CHAPTER B

Generating Functions of Partitions

For a complex sequence $\{\alpha_n | n = 0, 1, 2, \cdots\}$, its generating function with a complex variable q is defined by

$$A(q) := \sum_{n=0}^{\infty} \alpha_n \, q^n \quad \rightleftharpoons \quad \alpha_n = [q^n] \, A(q).$$

When the sequence has finite non-zero terms, the generating function reduces to a polynomial. Otherwise, it becomes an infinite series. In that case, we suppose in general $|q| < 1$ from now on.

B1. Basic generating functions of partitions

Given three complex indeterminates x, q and n with $|q| < 1$, the shifted factorial is defined by

$$\begin{aligned} (x;\, q)_\infty &= \prod_{k=0}^{\infty} (1 - xq^k) \\ (x;\, q)_n &= \frac{(x;\, q)_\infty}{(q^n x;\, q)_\infty}. \end{aligned}$$

When n is a natural number in particular, it reduces to

$$(x;\, q)_0 = 1 \quad \text{and} \quad (x;\, q)_n = \prod_{k=0}^{n-1} (1 - q^k x) \quad \text{for} \quad n = 1, 2, \cdots.$$

We shall frequently use the following abbreviated notation for shifted factorial fraction:

$$\left[\begin{matrix} a, & b, & \cdots, & c \\ \alpha, & \beta, & \cdots, & \gamma \end{matrix} \,\middle|\, q \right]_n = \frac{(a; q)_n (b; q)_n \cdots (c; q)_n}{(\alpha; q)_n (\beta; q)_n \cdots (\gamma; q)_n}.$$

B1.1. Partitions with parts in $\mathbb{S}$. We first investigate the generating functions of partitions with parts in $\mathbb{S}$, where the basic set $\mathbb{S} \subseteq \mathbb{N}$ with $\mathbb{N}$ being the set of natural numbers.

Let $\mathbb{S}$ be a set of natural numbers and $p(n|\mathbb{S})$ denote the number of partitions of n into elements of $\mathbb{S}$ (*or in other words, the parts of partitions belong to* $\mathbb{S}$). Then the univariate generating function is given by

$$\sum_{n=0}^{\infty} p(n|\mathbb{S})\, q^n \;=\; \prod_{k\in\mathbb{S}} \frac{1}{1-q^k}. \tag{B1.1a}$$

If we denote further by $p_\ell(n|\mathbb{S})$ the number of partitions with exactly ℓ-parts in $\mathbb{S}$, then the bivariate generating function is

$$\sum_{\ell,n\geq 0} p_\ell(n|\mathbb{S})\, x^\ell\, q^n \;=\; \prod_{k\in\mathbb{S}} \frac{1}{1-xq^k}. \tag{B1.1b}$$

Proof. For $|q| < 1$, we can expand the right member of the equation (B1.1a) according to the geometric series

$$\prod_{k\in\mathbb{S}} \frac{1}{1-q^k} \;=\; \prod_{k\in\mathbb{S}} \sum_{m_k=0}^{\infty} q^{km_k} \;=\; \sum_{\substack{m_k\geq 0\\ k\in\mathbb{S}}} q^{\sum_{k\in\mathbb{S}} km_k}.$$

Extracting the coefficient of q^n from both sides, we obtain

$$[q^n]\prod_{k\in\mathbb{S}} \frac{1}{1-q^k} \;=\; [q^n]\sum_{\substack{m_k\geq 0\\ k\in\mathbb{S}}} q^{\sum_{k\in\mathbb{S}} km_k} \;=\; \sum_{\substack{\sum_{k\in\mathbb{S}} km_k=n\\ m_k\geq 0:\ k\in\mathbb{S}}} 1.$$

The last sum is equal to the number of solutions of the Diophantine equation

$$\sum_{k\in\mathbb{S}} km_k = n$$

which enumerates the partitions $\big\{1^{m_1}, 2^{m_2}, \cdots, n^{m_n}\big\}$ of n into parts in $\mathbb{S}$.

This completes the proof of (B1.1a). The bivariate generating function (B1.1b) can be verified similarly.

In fact, consider the formal power series expansion

$$\prod_{k\in\mathbb{S}} \frac{1}{1-xq^k} \;=\; \prod_{k\in\mathbb{S}} \sum_{m_k=0}^{\infty} x^{m_k} q^{km_k} \;=\; \sum_{\substack{m_k\geq 0\\ k\in\mathbb{S}}} x^{\sum_{k\in\mathbb{S}} m_k} q^{\sum_{k\in\mathbb{S}} km_k}$$

in which the coefficient of $x^\ell q^n$ reads as

$$\begin{aligned}[x^\ell q^n]\prod_{k\in\mathbb{S}}\frac{1}{1-xq^k} &= [x^\ell q^n]\sum_{\substack{m_k\geq 0\\ k\in\mathbb{S}}} x^{\sum_{k\in\mathbb{S}} m_k} q^{\sum_{k\in\mathbb{S}} km_k}\\ &= \sum_{\substack{\left.\begin{array}{l}\sum_{k\in\mathbb{S}} m_k=\ell\\ \sum_{k\in\mathbb{S}} km_k=n\end{array}\right\}\\ m_k\geq 0:\, k\in\mathbb{S}}} 1.\end{aligned}$$

The last sum enumerates the solutions of the system of Diophantine equations

$$\begin{cases}\displaystyle\sum_{k\in\mathbb{S}} m_k &= \ell\\ \displaystyle\sum_{k\in\mathbb{S}} km_k &= n\end{cases}$$

which are the number of partitions $\{1^{m_1}, 2^{m_2}, \cdots, n^{m_n}\}$ of n with exactly ℓ-parts in $\mathbb{S}$. □

B1.2. Partitions into distinct parts in $\mathbb{S}$. Next we study the generating functions of partitions into distinct parts in $\mathbb{S}$.

If we denote by $Q(n|\mathbb{S})$ and $Q_\ell(n|\mathbb{S})$ the corresponding partition numbers with distinct parts from $\mathbb{S}$, then their generating functions read respectively as

$$\sum_{n=0}^{\infty} Q(n|\mathbb{S})\,q^n = \prod_{k\in\mathbb{S}}\big(1+q^k\big) \tag{B1.2a}$$

$$\sum_{\ell,n\geq 0} Q_\ell(n|\mathbb{S})\,x^\ell q^n = \prod_{k\in\mathbb{S}}\big(1+xq^k\big). \tag{B1.2b}$$

Proof. For the first identity, observing that

$$1+q^k = \sum_{m_k=0,1} q^{km_k}$$

we can reformulate the product on the right hand side as

$$\prod_{k\in\mathbb{S}}\big(1+q^k\big) = \prod_{k\in\mathbb{S}}\sum_{m_k=0,1} q^{km_k} = \sum_{\substack{m_k=0,1\\ k\in\mathbb{S}}} q^{\sum_{k\in\mathbb{S}} km_k}.$$

Extracting the coefficient of q^n, we obtain

$$[q^n]\prod_{k\in\mathbb{S}}\big(1+q^k\big) = [q^n]\sum_{\substack{m_k=0,1\\ k\in\mathbb{S}}} q^{\sum_{k\in\mathbb{S}} km_k} = \sum_{\substack{\sum_{k\in\mathbb{S}} km_k=n\\ m_k=0,1:\, k\in\mathbb{S}}} 1.$$

The last sum enumerates the solutions of Diophantine equation

$$\sum_{k\in\mathbb{S}} km_k = n \quad \text{with} \quad m_k = 0,1$$

which is equal to $Q(n|\mathbb{S})$, the number of partitions of n into distinct parts in $\mathbb{S}$.

Instead, we can proceed similarly for the second formula as follows:

$$\prod_{k\in\mathbb{S}}(1+xq^k) \;=\; \prod_{k\in\mathbb{S}}\sum_{m_k=0,1} x^{m_k}q^{km_k} \;=\; \sum_{\substack{m_k=0,1\\ k\in\mathbb{S}}} x^{\sum_{k\in\mathbb{S}} m_k} q^{\sum_{k\in\mathbb{S}} km_k}.$$

The coefficient of $x^\ell q^n$ leads us to the following

$$\begin{aligned}
[x^\ell q^n]\prod_{k\in\mathbb{S}}(1+xq^k) \;&=\; [x^\ell q^n]\sum_{\substack{m_k=0,1\\ k\in\mathbb{S}}} x^{\sum_{k\in\mathbb{S}} m_k} q^{\sum_{k\in\mathbb{S}} km_k}\\
&=\; \sum_{\substack{\left.\begin{array}{l}\sum_{k\in\mathbb{S}} m_k = \ell\\ \sum_{k\in\mathbb{S}} km_k = n\end{array}\right\}\\ m_k=0,1:\; k\in\mathbb{S}}} 1.
\end{aligned}$$

The last sum equals the number of solutions of the system of Diophantine equations

$$\left.\begin{aligned}\sum_{k\in\mathbb{S}} m_k &= \ell\\ \sum_{k\in\mathbb{S}} km_k &= n\end{aligned}\right\} \quad \text{with} \quad m_k = 0,1$$

which correspond to the partitions $\{1^{m_1}, 2^{m_2}, \cdots, n^{m_n}\}$ of n with exactly ℓ distinct parts in $\mathbb{S}$. $\square$

B1.3. Classical generating functions. When $\mathbb{S} = \mathbb{N}$, the set of natural numbers, the corresponding generating functions may be displayed, respectively, as

$$\frac{1}{(q;\, q)_\infty} = \prod_{m=1}^{\infty} \frac{1}{1-q^m} = \sum_{n=0}^{\infty} p(n)\, q^n \tag{B1.3a}$$

$$\frac{1}{(qx;\, q)_\infty} = \prod_{m=1}^{\infty} \frac{1}{1-xq^m} = \sum_{\ell,n\geq 0} p_\ell(n)\, x^\ell q^n \tag{B1.3b}$$

$$(-q;\, q)_\infty = \prod_{m=1}^{\infty} (1+q^m) = \sum_{n=0}^{\infty} Q(n)\, q^n \tag{B1.3c}$$

$$(-qx;\, q)_\infty = \prod_{m=1}^{\infty} (1+xq^m) = \sum_{\ell,n\geq 0} Q_\ell(n)\, x^\ell q^n. \tag{B1.3d}$$

Manipulating the generating function of the partitions into odd numbers in the following manner

$$\begin{aligned}\prod_{k=1}^{\infty} \frac{1}{1-q^{2k-1}} &= \prod_{k=1}^{\infty} \frac{1}{1-q^k} \times \prod_{k=1}^{\infty} (1-q^{2k}) \\ &= \prod_{k=1}^{\infty} \frac{1-q^{2k}}{1-q^k} = \prod_{k=1}^{\infty} (1+q^k)\end{aligned}$$

we see that it results in the generating function of the partitions into distinct parts. We have therefore proved the following theorem due to Euler. *The number of partitions of n into odd numbers equals to the number of partitions of n into distinct parts.*

B2. Classical partitions and the Gauss formula

B2.1. Proposition. Let $p_m(n)$ be the number of partitions into exactly m parts (*or dually, partitions with the largest part equal to m*). Its generating function reads as

$$\sum_{n=0}^{\infty} p_m(n)\, q^n = \frac{q^m}{(1-q)(1-q^2)\cdots(1-q^m)}. \tag{B2.1}$$

Proof. For $\mathbb{S}=\mathbb{N}$, the generating function of $\{p_\ell(n|\mathbb{N})\}$ reads as

$$\begin{aligned}\sum_{\ell,n\geq 0} p_\ell(n)\,x^\ell q^n &= \sum_{\ell\geq 0} x^\ell \sum_{n\geq 0} p_\ell(n) q^n\\ &= \prod_{k=1}^{\infty}\frac{1}{1-xq^k} = \frac{1}{(qx;q)_\infty}.\end{aligned}$$

Extracting the coefficient of x^m, we get

$$\sum_{n=0}^{\infty} p_m(n)\,q^n = [x^m]\frac{1}{(qx;q)_\infty}.$$

For $|q|<1$, the function $1/(qx;q)_\infty$ is analytic at $x=0$. We can therefore expand it in MacLaurin series:

$$\frac{1}{(qx;q)_\infty} = \sum_{\ell=0}^{\infty} A_\ell(q)x^\ell \tag{B2.2}$$

where the coefficients $\{A_\ell(q)\}$ are independent of x to be determined. Performing the replacement $x \to x/q$, we can restate the expansion just displayed as

$$\frac{1}{(x;q)_\infty} = \sum_{\ell=0}^{\infty} A_\ell(q)x^\ell q^{-\ell}. \tag{B2.3}$$

It is evident that (B2.2) equals $(1-x)$ times (B2.3), which results in the functional equation

$$\sum_{\ell=0}^{\infty} A_\ell(q)x^\ell = (1-x)\sum_{\ell=0}^{\infty} A_\ell(q)x^\ell q^{-\ell}.$$

Extracting the coefficient of x^m from both expansions, we get

$$A_m(q) = A_m(q)q^{-m} - A_{m-1}(q)q^{1-m}$$

which is equivalent to the following recurrence relation

$$A_m(q) = \frac{q}{1-q^m}A_{m-1}(q) \quad \text{where} \quad m=1,2,\cdots.$$

Iterating this recursion for m-times, we find that

$$A_m(q) = \frac{q^m\,A_0(q)}{(1-q^m)(1-q^{m-1})\cdots(1-q)} = \frac{q^m}{(q;q)_m}\,A_0(q).$$

Noting that $A_0(q)=1$, we get finally

$$\sum_{n=0}^{\infty} p_m(n)\,q^n = [x^m]\frac{1}{(qx;q)_\infty} = \frac{q^m}{(q;q)_m}.$$

This completes the proof of Proposition B2.1. □

A combinatorial proof. Let $p(n|\lambda_1 = m)$ be the number of partitions of n with the first part λ_1 equal to m. Then $p_m(n) = p(n|\lambda_1 = m)$ because the partitions enumerated by $p_m(n)$ are conjugate with those enumerated by $p(n|\lambda_1 = m)$. Therefore they have the same generating functions:

$$\sum_{n=0}^{\infty} p_m(n)\, q^n \;=\; \sum_{n=0}^{\infty} p(n|\lambda_1 = m)\, q^n.$$

All the partition of n enumerated by $p(n|\lambda_1 = m)$ have the first part $\lambda_1 = m$ in common and the remaining parts constitute the partitions of $n-m$ with each part $\leq m$. Therefore we have

$$\begin{aligned}
\sum_{n=0}^{\infty} p(n|\lambda_1 = m)\, q^n \;&=\; \sum_{n=m}^{\infty} p(n-m|\lambda_1 \leq m)\, q^n \;=\; q^m \sum_{n=0}^{\infty} p(n|\lambda_1 \leq m)\, q^n \\
&=\; q^m \sum_{n=0}^{\infty} p(n|\{1, 2, \cdots, m\})\, q^n \;=\; \frac{q^m}{(q;q)_m}
\end{aligned}$$

where the first line is justified by replacement $n \to n+m$ on summation index, while the second is a consequence of (B1.1a).

This confirms again the generating function (B2.1). $\square$

B2.2. Proposition. Let $p^m(n)$ be the number of partitions into $\leq m$ parts *(or dually, partitions into parts $\leq m$).* Then we have the generating function

$$\sum_{n=0}^{\infty} p^m(n)\, q^n \;=\; \frac{1}{(1-q)(1-q^2)\cdots(1-q^m)} \tag{B2.4}$$

which yields a finite summation formula

$$\frac{1}{(1-q)(1-q^2)\cdots(1-q^m)} \;=\; 1 + \sum_{k=1}^{m} \frac{q^k}{(1-q)(1-q^2)\cdots(1-q^k)}.$$

Proof. Notice that $p^m(n)$, the number of partitions into $\leq m$ parts is equal to the number of partitions into parts $\leq m$ in view of conjugate partitions. We get immediately from (B1.1a) the generating function (B2.4).

The classification of the partitions of n into $\leq m$ parts with respect to the number k of parts yields

$$p^m(n) = p_0(n) + p_1(n) + p_2(n) + \cdots + p_m(n).$$

The corresponding generating function results in

$$\sum_{n=0}^{\infty} p^m(n)\, q^n \;=\; \sum_{k=0}^{m}\sum_{n=0}^{\infty} p_k(n)\, q^n \;=\; \sum_{k=0}^{m} \frac{q^k}{(q;q)_k}.$$

Recalling the first generating function expression (B2.4), we get the second formula from the last relation. □

B2.3. Gauss' classical partition identity.

$$\prod_{n=0}^{\infty} \frac{1}{1-xq^n} \;=\; 1+\sum_{m=1}^{\infty} \frac{x^m}{(1-q)(1-q^2)\cdots(1-q^m)}. \tag{B2.5}$$

Proof. In fact, we have already established this identity from the demonstration of the last theorem, where it has been displayed explicitly in (B2.3).

Alternatively, classifying all the partitions with respect to the number of parts, we can manipulate the bivariate generating function

$$\begin{aligned}\frac{1}{(xq;q)_\infty} \;&=\; \sum_{\ell,n=0}^{\infty} p_\ell(n) x^\ell q^n \;=\; \sum_{\ell=0}^{\infty} x^\ell \sum_{n=0}^{\infty} p_\ell(n) q^n \\ &=\; \sum_{\ell=0}^{\infty} \frac{x^\ell q^\ell}{(1-q)(1-q^2)\cdots(1-q^\ell)}\end{aligned}$$

which is equivalent to Gauss' classical partition identity. □

B2.4. Theorem. Let $p_\ell(n|m)$ be the number of partitions of n with exactly ℓ-parts $\leq m$. Then we have its generating function

$$\sum_{\ell,n=0}^{\infty} p_\ell(n|m)\, x^\ell q^n \;=\; \frac{1}{(1-qx)(1-q^2x)\cdots(1-q^mx)}.$$

The classification with respect to the maximum part k of partitions produces another identity

$$\frac{1}{(1-qx)(1-q^2x)\cdots(1-q^mx)} = 1 + x\sum_{k=1}^{m} \frac{q^k}{(1-qx)(1-q^2x)\cdots(1-q^kx)}.$$

Proof. The first generating function follows from (B1.1b).

From the first generating function, we see that the bivariate generating function of partitions into parts $\leq k$ reads as

$$\sum_{\ell,n=0}^{\infty} p_\ell(n|k)\, x^\ell q^n \;=\; \frac{1}{(qx;q)_k}.$$

Putting an extra part $\lambda_1 = k$ with enumerator xq^k over the partitions enumerated by the last generating function, we therefore derive the bivariate generating function of partitions into ℓ parts with the first one $\lambda_1 = k$ as follows:

$$\sum_{\ell,n=0}^{\infty} p_\ell(n|\lambda_1 = k)\, x^\ell q^n \;=\; \frac{xq^k}{(qx;q)_k}.$$

Classifying the partitions of n into exactly ℓ parts with each parts $\leq m$ according to the first part $\lambda_1 = k$, we get the following expression

$$\begin{aligned} \sum_{\ell,n=0}^{\infty} p_\ell(n|m)\, x^\ell q^n \;&=\; \sum_{k=0}^{m}\sum_{\ell,n=0}^{\infty} p_\ell(n|\lambda_1 = k) x^\ell q^n \\ &=\; 1 + x\sum_{k=1}^{m} \frac{q^k}{(qx;q)_k} \end{aligned}$$

which is the second identity. $\square$

B3. Partitions into distinct parts and the Euler formula

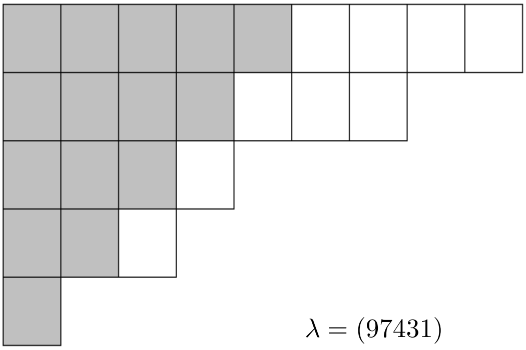
B3.1. Theorem. Let $Q_m(n)$ be the number of partitions into exactly m distinct parts. Its generating function reads as

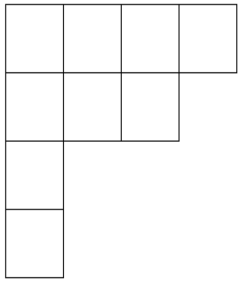
$$\sum_{n=0}^{\infty} Q_m(n)\, q^n \;=\; \frac{q^{\binom{1+m}{2}}}{(1-q)(1-q^2)\cdots(1-q^m)}. \tag{B3.1}$$

Proof. Let $\lambda = (\lambda_1 > \lambda_2 \cdots > \lambda_m > 0)$ be a partition enumerated by $Q_m(n)$. Based on λ, define another partition $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m \geq 0)$ by

$$\mu_k := \lambda_k - (m-k+1) \quad \text{for} \quad k = 1, 2, \cdots, m. \tag{B3.2}$$

It is obvious that μ is a partition of $|\lambda| - \binom{1+m}{2}$ into $\leq m$ parts. As an example, the following figures show this correspondence between two partitions $\lambda = (97431)$ and $\mu = (4311)$.





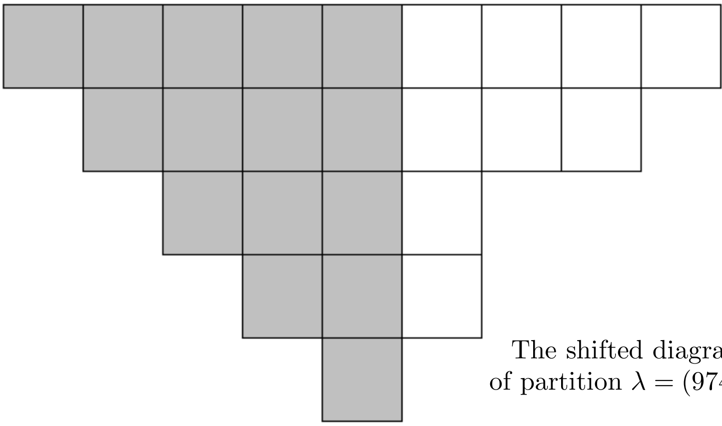
$\lambda = (97431)$ $\mu = (4311)$

It is not difficult to verify that the mapping (B3.2) is a bijection between the partitions of n with exactly m distinct parts and the partitions of $n - \binom{1+m}{2}$ with $\leq m$ parts. Therefore the generating function of $\{Q_m(n)\}_n$ is equal to that of $\{p^m\big(n - \binom{1+m}{2}\big)\}_n$, the number of partitions of $n - \binom{1+m}{2}$ with the number of parts $\leq m$:

$$\begin{aligned}\sum_{n=0}^{\infty} Q_m(n)\, q^n \;&=\; \sum_{n=0}^{\infty} p^m\Big(n - \tbinom{1+m}{2}\Big)\, q^n \\ &= q^{\binom{1+m}{2}} \sum_{n=0}^{\infty} p^m(n)\, q^n \;=\; \frac{q^{\binom{1+m}{2}}}{(q;q)_m}\end{aligned}$$

thanks for the generating function displayed in (B2.4). This completes the proof of Theorem B3.1. $\square$

Instead of the ordinary Ferrers diagram, we can draw a shifted diagram of λ as follows (see the figure). Under the first row of λ_1 squares, we put λ_2 squares lined up vertically from the second column. For the third row, we put λ_3 squares beginning from the third column. Continuing in this way, the last row of λ_m squares will be lined up vertically from the m-th column.



The shifted diagram
of partition $\lambda = (97431)$

From the shifted diagram of λ, we see that all the partitions enumerated by $Q_m(n)$ have one common triangle on the left whose weight is $\binom{1+m}{2}$. The remaining parts right to the triangle are partitions of $n - \binom{m+1}{2}$ with $\leq m$ parts. This reduces the problem of computing the generating function to the case just explained.

B3.2. Classifying all the partitions with distinct parts according to the number of parts, we get Euler's classical partition identity

$$\prod_{n=0}^{\infty}(1 - xq^n) \;=\; 1 + \sum_{m=1}^{\infty} \frac{(-1)^m\, x^m q^{\binom{m}{2}}}{(1-q)(1-q^2)\cdots(1-q^m)} \tag{B3.3}$$

which can also be verified through the correspondence between partitions into distinct odd parts and self-conjugate partitions.

PROOF. Considering the bivariate generating function of $Q_m(n)$, we have

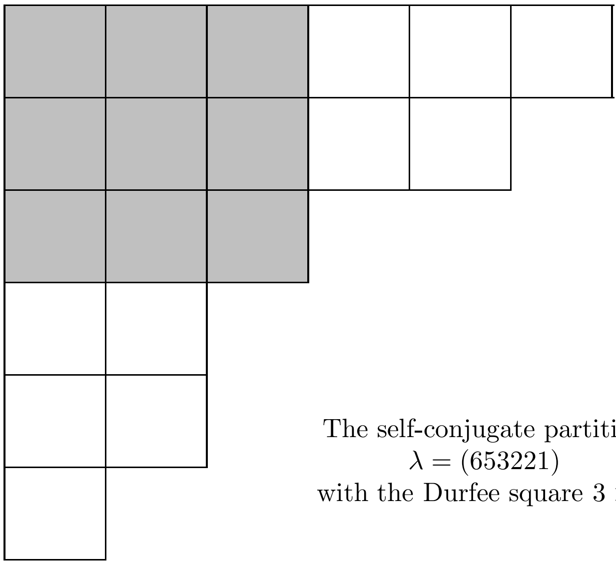
$$\prod_{k=1}^{\infty}(1 + xq^k) \;=\; \sum_{m=0}^{\infty} x^m \sum_{n=0}^{\infty} Q_m(n) q^n.$$

Recalling (B3.1) and then noting that $Q_0(n) = \delta_{0,n}$, we deduce that

$$\prod_{k=1}^{\infty}(1+xq^k) \;=\; 1+\sum_{m=1}^{\infty}\frac{x^m q^{\binom{1+m}{2}}}{(q;q)_m}$$

which becomes the Euler identity under parameter replacement $x \to -x/q$.

In view of Euler's Theorem A2.1, we have a bijection between the partitions into distinct odd parts and the self-conjugate partitions.



The self-conjugate partition
$\lambda = (653221)$
with the Durfee square 3×3

For a self-conjugate partition with the main diagonal length equal to m (which corresponds exactly to the length of partitions into distinct odd parts), it consists of three pieces: the first piece is the square of $m \times m$ on the top-left with bivariate enumerator $x^m q^{m^2}$, the second piece right to the square is a partition with $\leq m$ parts enumerated by $1/(q;q)_m$ and the third piece under the square is in effect the conjugate of the second one. Therefore the partitions right to the square and under the square $m \times m$ are altogether enumerated by $1/(q^2;q^2)_m$.

Classifying the self-conjugate partitions according to the main diagonal length m, multiplying both generating functions together and summing m

over $0 \le m \le \infty$, we find the following identity:

$$\prod_{n=0}^{\infty}(1+xq^{1+2n}) \;=\; \sum_{m=0}^{\infty}\frac{x^m q^{m^2}}{(q^2;q^2)_m}$$

where the left hand side is the bivariate generating function of the partitions into odd distinct parts.

It is trivial to verify that under replacements

$$x \to -xq^{-1/2} \quad\text{and}\quad q \to q^{1/2}$$

the last formula is exactly the identity displayed in (B3.3). $\square$

Unfortunately, there does not exist the closed form for the generating function of $Q^m(n)$, numbers of partitions into $\le m$ distinct parts.

B3.3. Dually, if we classify the partitions into distinct parts $\le m$ according to their maximum part. Then we can derive the following finite and infinite series identities

$$\prod_{j=1}^{m}(1+q^j x) \;=\; 1 + x\sum_{k=1}^{m} q^k \prod_{i=1}^{k-1}(1+q^i x) \tag{B3.4a}$$

$$\prod_{j=1}^{\infty}(1+q^j x) \;=\; 1 + x\sum_{k=1}^{\infty} q^k \prod_{i=1}^{k-1}(1+q^i x). \tag{B3.4b}$$

Proof. For the partitions into distinct parts with the maximum part equal to k, their bivariate generating function is given by

$$q^k x \prod_{i=1}^{k-1}(1+q^i x) \quad \text{which reduces to 1 for } k=0.$$

Classifying the partitions into distinct parts $\le m$ according to their maximum part k with $0 \le k \le m$, we get

$$(-qx;q)_m \;=\; 1 + x\sum_{k=1}^{m} q^k (-qx;q)_{k-1}.$$

The second identity follows from the first one with $m \to \infty$. $\square$

B4. Partitions and the Gauss q-binomial coefficients

B4.1. Lemma. Let $p_\ell(n|m)$ and $p^\ell(n|m)$ be the numbers of partitions of n into ℓ and $\leq \ell$ parts, respectively, with each part $\leq m$. We have the generating functions:

$$\sum_{\ell,n\geq 0} p_\ell(n|m)\, x^\ell q^n = \frac{1}{(1-xq)(1-xq^2)\cdots(1-xq^m)} \tag{B4.1a}$$

$$\sum_{\ell,n\geq 0} p^\ell(n|m)\, x^\ell q^n = \frac{1}{(1-x)(1-xq)\cdots(1-xq^m)}. \tag{B4.1b}$$

The first identity (B4.1a) is a special case of the generating function shown in (B1.1b).

On account of the length of partitions, we have

$$p^\ell(n|m) = p_0(n|m) + p_1(n|m) + \cdots + p_\ell(n|m).$$

Manipulating the triple sum and then applying the geometric series, we can calculate the corresponding generating function as follows:

$$\begin{aligned}
\sum_{\ell,n\geq 0} p^\ell(n|m)\, x^\ell q^n &= \sum_{\ell,n\geq 0}\sum_{k=0}^{\ell} p_k(n|m) x^\ell q^n \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} p_k(n|m) q^n \sum_{\ell=k}^{\infty} x^\ell \\
&= \frac{1}{1-x}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} p_k(n|m) x^k q^n.
\end{aligned}$$

The last expression leads us immediately to the second bivariate generating function (B4.1b) in view of the first generating function (B4.1a). □

B4.2. The Gauss q-binomial coefficients as generating functions. Let $p_\ell(n|m)$ and $p^\ell(n|m)$ be as in Lemma B4.1. The corresponding univariate generating functions read respectively as

$$\sum_{n\geq 0} p_\ell(n|m)\, q^n = \begin{bmatrix}\ell+m-1\\ m-1\end{bmatrix} q^\ell \tag{B4.2a}$$

$$\sum_{n\geq 0} p^\ell(n|m)\, q^n = \begin{bmatrix}\ell+m\\ m\end{bmatrix} \tag{B4.2b}$$

where the q-Gauss binomial coefficient is defined by

$$\begin{bmatrix} m+n \\ m \end{bmatrix}_q = \frac{(q;\,q)_{m+n}}{(q;\,q)_m(q;\,q)_n}.$$

PROOF. For these two formulae, it is sufficient to prove only one identity because

$$p_\ell(n|m) = p^\ell(n|m) - p^{\ell-1}(n|m).$$

In fact, supposing that (B4.2b) is true, then (B4.2a) follows in this manner:

$$\begin{aligned}\sum_{n\geq 0} p_\ell(n|m)\, q^n &= \sum_{n\geq 0} p^\ell(n|m)\, q^n - \sum_{n\geq 0} p^{\ell-1}(n|m)\, q^n \\ &= \begin{bmatrix} \ell+m \\ m \end{bmatrix}_q - \begin{bmatrix} \ell-1+m \\ m \end{bmatrix}_q = q^\ell \begin{bmatrix} \ell+m-1 \\ m-1 \end{bmatrix}_q.\end{aligned}$$

Now we should prove (B4.2b). Extracting the coefficient of x^ℓ from the generation function (B4.1b), we get

$$\sum_{n=0}^{\infty} p^\ell(n|m)\, q^n = [x^\ell]\frac{1}{(x;q)_{m+1}}.$$

Observing that the function $1/(x;q)_{m+1}$ is analytic at $x=0$ for $|q|<1$, we can expand it into MacLaurin series:

$$\frac{1}{(x;q)_{m+1}} = \sum_{k=0}^{\infty} B_k(q)x^k$$

where the coefficients $\{B_k(q)\}$ are independent of x to be determinated. Reformulating it under replacement $x \to qx$ as

$$\frac{1}{(qx;q)_{m+1}} = \sum_{k=0}^{\infty} B_k(q)x^k q^k$$

and then noting further that both fractions just displayed differ in factors $(1-x)$ and $(1-xq^{m+1})$, we have accordingly the following:

$$(1-x)\sum_{k=0}^{\infty} B_k(q)x^k = (1-xq^{m+1})\sum_{k=0}^{\infty} B_k(q)x^k q^k.$$

Extracting the coefficient of x^ℓ from both sides we get

$$B_\ell(q) - B_{\ell-1}(q) = q^\ell\, B_\ell(q) - q^{m+\ell}\, B_{\ell-1}(q)$$

which is equivalent to the following recurrence relation

$$B_\ell(q) = B_{\ell-1}(q)\frac{1-q^{m+\ell}}{1-q^\ell} \quad \text{for} \quad \ell = 1, 2, \cdots.$$

Iterating this relation ℓ-times, we find that

$$B_\ell(q) \;=\; B_0(q)\frac{(q^{m+1};q)_\ell}{(q;q)_\ell} \;=\; \begin{bmatrix} m+\ell \\ \ell \end{bmatrix}_q$$

where $B_0(q)=1$ follows from setting $x=0$ in the generating function

$$\frac{1}{(xq;q)_{m+1}} \;=\; \sum_{k=0}^{\infty} B_k(q)x^k.$$

Therefore we conclude the proof. □

B4.3. Theorem. Classifying the partitions according to the number of parts, we derive immediately two q-binomial identities (*finite and infinite*):

$$\sum_{\ell=0}^{n} q^\ell \begin{bmatrix} \ell+m \\ m \end{bmatrix} \;=\; \begin{bmatrix} m+n+1 \\ n \end{bmatrix} \tag{B4.3a}$$

$$\sum_{\ell=0}^{\infty} x^\ell \begin{bmatrix} \ell+m \\ m \end{bmatrix} \;=\; \prod_{k=0}^{m} \frac{1}{1-xq^k}. \tag{B4.3b}$$

Proof. In view of (B4.2a) and (B4.2b), the univariate generating functions for the partitions into parts $\le m+1$ with the lengths equal to ℓ and $\le n$ are respectively given by the q-binomial coefficients $q^\ell \begin{bmatrix} \ell+m \\ m \end{bmatrix}$ and $\begin{bmatrix} m+n+1 \\ n \end{bmatrix}$. Classifying the partitions enumerated by the latter according to the number of parts ℓ with $0 \le \ell \le n$, we establish the first identity.

By means of (B4.1b), we have

$$\prod_{k=0}^{m} \frac{1}{1-xq^k} \;=\; \sum_{\ell=0}^{\infty} x^\ell \sum_{n=0}^{\infty} p^\ell(n|m)\, q^n \;=\; \sum_{\ell=0}^{\infty} x^\ell \begin{bmatrix} m+\ell \\ \ell \end{bmatrix}$$

which is the second q-binomial identity. □

B5. Partitions into distinct parts and finite q-differences

Similarly, let $Q_\ell(n|m)$ be the number of partitions of n into exactly ℓ distinct parts with each part $\leq m$. Then we have generating functions

$$\begin{aligned} \begin{bmatrix} m \\ \ell \end{bmatrix} q^{\binom{1+\ell}{2}} &= \sum_{n\geq 0} Q_\ell(n|m)\, q^n && \text{(B5.1)} \\ \prod_{k=1}^{m}(1+xq^k) &= \sum_{\ell,n\geq 0} Q_\ell(n|m)\, x^\ell q^n && \text{(B5.2)} \end{aligned}$$

whose combination leads us to Euler's finite q-differences

$$(x;q)_n = \prod_{\ell=0}^{n-1}(1-xq^\ell) = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} x^k. \qquad \text{(B5.3)}$$

Following the second proof of Theorem B3.1, we can check without difficulty that the shifted Ferrers diagrams of the partitions into ℓ-parts $\leq m$ are unions of the same triangle of length ℓ enumerated by $q^{\binom{\ell+1}{2}}$ and the ordinary partitions into parts $\leq m-\ell$ with length $\leq \ell$ whose generating function reads as the q-binomial coefficient $\begin{bmatrix} m \\ \ell \end{bmatrix}$. The product of them gives the generating function for $\{Q_\ell(n|m)\}_n$.

The second formula is a particular case of (B1.2b). Its combination with the univariate generating function just proved leads us to the following:

$$(-qx;q)_m = \sum_{\ell=0}^{m}\sum_{n\geq 0} Q_\ell(n|m)\, q^n\, x^\ell = \sum_{\ell=0}^{m} x^\ell \begin{bmatrix} m \\ \ell \end{bmatrix} q^{\binom{1+\ell}{2}}.$$

Replacing x by $-x/q$ in the above, we get Euler's q-difference formula:

$$(x;q)_m = \sum_{\ell=0}^{m}(-1)^\ell \begin{bmatrix} m \\ \ell \end{bmatrix} q^{\binom{\ell}{2}} x^\ell.$$

Remark The last formula is called the *Euler q-difference formula* because if we put $x := q^{-n}$, the finite sum results in

$$\sum_{\ell=0}^{m}(-1)^\ell \begin{bmatrix} m \\ \ell \end{bmatrix} q^{\binom{\ell}{2}-\ell n} = (q^{-n};q)_m = \begin{cases} 0, & 0\leq n<m \\ (-1)^n q^{-\binom{n+1}{2}}(q;q)_n, & n=m \end{cases}$$

just like the ordinary finite differences of polynomials.

Keep in mind of the q-binomial limit

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q^{1+n-k};q)_k}{(q;q)_k} \longrightarrow \frac{1}{(q;q)_k} \quad \text{as} \quad n \to \infty.$$

Letting $n \to \infty$ in Euler's q-finite differences, we recover again the Euler classical partition identity

$$(x;\ q)_\infty \quad = \quad \sum_{k=0}^{\infty} \frac{(-1)^k\ x^k}{(q;\ q)_k} q^{\binom{k}{2}}$$

where Tannery's theorem has been applied for the limiting process.