and Rasin (independently), in 1979.

THEOREM 4.1. (Clifford [3] Theorem 7.4, Rasin [28] Theorem 1) Let X be a non-empty set and fix $z \in X$. Let $Y = \{p_{x,y} : x,y \in X\setminus\{z\}\}$ be a set indexed by pairs of elements from X different from z and let G be the free group on $Z = X \cup Y$. Let $p_{z,x} = p_{x,z} = 1$, the identity of G, for all $x \in X$, and let $P = (p_{x,y})$ be the $X \times X$ matrix with $(x,y)^{th}$ entry equal to $p_{x,y}$. Then $F\mathcal{CS}(X) = (\mathcal{M}(X,G,X;P),\theta)$ where $x\theta = (x,x,x)$, for all $x \in X$.

NOTATION Let $\mathcal{E}$ denote the set of endomorphisms ω of G for which there exist mappings φ and ψ of X into itself such that, for all $x,y \in X$,

$$p_{x,y}\omega = p_{z\psi,z\varphi}(p_{x\psi,z\varphi})^{-1}p_{x\psi,y\varphi}(p_{z\psi,y\varphi})^{-1}.$$

Let $\mathcal{N}$ denote the set of normal subgroups of G which are invariant under all elements of $\mathcal{E}$. It is easily verified that $\mathcal{N}$ is a sublattice of the lattice of normal subgroups of G.

THEOREM 4.2. (Rasin [28], Theorem 3) The interval $[\mathcal{RB},\mathcal{CS}]$ is anti-isomorphic to the lattice $\mathcal{N}$.

Because of this result, most of the advances to date in the study of $\mathcal{L}(\mathcal{CS})$ have involved the study of the structure of G and $\mathcal{N}$.

NOTATION Let $\mathcal{C}$ denote the variety of all completely simple semigroups with the property that the product of any two idempotents lies in the centre of the $\mathcal{H}$-class containing it. This variety is defined by the identity

$$ax^0a^0ya = aya^0x^0a.$$

For any $\mathcal{V} \in \mathcal{L}(\mathcal{CS})$, let $\mathcal{I}(\mathcal{V})$ denote the class of all idempotent generated members of $\mathcal{V}$ and let $\langle\mathcal{I}(\mathcal{V})\rangle$ denote the variety of completely simple semigroups generated by $\mathcal{I}(\mathcal{V})$.

The largest ideal of $\mathcal{L}(\mathcal{CS})$ to have been given a fairly precise characterization is $\mathcal{L}(\mathcal{C})$.

THEOREM 4.3. (Petrich and Reilly [20], Theorem 3.11) The mapping

$$\zeta : \mathcal{A} \longrightarrow (\mathcal{A} \cap \mathcal{RB},\ \langle\mathcal{IA}\rangle \cap \mathcal{AG},\ \mathcal{A} \cap \mathcal{G}) \qquad (\mathcal{A} \in \mathcal{L}(\mathcal{C}))$$

is an isomorphism of $\mathcal{L}(\mathcal{C})$ onto the subdirect product

$\{(\mathcal{U},\mathcal{V},\mathcal{W}) \in \mathcal{L}(\mathcal{RB}) \times \mathcal{L}(\mathcal{AG}) \times \mathcal{L}(\mathcal{G}): \mathcal{V} \subseteq \mathcal{W},\ \mathcal{U} \neq \mathcal{RB} \implies \mathcal{V} = \mathcal{T}\}$.

Despite the "simple" characterízation of completely simple semigroups provided by the Rees Theorem, the structure of $\mathcal{L}(\mathcal{CS})$ outside of the ideal $\mathcal{L}(\mathcal{C})$, remains a mystery.

In order to probe deeper into the structure of $\mathcal{L}(\mathcal{CR})$, we take advantage of the recent techniques for investigating congruences that were described in earlier sections.

In Theorem 3.10, we saw that the relations T, T_ℓ and T_r are complete congruences on $\mathcal{C}(S)$, for any completely regular semigroup S, but that K need not be. We now have:

THEOREM 4.4.(Polák [25] Theorem 1, Pastijn [12] Theorem 11) K is a complete congruence on Γ.

Thus K, T, T_ℓ and T_r are all complete congruences on Γ. Under the antiisomorphism π^{-1}, these carry over to complete congruences on $\mathcal{L}(\mathcal{CR})$:

$$\mathcal{U}\ K\ \mathcal{V} \iff \rho_{\mathcal{U}}\ K\ \rho_{\mathcal{V}}, \quad \mathcal{U}\ T\ \mathcal{V} \iff \rho_{\mathcal{U}}\ T\ \rho_{\mathcal{V}}$$
$$\mathcal{U}\ T_\ell\ \mathcal{V} \iff \rho_{\mathcal{U}}\ T_\ell\ \rho_{\mathcal{V}}, \quad \mathcal{U}\ T_r\ \mathcal{V} \iff \rho_{\mathcal{U}}\ T_r\ \rho_{\mathcal{V}}$$

The classes of any complete congruence are intervals and so it is convenient to denote the intervals for these four congruences as follows:

$$\mathcal{V}K = [\mathcal{V}_K, \mathcal{V}^K], \qquad \mathcal{V}T = [\mathcal{V}_T, \mathcal{V}^T]$$
$$\mathcal{V}T_\ell = [\mathcal{V}_{T_\ell}, \mathcal{V}^{T_\ell}], \quad \mathcal{V}T_r = [\mathcal{V}_{T_r}, \mathcal{V}^{T_r}]$$

THEOREM 4.5.(Polák [25] Theorem 1 and [26] Theorem 1.6, Pastijn [12] Theorem 8) The mappings

$$\mathcal{V} \longrightarrow \mathcal{V}^K, \quad \mathcal{V} \longrightarrow \mathcal{V}_{T_\ell}, \quad \mathcal{V} \longrightarrow \mathcal{V}_{T_r} \qquad (\mathcal{V} \in \mathcal{L}(\mathcal{CR}))$$

are complete endomorphisms of $\mathcal{L}(\mathcal{CR})$ inducing the congruences K, T_ℓ and T_r.

Somewhat surprisingly, the mapping

$$\mathcal{V} \longrightarrow \mathcal{V}_T \qquad (\mathcal{V} \in \mathcal{L}(\mathcal{CR}))$$

is not an endomorphism of $\mathcal{L}(\mathcal{CR})$ (see Petrich and Reilly [22], Proposition 7.6). In addition, the mappings associated with the other ends of the intervals of K, T, T_ℓ and T_r are not endomorphisms. An interesting and useful fact is that the upper ends of the intervals of K, T_ℓ and T_r can be described in terms of Mal'cev products (Pastijn [12] Lemma 3, Theorem 13):

$$\mathcal{V}^K = \mathcal{R}\mathcal{B}\circ(\mathcal{V}\vee\mathcal{S}), \quad \mathcal{V}^{T_\ell} = \mathcal{L}\mathcal{G}\circ\mathcal{V}, \quad \mathcal{V}^{T_r} = \mathcal{R}\mathcal{G}\circ\mathcal{V}.$$

An alternative expression for $\mathcal{V}^K$ is $\mathcal{V}^K = \mathcal{B}\circ\mathcal{V}$.

One approach used in the study of $\mathcal{L}(\mathcal{CR})$ has been to describe certain intervals of the form $[\mathcal{U} \wedge \mathcal{V}, \mathcal{U} \vee \mathcal{V}]$, for suitable $\mathcal{U},\mathcal{V} \in \mathcal{L}(\mathcal{CR})$, as particular subdirect products of the intervals $[\mathcal{U} \wedge \mathcal{V}, \mathcal{U}]$ and $[\mathcal{U} \wedge \mathcal{V}, \mathcal{V}]$.

We begin by studying the circumstances under which an interval of the form $[a \wedge b, a \vee b]$ in a lattice may be isomorphic to the product $[a \wedge b, a] \times [a \wedge b, b]$ with a view to applying this to the lattice $\mathcal{L}(\mathcal{CR})$.

For any complete congruence λ on a complete lattice L and any $a \in L$, the class $a\lambda$ is an interval. We define a_λ and a^λ by $a\lambda = [a_\lambda, a^\lambda]$. The following discussion is taken from (Petrich and Reilly [23]).

<u>LEMMA</u> 4.6. Let κ and τ be congruences on a lattice L and $a,b \in L$. The following statements are equivalent.

(i) $a \;\kappa\; a \wedge b \;\tau\; b$. (ii) $a \;\tau\; a \vee b \;\kappa\; b$.

<u>Proof</u>. If (i) holds then

$a = a \vee (a \wedge b) \;\;\tau\;\; a \vee b$ and $b = (a \wedge b) \vee b \;\;\kappa\;\; a \vee b$

which gives (i). The proof that (ii) implies (i) is similar.

<u>DEFINITION</u> If L,a,b,κ and τ satisfy (i) and (ii) in Lemma 6.1, then we will say that a and b are $\kappa\tau$-<u>neighbours</u>. Congruences κ and τ on a lattice L are said to be <u>disjoint</u> if $\kappa \cap \tau = \epsilon$.

<u>LEMMA</u> 4.7. Let κ and τ be disjoint complete congruences on a complete lattice L and let $a \in L$. Then

$$a = a_\kappa \vee a_\tau = a^\kappa \wedge a^\tau .$$

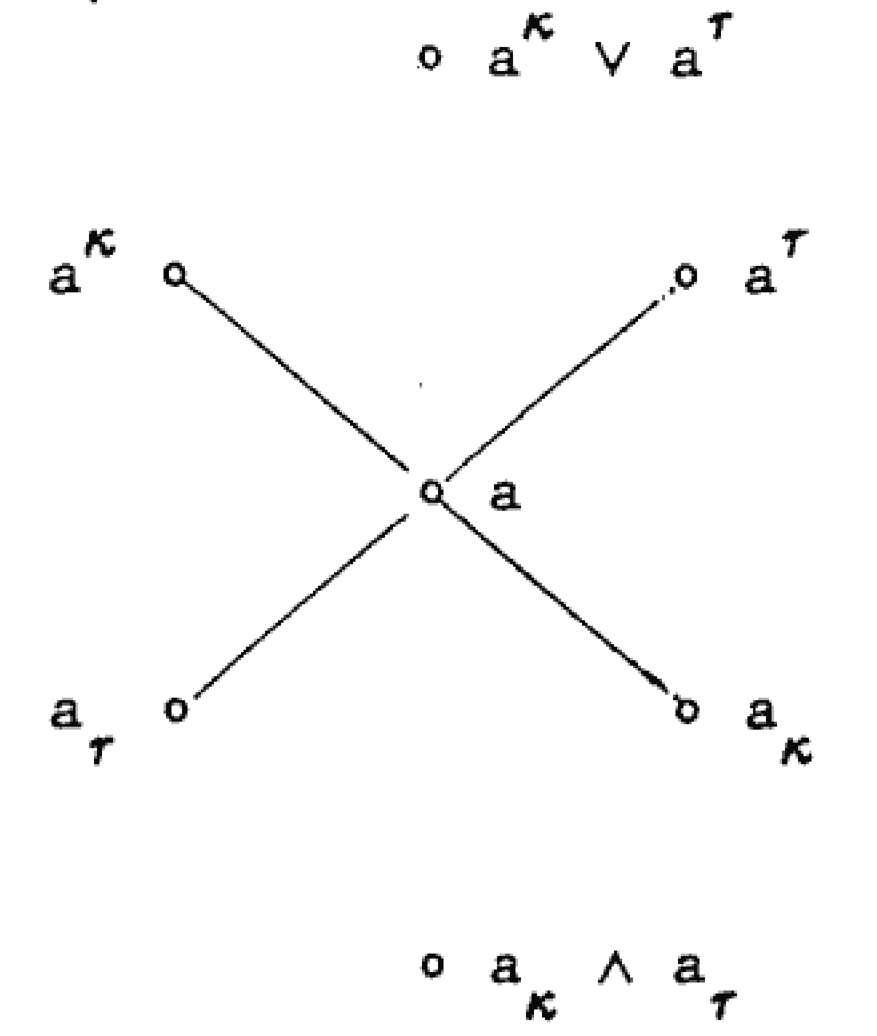
<u>Proof.</u> Since κ and τ are congruences, we have

$$a_\kappa \vee a_\tau \;\;\kappa\;\; a \vee a_\tau = a \quad \text{and} \quad a_\kappa \vee a_\tau \;\;\tau\;\; a_\kappa \vee a = a$$

so that $a_\kappa \vee a_\tau \;(\kappa \cap \tau)\; a$. But κ and τ are disjoint. Therefore $a = a_\kappa \vee a_\tau$. The second equality follows by duality.

<u>COROLLARY</u> 4.8. Let κ and τ be disjoint complete congruences on a complete lattice L and let $a \in L$. Then a^κ, a^τ are

$\kappa\tau$-neighbours and a_κ, a_τ are $\tau\kappa$-neighbours.



Proof. By Lemma 4.7, we have

$$a^\kappa \wedge a^\tau = a \;\; \kappa \;\; a^\kappa \qquad \text{and} \qquad a^\kappa \wedge a^\tau = a \;\; \tau \;\; a^\tau$$

from which we deduce the first claim. The second claim follows similarly using Lemmas 4.6 and 4.7.

We are now ready for the main lattice theoretic observation. One of the striking features of this result is the fact that neither modularity nor neutrality appear in the hypotheses.

THEOREM 4.9. Let κ, τ be disjoint congruences on a lattice L and $a, b \in L$ be $\kappa\tau$-neighbours. Then the mappings

$$\varphi\colon z \longrightarrow (z \wedge a, z \wedge b), \qquad \psi\colon (x,y) \longrightarrow x \vee y$$

are mutually inverse isomorphisms between $[a \wedge b, a \vee b]$ and $[a \wedge b, a] \times [a \wedge b, b]$.

Applying these lattice theoretic considerations to congruences, we obtain:

THEOREM 4.10. (Pastijn and Trotter [15], Theorems 5.1 and 5.2) Let $\rho \in \Gamma$.

(i) The mappings

$$\theta \longrightarrow (\theta \cap \rho^K, \; \theta \cap \rho^T), \qquad (\xi, \eta) \longrightarrow \xi \vee \eta$$

are mutually inverse isomorphisms between $[\rho, \rho^K \vee \rho^T]$ and $[\rho, \rho^K] \times [\rho, \rho^T]$.

(ii) The mappings

$$\theta \longrightarrow (\theta \vee \rho_K, \; \theta \vee \rho_T), \qquad (\xi, \eta) \longrightarrow \xi \cap \eta$$

are mutually inverse isomorphisms between $[\rho_K \cap \rho_T, \rho]$ and $[\rho_K, \rho] \times [\rho_T, \rho]$.

Proof. (i) From Lemma 3.1, we know that K and T are disjoint complete congruences on Γ. It follows from Corollary 4.8 that ρ^K and ρ^T are KT-neighbours and the claim follows by Theorem 4.9.

(ii) This follows from (i) by duality.

In order to provide some specific illustrations of the preceding discussions in terms of varieties rather than fully invariant congruences, we need to know some specific values for the upper end points of some of the K- and T-classes.

LEMMA 4.11. (i) $\mathcal{T}^K = \mathcal{B}$, $\mathcal{G}^K = \mathcal{R}e\mathcal{G}^K = \mathcal{O}\mathcal{G}$.
(ii) $\mathcal{T}^T = \mathcal{G}$, $\mathcal{R}\mathcal{B}^T = \mathcal{R}e\mathcal{G}^T = \mathcal{C}\mathcal{S}$.

Proof. (i) See (Polák [25], Theorem 2).
(ii) See (Petrich and Reilly [21], Section 9).

We can now give some examples of applications in $\mathcal{L}(\mathcal{C}\mathcal{R})$.

LEMMA 4.12. (i) (Petrich [16], Theorem) The mappings

$$\mathcal{V} \longrightarrow (\mathcal{V} \cap \mathcal{B}, \mathcal{V} \cap \mathcal{G}), \qquad (\mathcal{U}, \mathcal{W}) \longrightarrow \mathcal{U} \vee \mathcal{W}$$

are mutually inverse isomorphisms between $\mathcal{L}(\mathcal{O}\mathcal{C}\mathcal{G})$ and $\mathcal{L}(\mathcal{B}) \times \mathcal{L}(\mathcal{G})$.

(ii) (Hall and Jones [9], Corollary 5.5 and Rasin [30], Proposition 1) The mappings

$$\mathcal{V} \longrightarrow (\mathcal{V} \cap \mathcal{B}, \mathcal{V} \cap \mathcal{C}\mathcal{S}), \qquad (\mathcal{U}, \mathcal{W}) \longrightarrow \mathcal{U} \vee \mathcal{W}$$

are mutually inverse isomorphisms between $[\mathcal{R}\mathcal{B}, L\mathcal{O}\mathcal{C}\mathcal{G}]$ and $[\mathcal{R}\mathcal{B}, \mathcal{B}] \times [\mathcal{R}\mathcal{B}, \mathcal{C}\mathcal{S}]$.

(iii) (Reilly [32], Theorem 4.9) The mappings

$$\mathcal{V} \longrightarrow (\mathcal{V} \cap \mathcal{O}\mathcal{G}, \mathcal{V} \cap \mathcal{C}\mathcal{S}) \qquad (\mathcal{U}, \mathcal{W}) \longrightarrow \mathcal{U} \vee \mathcal{W}$$

are mutually inverse isomorphisms between $[\mathcal{R}e\mathcal{G}, CL\mathcal{O}\mathcal{C}\mathcal{G}]$ and $[\mathcal{R}e\mathcal{G}, \mathcal{O}\mathcal{G}] \times [\mathcal{R}e\mathcal{G}, \mathcal{C}\mathcal{S}]$.

5. POLAK'S THEOREM

The following subset has a special role to play in the study of $\mathcal{L}(\mathcal{CR})$:

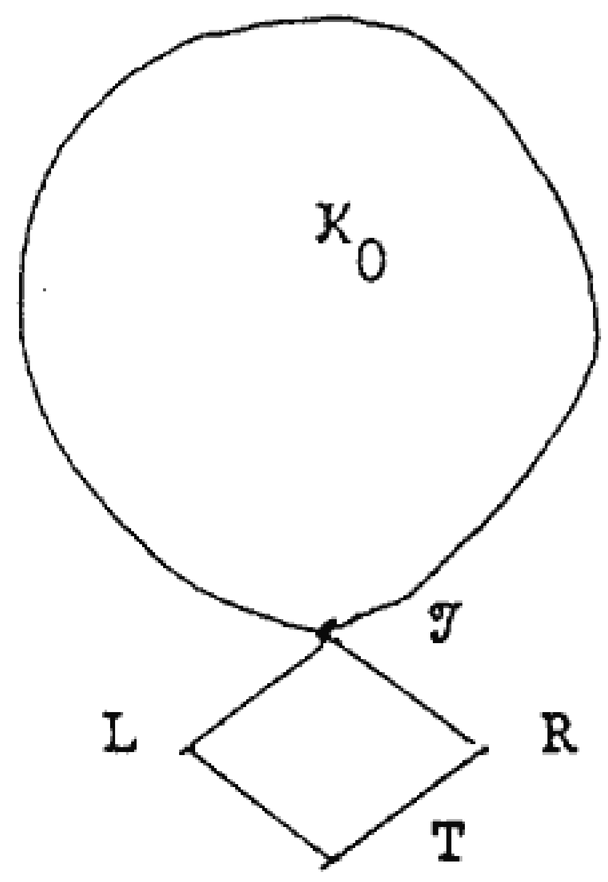
$$\mathcal{K}_0 = \{\mathcal{V}_K : \mathcal{V} \in \mathcal{L}(\mathcal{CR})\}$$

Examples of members of $\mathcal{K}_0$ are plentiful and include all group varieties and all non-orthodox varieties of completely simple semigroups.

Since K is a complete congruence on $\mathcal{L}(\mathcal{CR})$ and $\mathcal{K}_0$ contains exactly one representative from each K-class, we may consider $\mathcal{K}_0$ as being a lattice with the lattice structure inherited from $\mathcal{L}(\mathcal{CR})/K$. Thus, for $\mathcal{U}$, $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$, $\mathcal{U} \le \mathcal{V}$ if and only if $\mathcal{U}K \le \mathcal{V}K$.

We now adjoin three elements to the bottom of $\mathcal{K}_0$ (below the trivial variety $\mathcal{T}$) and extend the order on $\mathcal{K}_0$ to $\mathcal{K} = \mathcal{K}_0 \cup \{L,T,R\}$ as indicated in the diagram below so that $\mathcal{K}$ becomes a lattice with $\mathcal{K}_0$ as a sublattice.

$\mathcal{K}$ =



Before proceeding, we require some additional notation:

$\mathcal{LNB}$ = the variety of left normal bands = $[x^2 = x,\ xyz = xzy]$

$\mathcal{RNB}$ = the variety of right normal bands = $[x^2 = x,\ xyz = yxz]$.

For $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$, let the mapping

$$\mathcal{V} \longrightarrow \mathcal{V}_K^* \qquad (\mathcal{V} \in [\mathcal{S},\mathcal{CR}])$$

be defined by the following:

$$\mathcal{V}_K^* = \begin{cases} \mathcal{V}_K & \text{if } \mathcal{V} \ne \mathcal{S},\mathcal{LNB},\mathcal{RNB} \\ L & \text{if } \mathcal{V} = \mathcal{LNB} \\ T & \text{if } \mathcal{V} = \mathcal{S} \\ R & \text{if } \mathcal{V} = \mathcal{RNB}. \end{cases}$$

We wish to combine the above mapping with mappings associated with T_ℓ and T_r. Towards this end we introduce "products" of T_ℓ and T_r. Let

$$\Theta = \langle T_\ell, T_r \mid T_\ell^2 = T_\ell,\ T_r^2 = T_r \rangle \quad \text{and} \quad \Theta^1 = \Theta \cup \{1\}$$

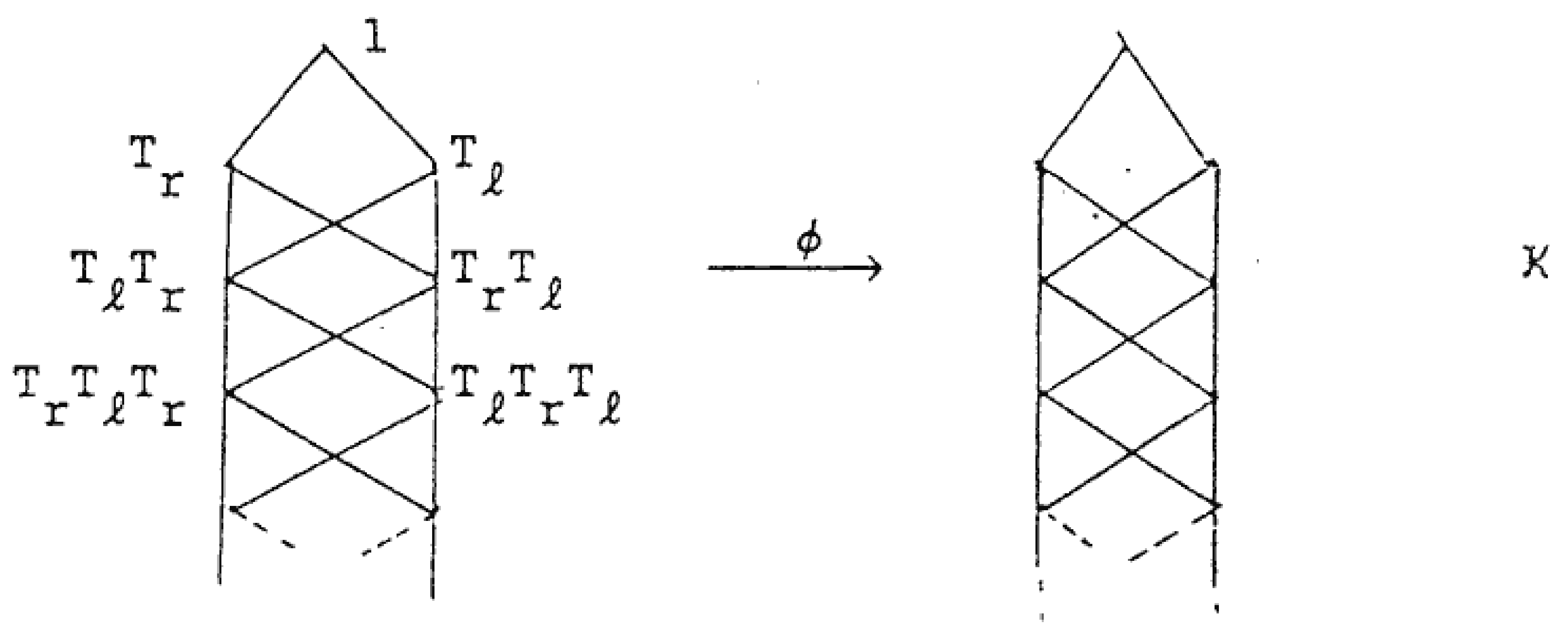
be the monoid with generators T_ℓ and T_r subject to the relations $T_\ell^2 = T_\ell$, $T_r^2 = T_r$. It is easy to see that every element of Θ can be written uniquely in canonical form as

$$\tau = T_1 T_2 \ldots T_n \quad \text{where } T_i \in \{T_\ell, T_r\},\ T_i \neq T_{i+1}$$

For such an element τ, let $|\tau| = n$, $h(\tau) = T_1$ and $t(\tau) = T_n$. Define a relation $\leq$ on Θ^1 by

$$\sigma \leq \tau \iff |\sigma| > |\tau| \text{ or } \sigma = \tau \text{ or } \tau = 1.$$

Then $(\Theta^1, \leq)$ is the partially ordered set depicted on the left of the diagram:



We also extend the definitions of $\mathcal{V}_{T_\ell}$ and $\mathcal{V}_{T_r}$ to cover $\mathcal{V}_\tau$ for any $\tau \in \Theta^1$ by defining $\mathcal{V}_1 = \mathcal{V}$ and otherwise defining $\mathcal{V}_\tau$ inductively as follows: for $\tau = T_1 T_2 \ldots T_n \in \Theta$ and $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$ let

$$\mathcal{V}_\tau = (\mathcal{V}_{T_1 \ldots T_{n-1}})_{T_n}.$$

Our main interest is in certain mappings of Θ^1 into $\mathcal{K}$.

Let Φ denote the set of all $\phi \in \mathcal{K}^{\Theta^1}$ satisfying the following conditions:

D(i) $1\phi \in \mathcal{K}_0$,

D(ii) ϕ is order preserving,

D(iii) $\tau \in \Theta,\ \tau\phi = L \implies t(\tau) = T_r$,

D(iv) $\tau \in \Theta,\ \tau\phi = R \implies t(\tau) = T_\ell$, D(v) $\sigma \in \Theta^1, \tau \in \Theta, \sigma\phi \in K_0$ and either $\sigma = \emptyset$

$$\text{or } t(\sigma) \neq h(\tau) \implies (\sigma\tau)\phi \supseteq (\sigma\phi)_{\tau K^*}.$$

THEOREM 5.1. (Polák [26], Theorem 3.6) Φ is a complete lattice (with respect to the component-wise order).

Polák's main theorem concerns those subvarieties of $\mathcal{CR}$ that contain the variety of semilattices.

THEOREM 5.2. (Polák [26], Theorem 3.6) For any $\mathcal{V} \in [\mathcal{S},\mathcal{CR}]$, let $\chi_{\mathcal{V}} \in K^{\Theta^1}$ be defined by:

$$\tau\chi_{\mathcal{V}} = \begin{cases} \mathcal{V}_K & \text{if } \tau = \emptyset \\ \mathcal{V}_{\tau K^*} & \text{otherwise} \end{cases}$$

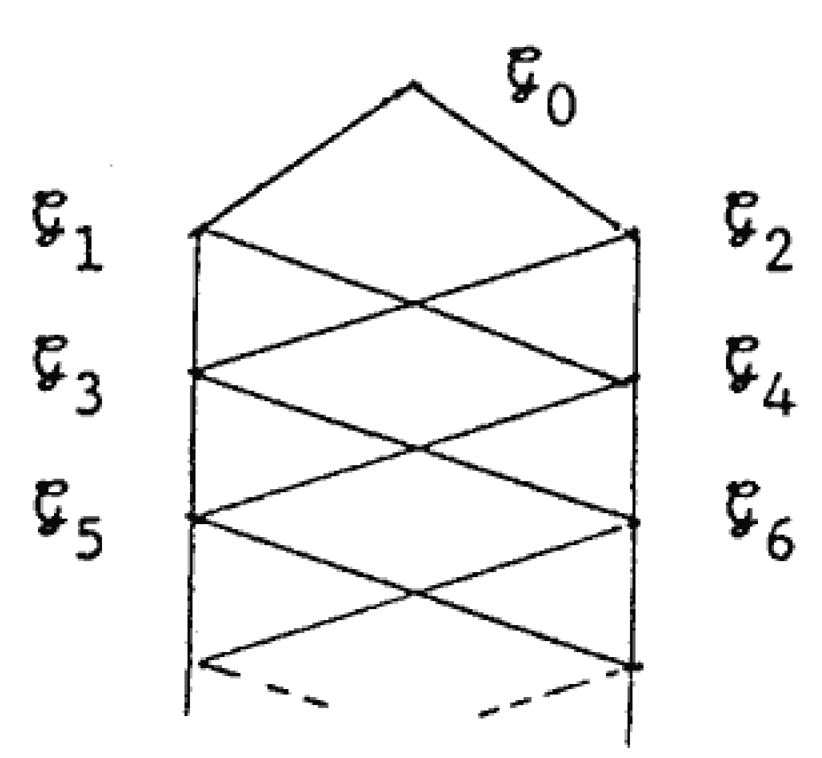
Then the mapping

$$\chi: \mathcal{V} \longrightarrow \chi_{\mathcal{V}} \qquad (\mathcal{V} \in [\mathcal{S},\mathcal{CR}])$$

is an isomorphism of $[\mathcal{S},\mathcal{CR}]$ onto Φ.

Many interesting subsidiary facts and applications of this theorem can be found in Polák's three papers [25], [26] and [27].

A case to which Polák's Theorem can be quickly applied to give new information, is the lattice $\mathcal{L}(\mathcal{OG})$ of subvarieties of the variety $\mathcal{OG}$ of orthodox completely regular semigroups It is not hard to show that $\mathcal{OG}_K = \mathcal{G}$, where $\mathcal{G}$ denotes the variety of groups. Therefore, for any $\mathcal{V} \in \mathcal{L}(\mathcal{OG})$, the partially ordered set of values of $\chi_{\mathcal{V}}$ may be depicted as follows:



where $\mathcal{G}_0 \in \mathcal{L}(\mathcal{G})$, the lattice of varieties of groups and, for each $n \geq 1$, $\mathcal{G}_n \in \mathcal{L}(\mathcal{G}) \cup \{ L , T , R \}$. From this it is easy to deduce the following result.

THEOREM 5.3. (Polák [26], Theorem 4.2) $\mathcal{L}(\mathcal{OG})$ is a subdirect product of countably many copies of $\mathcal{L}(\mathcal{G})$ and a single copy of $\mathcal{L}(\mathcal{B})$.

One question about $\mathcal{L}(\mathcal{CR})$ that remained unanswered for a considerable time was whether or not it is a modular lattice. Various results had been obtained concerning various sublattices of $\mathcal{L}(\mathcal{CR})$ (see, for example, Rasin [29] for the lattice of varieties of completely simple semigroups and Hall and Jones [9] for the lattice of varieties of completely regular semigroups for which $\mathcal{H}$ is a congruence). The question was finally answered in full generality with the aid of Polák's Theorem by Pastijn:

THEOREM 5.4. (Pastijn [12], Theorem 18) $\mathcal{L}(\mathcal{CR})$ is modular.

Verifications of the modularity of $\mathcal{L}(\mathcal{CR})$ that are not dependent on Polák's Theorem have been obtained by Pastijn [13] and Petrich and Reilly [23].

Since the lattice of group varieties is a sublattice of $\mathcal{L}(\mathcal{CR})$ it follows that $\mathcal{L}(\mathcal{CR})$ is not distributive. However, even in a non-distributive lattice, there may be elements which have properties that are normally associated with distributivity. More exactly, an element a in a lattice L is neutral if the mapping

$$x \longrightarrow (x \wedge a, x \vee a)$$

is a monomorphism of L onto a subdirect product of (a] and [a) (where (a] and [a) denote the ideal and filter of L, respectively, generated by a).

The usefulness of a neutral element a in a lattice L is that it makes it possible to convert certain types of problems on the whole lattice L to (hopefully simpler) problems on the (hopefully simpler) sublattices (a] and [a). One nice feature of modular lattices is that, by virtue of the lemma below, in order to establish that an element is neutral it is not necessary to verify all the conditions in the definition each time.

LEMMA 5.5. ([7]) For any element a in a modular lattice L, the following statements are equivalent:

(i) a is neutral in L;

(ii) the mapping

$$\mu_a : x \longrightarrow x \wedge a \qquad (x \in L)$$

is an endomorphism of L;

(iii) the mapping

$$\nu_a : x \longrightarrow x \vee a \qquad (x \in L)$$

is an endomorphism of L.

Prior to Polák's Theorem, a few simple examples of neutral elements in $\mathcal{L}(\mathcal{CR})$ were known. For example, Hall and Jones [9] had shown that the variety $\mathcal{S}$ of semilattices is neutral and Jones [11] extended the list to include all subvarieties of the variety $\mathcal{NB}$ of normal bands.

Also Jones [11] had shown that $\mu_{\mathcal{G}}$ and $\mu_{\mathcal{CS}}$ are homomorphisms so that, by the preceding theorem and lemma, we may conclude immediately that $\mathcal{G}$ and $\mathcal{CS}$ are both neutral in $\mathcal{L}(\mathcal{CR})$. But now, with the techniques available on account of Polák's Theorem it is possible to determine many more neutral elements and to approach the search for neutral elements in a much more systematic way.

The following is a partial listing of the varieties that are now known to be neutral in $\mathcal{L}(\mathcal{CR})$:(for details, see Hall and Jones [9], Jones [11] and Reilly [33])

$\mathcal{G}$, $\mathcal{CS}$, $\mathcal{AG}$, $\mathcal{B}$, $\mathcal{OG}$, $\mathcal{LOCG}$,

$\mathcal{CS}(\mathcal{AG})$ - the variety of completely simple semigroups with abelian subgroups.

$\mathcal{OG}(\mathcal{AG})$ - the variety of orthodox completely regular semigroups with abelian subgroups.

$\mathcal{LOCG}(\mathcal{AG})$ - the variety of locally orthodox cryptogroups with abelian subgroups.

$\mathcal{L}(\mathcal{B})$ - all subvarieties of $\mathcal{B}$

$\mathcal{L}(\mathcal{OG}(\mathcal{AG}))$ - all subvarieties of $\mathcal{OG}(\mathcal{AG})$.

$\mathcal{L}(\mathcal{LOCG}(\mathcal{AG}))$ - all subvarieties of $\mathcal{LOCG}(\mathcal{AG})$.

Some partial results can also be obtained, such as the following.

COROLLARY 5.6. (Reilly [33], Corollary 5.8) The variety $\mathcal{CG}$ is neutral in $\mathcal{L}(\mathcal{CG}^K)$.

Since $\mathcal{G} \subseteq \mathcal{CG}$, we must also have $\mathcal{OG} = \mathcal{G}^K \subseteq \mathcal{CG}^K$ and therefore also

$\mathcal{CG} \vee \mathcal{OG} \subseteq \mathcal{CG}^K$. From this it can be shown that $\mathcal{CG}$ is neutral in $\mathcal{L}(\mathcal{CG} \vee \mathcal{OG})$.

An important feature of the next theorem is the fact that certain varieties are expressible as joins of well known varieties.

LEMMA 5.7. (i) $\mathcal{B} \vee \mathcal{G} = \mathcal{OCG}$. (ii) $\mathcal{B} \vee \mathcal{CS} = \mathcal{LOCG}$.

(iii) $\mathcal{OG} \vee \mathcal{CS} = \mathcal{CLOCG}$.

Proof. (i) See (Petrich [16], Lemma 1).

(ii) See (Hall and Jones [9], Corollary 5.4).

(iii) See (Hall and Jones [9], Theorem 5.3 and Reilly [32], Proposition 5.3).

COROLLARY 5.8. (i) (Petrich [16], Theorem) The mappings

$$\mathcal{V} \longrightarrow (\mathcal{V} \cap \mathcal{B},\ \mathcal{V} \cap \mathcal{G}), \qquad (\mathcal{U},\mathcal{W}) \longrightarrow \mathcal{U} \vee \mathcal{W}$$

are mutually inverse isomorphisms between $\mathcal{L}(\mathcal{OCG})$ and $\mathcal{L}(\mathcal{B}) \times \mathcal{L}(\mathcal{G})$.

(ii) (Hall and Jones [9], Corollary 5.5, Rasin [30], Proposition 1) The mappings

$$\mathcal{V} \longrightarrow (\mathcal{V} \cap \mathcal{B},\ \mathcal{V} \cap \mathcal{CS}), \qquad (\mathcal{U},\mathcal{W}) \longrightarrow \mathcal{U} \vee \mathcal{W}$$

are mutually inverse isomorphisms between $\mathcal{L}(\mathcal{LOCG})$ and the subdirect product of $\mathcal{L}(\mathcal{B})$ and $\mathcal{L}(\mathcal{CS})$ consisting of all those pairs $(\mathcal{U}, \mathcal{W})$ with $\mathcal{U} \cap \mathcal{B} = \mathcal{W} \cap \mathcal{CS}$.

(iii) (Reilly [33], Theorem 5.9) The mappings

$$\mathcal{V} \longrightarrow (\mathcal{V} \cap \mathcal{CG},\ \mathcal{V} \cap \mathcal{OG}), \qquad (\mathcal{U}, \mathcal{W}) \longrightarrow \mathcal{U} \vee \mathcal{W}$$

are mutually inverse isomorphisms between $\mathcal{L}(\mathcal{CG} \vee \mathcal{OG})$ and the subdirect product of $\mathcal{L}(\mathcal{CG})$ and $\mathcal{L}(\mathcal{OG})$ consisting of all those pairs $(\mathcal{U}, \mathcal{W})$ with $\mathcal{U} \cap \mathcal{OCG} = \mathcal{W} \cap \mathcal{OCG}$.

REFERENCES

[1] A.P.Birjukov, Varieties of idempotent semigroups, Algebra i Logika, 9(1970), 255-273 (Russian), AMS Translation.

[2] A.H.Clifford, Semigroups admitting relative inverses, Annals of Math. 42(1941), 1037-1049.

[3] A.H. Clifford, The free completely regular semigroup on a

set, J. Algebra 59 (1979), 434-451.

[4] A.H.Clifford and G.B.Preston, The algebraic theory of semigroups, Amer. Math. Surveys No. 7, 1961, Providence.

[5] R.Feigenbaum, Regular semigroup congruences, Semigroup Forum 17 (1979), 373-377.

[6] C.F.Fennemore, All varieties of bands, Math. Nachr. 48 (1971), I:237-252, II: 253-262.

[7] G.Grätzer, General lattice theory, Academic Press, New York, 1978.

[8] J.A. Gerhard, The lattice of equational classes of idempotent semigroups, J.Algebra 15(1970), 195-224.

[9] T.E.Hall and P.R. Jones, On the lattice of varieties of bands of groups, Pacific J. Math. 91(1980), 327-337.

[10] J.M.Howie, An Introduction to Semigroup Theory, Academic Press, London, 1976.

[11] P.R.Jones, On the lattice of varieties of completely regular semigroups, J. Austral. Math. Soc. (Series A) 35 (1983), 227-235.

[12] F.Pastijn, The lattice of completely regular semigroup varieties, preprint.

[13] F.Pastijn, Commuting fully invariant congruences on free completely regular semigroups, (manuscript).

[14] F.Pastijn and M. Petrich, Congruences on regular semigroups, Trans. Amer. Math. Soc. 295(1986), 607-633.

[15] F. Pastijn and P.G. Trotter, Lattices of completely regular semigroup varieties, Pacific J. Math. 119 (1985), 191-214.

[16] M.Petrich, Varieties of orthodox bands of groups, Pacific J. Math., 58(1975), 209-217.

[17] M.Petrich, A structure theorem for completely regular semigroups, Proc. Amer. Math. Soc. 99(1987), 617-622.

[18] M.Petrich, Congruences on completely regular semigroups, preprint.

[19] M.Petrich and N.R.Reilly, Bands of groups with universal properties, Monatshefte für Mathematik 94(1982), 45-67.

[20] M.Petrich and N.R.Reilly, All varieties of central completely simple semigroups, Trans. Amer. Math. Soc. 280(1983), 623-636.

[21] M. Petrich and N.R. Reilly, Semigroups generated by certain operators on varieties of completely regular semigroups, Pacific J. Math. 132 (1988), 151-175.

[22] M.Petrich and N.R.Reilly, Operators related to E-disjunctive and fundamental completely regular semigroups (to appear).

[23] M.Petrich and N.R.Reilly, The modularity of the lattice of varieties of completely regular semigroups and related representations (to appear).
[24] M.Petrich and N.R.Reilly, Completely regular semigroups (manuscript).
[25] L.Polák, On varieties of completely regular semigroups I, Semigroup Forum 32(1985), 97-123.
[26] L.Polák, On varieties of completely regular semigroups II, Semigroup Forum 36(1987), 253-284.
[27] L.Polák, On varieties of completely regular semigroups III, Semigroup Forum 37(1988), 1-30.
[28] V.V. Rasin, Free completely simple semigroups, Ural. Gos. Univ. Mat. Zap. 11 (1979), 140-151 (Russian).
[29] V.V.Rasin, On the lattice of varieties of completely simple semigroups, Semigroup Forum 17(1979), 113-122.
[30] V.V.Rasin, On the varieties of Cliffordian semigroups, Semigroup Forum 23(1981), 201-220.
[31] D.Rees, On semi-groups, Proc. Cambridge Phil. Soc. 36(1940), 387-400.
[32] N.R.Reilly, Varieties of completely regular semigroups, J. Austral. Math. Soc. (Series A) 38 (1985), 372-393.
[33] N.R.Reilly, Completely regular semigroups, Proc. of the Inter. Conf. on Algebra, Lisbon (1988).

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