Chapter IV

Some facts from representation theory

This deep and important theory cannot be developed in these notes. We just give some basic results and refer, for a systematic exposition, to books like [6], [7], [13].

1 Irreducible and indecomposable modules

We consider the space $\mathbb{F}^n$ of column vectors as a *left module* over the ring $\mathrm{Mat}_n(\mathbb{F})$ with respect to the usual product of matrices. Let A be a subset of $\mathrm{Mat}_n(\mathbb{F})$.

(1.1) Definition *A subspace W of $\mathbb{F}^n$ is A-*invariant *if $AW \leq W$, i.e., if:*

$$aw \in W, \quad \forall\ a \in A,\ \forall\ w \in W.$$

Clearly W is A-invariant if and only if it is $\mathbb{F}A$-invariant, where $\mathbb{F}A$ denotes the linear subspace of $\mathrm{Mat}_n(\mathbb{F})$ generated by A. Moreover, when A is a subring of $\mathrm{Mat}_n(\mathbb{F})$, then W is A-invariant if and only if it is a module over A.

(1.2) Lemma *Let $\mathbb{F} \leq \mathbb{K}$, a field extension. If $w_1, \dots, w_m$ are linearly independent vectors of $\mathbb{F}^n$, then they are linearly independent in $\mathbb{K}^n$.*

Proof There exists $P \in \mathrm{GL}_n(\mathbb{F})$ such that $Pw_j = e_j$, $1 \leq j \leq m$. So assume $\sum_{i=1}^m k_i w_i = 0$, with $k_i \in \mathbb{K}$. Multiplying by P we get $\sum_{i=1}^m k_i e_i = 0$, whence $k_1 = \cdots = k_m = 0$. ■

A subspace W of $\mathbb{F}^n$ can be extended to the subspace $W \otimes_{\mathbb{F}} \mathbb{K}$ of $\mathbb{K}^n$ defined as the subspace of $\mathbb{K}^n$ generated by any basis $\mathcal{B} = \{w_1, \dots, w_m\}$ of W, namely:

$$W \otimes_{\mathbb{F}} \mathbb{K} \ = \ \left\{ \sum_{j=1}^{m} k_j w_j \mid k_j \in \mathbb{K} \right\} \quad \text{(tensor product)}.$$

$\mathcal{B}$ is a basis of $W \otimes_{\mathbb{F}} \mathbb{K}$, by Lemma 1.2. Thus, if W is an A-module, also $W \otimes_{\mathbb{F}} \mathbb{K}$ becomes an A-module via the action:

$$a \sum_{j=1}^{m} k_j w_j = \sum_{j=1}^{m} k_j \, a w_j \quad \forall \, a \in A.$$

(1.3) Definition *Let A be a subring (or a subgroup) of $\mathrm{Mat}_n(\mathbb{F})$ and W be an A-invariant subspace of $\mathbb{F}^n$. The* A-module W *is said to be:*

(1) indecomposable*, if there is no decomposition $W = W_1 \oplus W_2$ into proper A-invariant subspaces W_1, W_2;*

(2) irreducible*, if the only A-invariant subspaces of W are $\{0_{\mathbb{F}^n}\}$ and W;*

(3) absolutely irreducible*, if $W \otimes_{\mathbb{F}} \mathbb{K}$ is irreducible for any field extension $\mathbb{K}$ of $\mathbb{F}$.*

Accordingly, a subring (or a subgroup) A of $\mathrm{Mat}_n(\mathbb{F})$ is said to be:

- *indecomposable*, if $\mathbb{F}^n$ is indecomposable as an A-module;
- *irreducible*, if $\mathbb{F}^n$ is irreducible as an A-module;
- *absolutely irreducible*, if $\mathbb{F}^n$ is absolutely irreducible as an A-module.

Clearly an irreducible group is indecomposable. The converse is not true in general, as shown in Example 1.5 below. It is true when G is finite and $\mathbb{F}$ has characteristic p where $p = 0$ or p does not divide $|G|$ (see Theorem 1.11).

(1.4) Example *The subgroup G of $\mathrm{GL}_2(\mathbb{R})$, generated by the matrix $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, is irreducible but not absolutely irreducible.*

Indeed g has no eigenvalue in $\mathbb{R}$. Thus $\mathbb{R}^2$ has no 1-dimensional G-submodule. But g has eigenvalues in $\mathbb{C}$. Thus, for example, $\left\langle \begin{pmatrix} 1 \\ i \end{pmatrix} \right\rangle$ is G-invariant in $\mathbb{C}^2$.

(1.5) Example *The subgroup $G = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \mid t \in \mathbb{F} \right\}$ of $\mathrm{GL}_2(\mathbb{F})$ is reducible, but indecomposable, for any field $\mathbb{F}$.*

G is reducible because $\langle e_1 \rangle$ is G-invariant. Suppose $\mathbb{R}^2 = \langle v_1 \rangle \oplus \langle v_2 \rangle$ where each $\langle v_i \rangle$ is G-invariant. Then v_1, v_2 should be a basis of eigenvectors of G. Since every $g \in G$ has only the eigenvalue 1, one gets $Gv_1 = v_1$, $Gv_2 = v_2$, whence the contradiction $G = I$.

(1.6) Example *The subgroup G of $\mathrm{GL}_2(\mathbb{R})$, generated by the matrices*

$$g_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is absolutely irreducible.

Indeed the only 1-dimensional g_1-invariant subspaces are its eigenspaces, namely $\langle e_1 \rangle$ and $\langle e_2 \rangle$, but they are not g_2-invariant.

(1.7) Lemma $\mathrm{Mat}_n(\mathbb{F})$ *is absolutely irreducible for any field* $\mathbb{F}$. *Moreover its center* Z *coincides with the field* $\mathbb{F}I_n$ *of scalar matrices.*

Proof Set $A = \mathrm{Mat}_n(\mathbb{F})$ and let $\{0\} \neq W$ be an A-invariant subspace of $\mathbb{K}^n$, where $\mathbb{K}$ is a field extension of $\mathbb{F}$. Take $0 \neq w \in W$. Then there exists a non-zero component α_i of w. From $e_{i,i} \in A$, it follows that $e_{i,i}w = \alpha_i e_i \in W$. Hence $e_i \in W$. Considering in A the permutation matrices $\pi_{(i,j)}$ we get that $\pi_{(i,j)}e_i = e_j \in W$ for $1 \leq j \leq n$. So W contains the canonical basis, whence $W = \mathbb{K}^n$.
By direct calculation one sees that a matrix commutes with all matrices $e_{ij} \in \mathrm{Mat}_n(\mathbb{F})$ if and only if it is scalar. ∎

(1.8) Theorem *Let G be one of the following classical groups:*

$$\mathrm{SL}_n(\mathbb{F}),\ \mathrm{SU}_n(\mathbb{F}),\ \mathrm{Sp}_n(\mathbb{F}),\ n = 2m,\ \Omega_n(\mathbb{F}, Q).$$

Then $\mathbb{F}G = \mathrm{Mat}_n(\mathbb{F})$, *except when* $G = \Omega_2(\mathbb{F}, Q)$. *In particular G is absolutely irreducible and its centralizer in* $\mathrm{Mat}_n(\mathbb{F})$ *consists of the scalar matrices.*

Proof One can see that in each case, provided $G \neq \Omega_2(\mathbb{F}, Q)$, the group G contains n^2 linearly independent matrices (for instance the generators of these groups given in the previous Chapter). Hence the subspace $\mathbb{F}G$ generated by G coincides with $\mathrm{Mat}_n(\mathbb{F})$, which is absolutely irreducible. ∎

(1.9) Lemma *(Schur's Lemma) Let* $A \leq \mathrm{Mat}_n(\mathbb{F})$ *be irreducible. Then*

$$C = C_{\mathrm{Mat}_n(\mathbb{F})}(A) := \{c \in \mathrm{Mat}_n(\mathbb{F}) \mid ca = ac,\ \forall\ a \in A\}$$

is a division algebra over $\mathbb{F}I_n$. *In particular, if commutative, C is a field.*

Proof It is easy to see that C is a subalgebra of $\mathrm{Mat}_n(\mathbb{F})$, which contains $Z = \mathbb{F}I_n$. Consider a non-zero matrix $c \in C$. The subspace $c\mathbb{F}^n$ is A-invariant, as:

$$a\,(c\mathbb{F}^n) = (ac)\mathbb{F}^n = (ca)\mathbb{F}^n = c(a\mathbb{F}^n) \leq c\mathbb{F}^n, \quad \forall\, a \in A.$$

$0_{\mathrm{Mat}_n(\mathbb{F})} \neq c \Longrightarrow c\mathbb{F}^n \neq \{0_{\mathbb{F}^n}\}$. It follows $c\mathbb{F}^n = \mathbb{F}^n$, by the irreducibility of A. Since the multiplication by c is surjective, it is injective. Thus c has inverse c^{-1}. Clearly $c^{-1} \in C$.
■

Up to here we considered the natural $\mathrm{Mat}_n(\mathbb{F})$-module $\mathbb{F}^n$. But we may also consider the left regular module ${}_{\mathrm{Mat}_n(\mathbb{F})}\,\mathrm{Mat}_n(\mathbb{F})$ and compare these two modules.

(1.10) Lemma *Let A be a subring of $\mathrm{Mat}_n(\mathbb{F})$, acting irreducibly on $\mathbb{F}^n$, and let $\{0\} \neq W \leq \mathrm{Mat}_n(\mathbb{F})$ be a minimal A-invariant subspace, in the regular action of $\mathrm{Mat}_n(\mathbb{F})$ on itself. Then there exists a vector e_i of the canonical basis such that $\mathbb{F}^n = We_i$. Moreover W is isomorphic to $\mathbb{F}^n$, as an A-module. In particular $dim_{\mathbb{F}}W = n$.*

Proof Choose $0 \neq w \in W$. Then w has a non-zero column we_i. The subspace We_i of $\mathbb{F}^n$ is such that $A(We_i) \leq We_i$. From $we_i \in We_i$ it follows $We_i \neq \{0\}$. Hence $We_i = \mathbb{F}^n$, by the irreducibility of A. Finally, the map $f : W \to \mathbb{F}^n$ defined by $w \mapsto we_i$ is an $\mathbb{F}$-isomorphism such that $f(aw) = af(w)$ for all $a \in A$. ■

(1.11) Theorem *(Maschke) Let $G \leq \mathrm{GL}_n(\mathbb{F})$ be a finite group, where $\mathbb{F}$ has characteristic 0 or a prime p which does not divide $|G|$. Then every G-invariant subspace W of $\mathbb{F}^n$ has a G-invariant complement.*

Proof Let $\mathbb{F}^n = W \oplus U$, where U is an $\mathbb{F}$-complement of W, and call $\pi : \mathbb{F}^n \to U$ the projection. Consider $\psi : \mathbb{F}^n \to \mathbb{F}^n$ defined by:

$$\psi(v) := \frac{1}{|G|}\sum_{x\in G} x^{-1}\pi(xv), \quad \forall\, v \in \mathbb{F}^n.$$

The image of ψ, namely $\psi(\mathbb{F}^n)$, is G-invariant, since for all $g \in G$ and $v \in \mathbb{F}^n$:

$$\psi(gv) := \frac{1}{|G|}\sum_{x\in G} x^{-1}\pi(xgv) = \frac{1}{|G|}g\sum_{x\in G}\left(g^{-1}x^{-1}\right)\pi(xgv) = g\psi(v).$$

Moreover, from $u - \pi(u) \in W$ for all $u \in \mathbb{F}^n$, it follows that:

$$v - \psi(v) = \frac{1}{|G|}\sum_{x\in G} x^{-1}xv - \frac{1}{|G|}\sum_{x\in G} x^{-1}\pi(xv) = \frac{1}{|G|}\sum_{x\in G} x^{-1}\left(xv - \pi(xv)\right) \in W.$$

Thus $v = (v - \psi(v)) + \psi(v)$ for all $v \in \mathbb{F}^n$, gives $\mathbb{F}^n = W + \psi(\mathbb{F}^n)$.
For all $w \in W$ and all $x \in G$ we have $\pi(xw) = 0$. So $\psi(w) = 0$, whence $\psi(v - \psi(v)) = 0$, for all v. This gives $\psi^2 = \psi$ and $W \cap \psi(\mathbb{F}^n) = \{0\}$. Indeed, from $w = \psi(v) \in W \cap \psi(V)$, we have $\psi(v) = \psi^2(v) = \psi(w) = 0$.
We conclude that $\psi(\mathbb{F}^n)$ is a G-invariant complement of W in $\mathbb{F}^n$. ■

2 Representations of groups

(2.1) Definition *Let H be an abstract group.*

(1) *A* representation *of H of degree n over $\mathbb{F}$ is a homomorphism $f : H \to \mathrm{GL}_n(\mathbb{F})$. The representation f is said to be* irreducible *if $\mathbb{F}^n$ is an irreducible $f(H)$-module.*

(2) *The* character *χ of f is the map $\chi : H \to \mathbb{F}$ such that*

$$\chi(h) := \mathrm{tr}(f(h)), \quad \forall\ h \in H.$$

(3) *Two representations $f_i : H \to \mathrm{GL}_n(\mathbb{F})$, $i = 1, 2$ are said to be* equivalent *if there exists $P \in \mathrm{GL}_n(\mathbb{F})$ such that*

$$Pf_1(h) = f_2(h)P, \quad \forall\ h \in H. \tag{2.2}$$

Since conjugate matrices have the same trace, equivalent representations have the same characters.

(2.3) Definition *Let H be an abstract group. The* group algebra *$\mathbb{F}H$ is defined as follows. The elements of H are a basis of $\mathbb{F}H$ as a vector space over $\mathbb{F}$. The product in $\mathbb{F}H$ is the extension, by linearity, of the product in H.*

In particular, by definition, the elements of $\mathbb{F}H$ are the formal linear combinations

$$\sum_{h \in H} \alpha_h h, \quad \alpha_h \in \mathbb{F}$$

with a finite number of non-zero coefficients. By definition, $\dim_{\mathbb{F}} \mathbb{F}H = |H|$.
The extension to $\mathbb{F}H$, by linearity, of any representation $f : H \to \mathrm{GL}_n(\mathbb{F})$ gives rise to an algebra homomorphism $f : \mathbb{F}H \to \mathrm{Mat}_n(\mathbb{F})$. Vice versa, if $f : \mathbb{F}H \to \mathrm{Mat}_n(\mathbb{F})$ is an algebra homomorphism, its restriction $f_H : H \to \mathrm{GL}_n(\mathbb{F})$ is a representation of H.

(2.4) Remark *If $f : H \to \mathrm{GL}_n(\mathbb{F})$ is a representation, then $\mathbb{F}^n$ is an H-module with respect to $hv := f(h)v$, for all $v \in \mathbb{F}^n$. Vice versa, if $\mathbb{F}^n$ is an $\mathbb{F}H$-module, the map $f : H \to \mathrm{GL}_n(\mathbb{F})$ such that $f(h) = \left(\, he_1 \, \middle| \, \dots \, \middle| \, he_n \, \right)$ is a representation.*

(2.5) Lemma *Two representations $f_1 : H \to \mathrm{GL}_n(\mathbb{F})$ and $f_2 : H \to \mathrm{GL}_n(\mathbb{F})$ are equivalent if and only if the corresponding $\mathbb{F}H$-modules $V_i = \mathbb{F}^n$ are isomorphic, $i = 1, 2$.*

Proof Suppose first that f_1 and f_2 equivalent and let $P \in \mathrm{GL}_n(\mathbb{F})$ be as in point (3) of Definition 2.1. Then the the multiplication by P, namely the map $\mu_P : V_1 \to V_2$, is an $\mathbb{F}H$-isomorphism. Indeed μ_P is $\mathbb{F}$-linear and, for all $v \in \mathbb{F}^n$ and all $h \in H$:

$$\mu_P(f_1(h)v) = Pf_1(h)v = f_2(h)Pv = f_2(h)\mu_P(v).$$

Vice versa, if there exists an $\mathbb{F}H$-isomorphism $\sigma : V_1 \to V_2$ and $P \in \mathrm{GL}_n(\mathbb{F})$ is the matrix of σ with respect to the canonical basis, then $Pf_1(h) = f_2(h)P$ for all $h \in H$. Thus f_1 and f_2 are equivalent. ∎

Given two representations $f_i : H \to \mathrm{GL}_{n_i}(\mathbb{F})$, $i = 1, 2$, we may consider their *sum*, namely the representation $f : H \to \mathrm{GL}_{n_1+n_2}(\mathbb{F})$, defined by:

$$f(h) := \begin{pmatrix} f_1(h) & 0 \\ 0 & f_2(h) \end{pmatrix}, \ \forall \ h \in H.$$

Set $M_i = \mathrm{Mat}_{n_i}(\mathbb{F})$. Clearly the subspace

$$M_1 \oplus M_2 := \left\{ \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \mid A_1 \in \mathrm{Mat}_{n_1}(\mathbb{F}), \ A_2 \in \mathrm{Mat}_{n_2}(\mathbb{F}) \right\}$$

is an $f(H)$-module. Moreover the projections

$$\pi_i : M_1 \oplus M_2 \to \mathrm{Mat}_{n_i}(\mathbb{F})$$

are $f(H)$-homomorphisms. In particular $f(H)\,\mathrm{Ker}\ \pi_i = \mathrm{Ker}\ \pi_i$, for $i = 1, 2$.

(2.6) Lemma *In the above notation, suppose that the representations*

$$f_i : H \to \mathrm{GL}_{n_i}(\mathbb{F}), \quad i = 1, 2$$

are irreducible and inequivalent. Let $0 \neq M$ be a minimal subspace of $M_1 \oplus M_2$ such that $f(H)M = M$. Then either $\pi_1(M) = 0$ or $\pi_2(M) = 0$.

Proof Suppose, by contradiction, $M \not\leq \mathrm{Ker}\ \pi_i$, for $i = 1, 2$. It follows that the $f(H)$-module $\mathrm{Ker}\ \pi_i \cap M$ is zero, $i = 1, 2$, by the minimality of M. Thus the restrictions

$$\pi_{i|M} : M \to \pi_i(M), \quad i = 1, 2$$

are $\mathbb{F}$-isomorphisms. In particular $n_1 = n_2 = \dim_{\mathbb{F}} M$. Again by the minimality of M, each $\pi_i(M)$ is a minimal $f_i(H)$-submodule of $\mathrm{Mat}_{n_i}(\mathbb{F})$. It follows from Lemma 1.10 of this Chapter, with $A = f_i(H)$, $W = \pi_i(M)$, that there exist $f_i(H)$ isomorphisms $\tau_i : \pi_i(M) \cong \mathbb{F}^{n_i}$, $i = 1, 2$. Thus $\tau_2\tau_1^{-1} : \mathbb{F}^{n_1} \to \mathbb{F}^{n_2}$ is a isomorphism of the $f_1(H)$-module $\mathbb{F}^{n_1}$ onto the $f_2(H)$-module $\mathbb{F}^{n_2}$, a contradiction. ■

Note that, if G is a group and V is a G-module, then $GW \leq W$ if and only if $GW = W$, for any subspace W of V. Indeed $W = 1_G W \leq GW$.

(2.7) Theorem *Let $f_i : H \to \mathrm{GL}_{n_i}(\mathbb{F})$ be irreducible pairwise inequivalent representations of a group H, with $\mathbb{F}$ algebraically closed. Suppose that $m_i \in \mathrm{Mat}_{n_i}(\mathbb{F})$, $1 \leq i \leq s$, are such that*

$$\sum_{i=1}^{s} \mathrm{tr}\,(m_i f_i(h)) = 0_{\mathbb{F}}, \ \forall\ h \in H.$$

Then each $m_i = 0_{\mathrm{Mat}_{n_i}(\mathbb{F})}$, for $i = 1, \ldots, s$.

Proof Induction on s. Suppose $s = 1$ and put $n = n_1$, $f = f_1$. The set

$$M = \{m \in \mathrm{Mat}_n(\mathbb{F}) \mid \mathrm{tr}(mf(h)) = 0, \ \forall\ h \in H\}$$

is a subspace. Moreover $f(H)M = M$ since for all $h_1, h \in H$, $m \in M$:

$$\mathrm{tr}(f(h_1)m\, f(h)) = \mathrm{tr}(f(h)\, f(h_1)m) = \mathrm{tr}(f(hh_1)m) = 0.$$

We want to show that $M = \{0_{\mathrm{Mat}_n(\mathbb{F})}\}$. If this is false, we may choose a non-zero subspace U of M of minimal dimension with respect to the property $f(H)U = U$. By Lemma 1.10 we have $\dim U = n$ and $Uv = \mathbb{F}^n$ for some v. If $\{u_1, \ldots u_n\}$ is a basis of U, then $\{u_1 v, \ldots u_n v\}$ is a basis of $\mathbb{F}^n$. Up to conjugation we may suppose that

$$\{u_1 v, \ldots u_n v\} = \{e_1, \ldots, e_n\} \quad \text{(canonical basis)}.$$

For all $w \in \mathbb{F}^n$ we consider the matrix A_w with columns $A_w e_i = u_i w$, i.e.,:

$$A_w = \left(\ u_1 w \ \middle|\ \ldots\ \middle|\ u_n w\ \right).$$

Let λ_w be an eigenvalue of A_w, with eigenvector $\sum_{i=1}^{n} \rho_i e_i \neq 0_{\mathbb{F}^n}$. Then:

$$0_{\mathbb{F}^n} = (A_w - \lambda_w I) \sum_{i=1}^{n} \rho_i e_i =$$

$$\sum_{i=1}^{n} \rho_i \left(A_w - \lambda_w I\right) e_i = \sum_{i=1}^{n} \rho_i \left(u_i w - \lambda_w u_i v\right) = \sum_{i=1}^{n} \rho_i u_i \left(w - \lambda_w v\right).$$

It follows that the vectors

$$u_1(w-\lambda_w v),\ \dots,\ u_n(w-\lambda_w v)$$

are linearly dependent. Hence the space $U(w-\lambda_w v)$, generated by them, has dimension less then n. Since it is $f(H)$-invariant, the irreducibility of $\mathbb{F}^n$ gives:

$$U(w-\lambda_w v)=\{0_{\mathbb{F}^n}\}, \forall\ w\in\mathbb{F}^n.$$

In particular $u_i(e_j-\lambda_{e_j}v)=0_{\mathbb{F}^n}$ for all i,j. Thus, setting $\lambda_{e_j}=\lambda_j$:

$$u_i e_j=\lambda_j u_i v=\lambda_j e_i,\quad 1\leq i,j\leq n. \tag{2.8}$$

This tells us:

$$u_i=\left(\ \lambda_1 e_i \,\middle|\, \dots \,\middle|\, \lambda_n e_i\ \right),\quad 1\leq i\leq n.$$

$$0=\mathrm{tr}(u_i\,\mathrm{id}_G)=\mathrm{tr}\,(u_i)=\lambda_i,\quad 1\leq i\leq n.$$

And now (2.8) gives that u_i has all columns equal to zero, hence $u_i=0_{\mathbb{F}^n}$ for all i-s, against the assumption that $u_1,\dots,u_n$ are linearly independent. We conclude $M=\{0_{\mathrm{Mat}_n(\mathbb{F})}\}$ and the first step of induction is proved.

Now suppose $s>1$. Set $n=\sum_{i=1}^s n_i$ and consider the sum $f:H\to\mathrm{GL}_n(\mathbb{F})$ of the representations f_i, defined by:

$$f(h):=\begin{pmatrix} f_1(h) & & \\ & \dots & \\ & & f_s(h)\end{pmatrix},\ \forall\ h\in H.$$

Let M be the following subset of $\mathrm{Mat}_{n_1}(\mathbb{F})\oplus\cdots\oplus\mathrm{Mat}_{n_s}(\mathbb{F})$:

$$M:=\left\{m=\begin{pmatrix} c_1 & & \\ & \dots & \\ & & c_s\end{pmatrix}\mid \mathrm{tr}\,(mf(h))=\sum_{i=1}^s \mathrm{tr}\,(c_i f_i(h))=0,\quad \forall\ h\in H\right\}.$$

Clearly M is an $f(H)$-invariant subspace and we want to show that $M=\{0_{\mathrm{Mat}_n(\mathbb{F})}\}$. If this is false, we may choose a non-zero subspace U of M of minimal dimension with respect to the property $f(H)U=U$. By the assumption that the representations $f_i: H\to\mathrm{GL}_{n_i}(\mathbb{F})$ are irreducible and pairwise inequivalent, Lemma 2.6 tells us that $\pi_i(U)=0_{\mathrm{Mat}_{n_i}(\mathbb{F})}$ for at least one i. We may suppose $i=1$. This means that, for all

$$\begin{pmatrix} u_1 & & \\ & \dots & \\ & & u_s\end{pmatrix}\in U$$

we have $u_1 = 0_{\mathrm{Mat}_{n_1}(\mathbb{F})}$. It follows

$$0_{\mathbb{F}} = \sum_{i=1}^{s} \mathrm{tr}\,(u_i f_i(h)) = \sum_{i=2}^{s} \mathrm{tr}\,(u_i f_i(h))\,, \quad \forall\ h \in H.$$

By induction $u_2 = \cdots = u_s = 0$, whence $U = \left\{0_{\mathrm{Mat}_n(\mathbb{F})}\right\}$, a contradiction. ■

(2.9) Corollary *Let $f_i : G \to \mathrm{GL}_{n_i}(\mathbb{F})$, $i \leq s$, be pairwise inequivalent, absolutely irreducible representations of a group G with k conjugacy classes. Then $s \leq k$.*

Proof We may suppose $\mathbb{F}$ algebraically closed. Choose representatives $g_1, \ldots, g_k$ of the conjugacy classes of G and consider the s vectors of $\mathbb{F}^k$:

$$v_1 = \begin{pmatrix} \mathrm{tr}\,(f_1(g_1)) \\ \cdots \\ \mathrm{tr}\,(f_1(g_k)) \end{pmatrix}, \ \ldots\ , \quad v_s = \begin{pmatrix} \mathrm{tr}\,(f_s(g_1)) \\ \cdots \\ \mathrm{tr}\,(f_s(g_k)) \end{pmatrix}.$$

Suppose $\sum_{i=1}^{s} \alpha_i v_i = 0_{\mathbb{F}^n}$ for some $\alpha_i \in \mathbb{F}$. It follows

$$\sum_{i=1}^{s} \alpha_i \,\mathrm{tr}\,(f_i(g_j)) = \sum_{i=1}^{s} \mathrm{tr}\,(\alpha_i f_i(g_j)) = 0_{\mathbb{F}}, \quad 1 \leq j \leq k.$$

Every $g \in G$ is conjugate to a g_j and $\mathrm{tr}(g) = \mathrm{tr}(g_j)$. Thus:

$$\sum_{i=1}^{s} \mathrm{tr}\,(\alpha_i\, f_i(g)) = 0_{\mathbb{F}}, \quad \forall\ g \in G.$$

By the previous Theorem $\alpha_i = 0$ for all $i \leq s$. This means that the vectors $v_1, \ldots, v_s$ are linearly independent in $\mathbb{F}^k$. We conclude $s \leq k$. ■

(2.10) Theorem *Let G be a subgroup of $\mathrm{GL}_n(\mathbb{F})$ and denote by $\mathbb{F}G$ the linear subspace of $\mathrm{Mat}_n(\mathbb{F})$ generated by G. The following conditions are equivalent:*

(1) *G is absolutely irreducible;*

(2) *$\mathbb{F}G = \mathrm{Mat}_n(\mathbb{F})$ (equivalently, $\dim_{\mathbb{F}} \mathbb{F}G = n^2$);*

(3) *G is irreducible and $C_{\mathrm{Mat}_n(\mathbb{F})}(G) = \mathbb{F}I_n$.*

Proof
(1) $\Longrightarrow$ (2) Substituting $\mathbb{F}$ with its algebraic closure, if necessary, we may suppose $\mathbb{F}$ algebraically closed. Let $g_1, \ldots, g_m$ be a basis of $\mathbb{F}G$ and consider the orthogonal space

$\mathbb{F}G^{\perp}$ with respect to the bilinear form $(g_1, g_2) = \operatorname{tr}(g_1 g_2)$ (see (3.7) in the Exercises of this Chapter). Since this form is non-degenerate, $FG^{\perp}$ has dimension $n^2 - m$. Thus, if $m < n^2$, there exists a non-zero matrix m such that $\operatorname{tr} mg = 0$ for all $g \in G$, in contrast with Theorem 2.7.

(2) $\implies$ (3) Any G-invariant subspace would be $\operatorname{Mat}_n(\mathbb{F})$- invariant, against the irreducibility of $\operatorname{Mat}_n(\mathbb{F})$. The last claim follows from the fact that the center of $\operatorname{Mat}_n(\mathbb{F})$ consists of scalar matrices.

(3) $\implies$ (1) [6, Theorem 29.13]. ■

Point (1) of the following Lemma explains why, in the study of classical groups, one is interested in the groups of isometries of non-degenerate forms. By point (2) an absolutely irreducible group can fix at most one form, necessarily non-degenerate, up to scalars.

(2.11) Lemma *Let $J \in \operatorname{Mat}_n(\mathbb{F})$ be such that $J^T = J^{\sigma}$ (σ a field automorphism), or $J^T = -J$ and let $G \leq \operatorname{GL}_n(\mathbb{F})$ be a group of isometries of J, namely*

$$g^T J g^{\sigma} = J, \quad \forall\, g \in G.$$

(1) *if $\det J = 0$, then G is reducible;*

(2) *if G is absolutely irreducible, and J' is such that $g^T J' g^{\sigma} = J'$ for all $g \in G$, then $J' = \lambda J$ for some $0 \neq \lambda \in \mathbb{F}$.*

Proof

(1) The 0-eigenspace W of J is non-zero. W is G^{σ}-invariant, since:

$$J g^{\sigma} w = \left(g^{-1}\right)^T J w = 0 \implies g^{\sigma} w \in W, \quad \forall\, g \in G,\ w \in W.$$

From the fact that W is G^{σ}-invariant, it follows that $W^{\sigma^{-1}}$ is G-invariant.

(2) $J g^{\sigma} J^{-1} = \left(g^{-1}\right)^T = J' g^{\sigma} J'^{-1}$ for all g gives $J'^{-1} J \in C_{\operatorname{Mat}_n(\mathbb{F})}\left(G^{\sigma}\right) = \mathbb{F} I_n$. ■

3 Exercises

(3.1) Exercise *Let $\mathbb{K}$ be a field extension of $\mathbb{F}$. Show that any subset $\{w_1, \ldots, w_m\}$ of $\mathbb{F}^n$ which is linearly independent over $\mathbb{F}$ is also linearly independent over $\mathbb{K}$.*

(3.2) Exercise *Let $G = \operatorname{GL}_4(\mathbb{F})$ and $W = \langle e_1, e_2 \rangle \leq \mathbb{F}^4$. Determine:*

i) *the stabilizer* G_W *of* W *in* G*;*

ii) *the kernel of the restriction map defined by* $h \mapsto h_W$ *for all* $h \in G_W$*;*

iii) *the group* $(G_W)^W$ *induced by* G_W *on* W*.*

(3.3) Exercise *Let* $W \leq \mathbb{F}^n$, $G \leq \mathrm{GL}_n(\mathbb{F})$*. Suppose that dim* $W = m > \frac{n}{2}$ *and that* G_W *acts irreducibly on* W*. Show that* W *is the only* G_W*-invariant subspace of dimension* m*. Deduce that* $C_G(G_W) \leq G_W$*.*

(3.4) Exercise *Show that* $C_{\mathrm{Mat}_2(\mathbb{F})}(\mathrm{SL}_2(\mathbb{F})) = \mathbb{F}I_2$*.*

(3.5) Exercise *Show, by induction on* n*, that* $C_{\mathrm{Mat}_n(\mathbb{F})}(\mathrm{SL}_n(\mathbb{F})) = \mathbb{F}I_n$*.*

Hint. For $n \geq 3$, start with any $(n-1)$-dimensional subspace W. Consider its stabilizer H in $\mathrm{SL}_n(\mathbb{F})$ and note that $H^W \cong \mathrm{GL}_{n-1}(\mathbb{F})$ acts irreducibly on W. Deduce that, for every $c \in C_{\mathrm{Mat}_n(\mathbb{F})}(H)$ and for every $w \in W$

$$cw = \lambda_c w, \quad \lambda_c \in \mathbb{F}.$$

Take another $(n-1)$-dimensional subspace $W' \neq W$. Again, for all $w' \in W'$:

$$cw' = \mu_c w', \quad \mu_c \in \mathbb{F}.$$

The conclusion follows easily from $\mathbb{F}^n = W + W'$.

(3.6) Exercise *Show that the map* $(\ ,\ ) : \mathrm{Mat}_n(\mathbb{F}) \times \mathrm{Mat}_n(\mathbb{F}) \to \mathbb{F}$ *defined by:*

$$(A, B) := \mathrm{tr}(AB) \tag{3.7}$$

is bilinear and that it is non-degenerate.

(3.8) Exercise *Let* $G = \mathrm{Sym}(3)$ *and set:*

$$\begin{aligned}
&\sigma = \mathrm{id} + (123) + (132) + (12) + (13) + (23),\\
&\tau = \mathrm{id} + (123) + (132) - (12) - (13) - (23),\\
&\rho_1 = \mathrm{id} + (12) - (13) - (123), \quad \rho_2 = \mathrm{id} + (23) - (13) - (132),\\
&\zeta_1 = \mathrm{id} + (12) - (23) - (132), \quad \zeta_2 = \mathrm{id} + (12) - (13) + (123) - (132).
\end{aligned}$$

i) *Show that, with respect to the product* $j(fg) := (jf)g$ *for* $j \in \{1,2,3\}$, $f, g \in G$ *:*

$$\mathbb{C}G = \mathbb{C}\sigma \oplus \mathbb{C}\tau \oplus (\mathbb{C}\rho_1 + \mathbb{C}\rho_2) \oplus (\mathbb{C}\zeta_1 + \mathbb{C}\zeta_2)$$

is a decomposition of the group algebra $\mathbb{C}G$ *into* 4 *minimal left ideals.*

ii) *Calculate explicitly the representations* f_i *of* G *afforded by these ideals and show that they are irreducible (Clearly it is enough to write* $f_i(12)$ *and* $f_i(13)$ *for* $i = 1, 2, 3$*).*

iii) *Show that* 3 *of them, say* f_1, f_2, f_3 *are inequivalent, of respective degrees* 1, 1, 2.

iv) *Conclude that* f_1, f_2, f_3 *are the only irreducible representations of* G *over* $\mathbb{C}$ *(use Corollary* 2.9*).*