1 Introduction

The role played by innovation in determining economic growth is commonly recognized. In recent years an ever increasing amount of literature has been devoted to the subject and many authors, starting from the contributions by Romer (1990), Grossman and Helpman (1991) and Aghion and Howitt (1992), have written about its relevance. Many of these works underlined the importance of Arrow's (1962) claim that the development of new ideas spurs growth and they brought innovation to be one of the most relevant topics in contemporaneous economic literature giving start to the so called Endogenous Growth Theory or New Growth Theory.

The results produced by this stream of literature are many and, as is well known, very articulated.[1] Here, we recall the role played by the intentional research activity carried out in the innovative R&D sector described in the works by Grossman and Helpman (1991) and Aghion and Howitt (1992). While, on the one hand, Grossman and Helpman draw particular attention to the fact that research efforts may result in an extension of the number of available consumption varieties; on the other hand, Aghion and Howitt underline that these efforts may, among other things, improve the quality of the varieties already available.

The view that innovation is one of the engines of economic growth in the forms suggested by the above mentioned authors is widely recognized. However, we think that some new insights might be gained by means of a joint analysis on the effects of different types of innovation, which, following Schumpeter (1934), we identify as *product* and *process innovation*. By considering both types of innovation in a general equilibrium framework, we should be able to give a more articulated description of the effects of the innovative activity on the economy. Hence, in this work we introduce *process innovation* in the general equilibrium framework proposed by Grossman and Helpman (1991, ch. 3) in order to study the complementary role this kind of innovation plays with product innovation, which increases the number of available consumption (or intermediate)

[1] These results are a too big a subject to be summarized in the present paper, and we refer the interested reader to the exhaustive descriptions by Grossman and Helpman (1991) and Aghion and Howitt (1992).

varieties, and which is the type of innovation considered by Grossman and Helpman (1991), while process innovation increase productivity levels and, thus, decrease variable production costs of new firms entering the final good sector.

The complementarity between process and product innovation is studied by Athey and Schmutzler (1995) in a different framework. Their aim is to underline the interactions between the short-run innovative activity of a firm and its organizational structure, which defines its long-run characteristics such as research capabilities and flexibility, which determines "how costly it will be to make changes to product designs and production processes, once the opportunities for innovation have been identified." (Athey and Schmutzler, 1995, p. 558). Moreover, Eswaran and Gallini (1996, p. 722) "examine the interactions between firms' product and process innovation decisions, and the role patent policy can play in directing technological change toward a socially efficient mix of innovations." Moreover, Eswaran and Gallini (1996, p. 723) show that there is a trade-off "between an entrant's incentives to engage in product and process innovation. The more differentiated the products are, the less is the entrant's marginal profit from competing against the pioneer through lower production costs, and vice versa."

Differently from the works mentioned in the previous paragraph, our work does not focus on the organizational structure of innovating firms, rather it focuses on the dynamic effects of contextual process and product innovations in a general equilibrium framework and it aims to give one possible explanation, among different existing ones, to the fact that firms producing at any moment in time are *heterogeneous* in their productivity levels.

In a certain sense, the particular form of process innovation we represent is related, even if it is different, to the argument of the learning-by-doing processes suggested by Romer (1990), who considers knowledge accumulation as a side effect of conventional production activity not resulting from deliberate research activity in the innovative sector. We analyze the effects of the complementarity between product and process innovations when product innovations take place as a deliberate effort of researchers employed in the R&D sector, and contextual process innovations take place as a side effect of the innovative activity in the same sector, resulting in higher productivity of new consumption goods. This is possible because researchers, when carrying out their activity, accumulate knowledge which leads them, at a certain point, to develop new patents of consumption goods characterized by more productive production processes. However, we explain in the paper that workers (firms) engaged in the R&D sector have incentives in pursuing process innovation, because the purchasing power of their wages is increased only if process innovation takes place and only in terms of the more productive goods which have been made available.

Indeed, the improvements in production processes which we introduce in this work are assumed to take place as a by product in the R&D sector and they are costlessly adopted by firms starting the production of newly developed consumption (or intermediate) varieties, while already producing firms continue to produce using older technology which was available when the products were developed, and, as a consequence, we will be able to obtain long-run equilibria characterized by many different kinds of firms with different production processes.[2] These assumptions allow us to capture the fact that often new producing firms are more productive than older ones, given that they start to produce when knowledge accumulated in the past allows new production techniques to be more efficient.

Therefore, in this work we adopt a growth approach in order to identify one potential mechanism whose action will result in productivity heterogeneity of firms. Hence, if firms are heterogeneous in their productivity levels, this should result in a variety of prices, and of demand and market shares which reflect firms' productivity differences. Moreover, these differences should, in turn, be reflected in patents' price differences, given that we would expect that patent prices of more profitable varieties are higher.

[2] This is a simplifying assumption, given that we could consider a more general case in which old firms could adopt the new more productive production processes provided that they sustain a certain switching or implementing cost. The presence of different *implementing costs* would imply that not all preexisting firms would be able to adopt the new more productive production processes once available. Thus, as in our simplified case, there would be equilibria characterized by many different kinds of firms with different production processes. Hence the nature of the results which we would obtain would be similar to those we obtain, with the sole difference that the distribution of firms among the available production processes would be more biased towards the new ones. However, we chose to adopt a simplified framework in which no technology change is possible in order to avoid the complications which would arise describing the switching processes, and given that it would not change the heterogeneous nature of our results.

In particular, we introduce process innovations in the setup proposed by Grossman and Helpman (1991) characterized by the assumption of consumers' love for variety by Dixit and Stiglitz (1977). This last assumption identifies consumers' love for variety as one of the causes of economic growth, because it pushes firms to innovate in order to satisfy consumers' demand for new varieties. Moreover, it allows us to introduce a further assumption related to how process innovation takes place, given that we assume that they are more likely to occur when the market is larger. In fact, in this case, researchers are induced to increase their efforts to find improvements in the available production technology given that, ceteris paribus, the number of more productive varieties on which relative prices would be reduced and purchasing power would be higher, would be larger. We could also justify this assumption with the argument that larger markets allow researchers to exploit increasing returns to scale.

Thus, the explicit purpose we try to assess in this work is to understand how the steady state outcomes by Grossman and Helpman (1991) are affected by process innovations that accompany product innovations, investigating the effects of this complementarity on the long run growth rate of the economy, productivity heterogeneity of firms, worker distribution among different sectors and on prices of all available varieties and patents.

Finally, our work will also try to address, or better mitigate, the scale effect problem which affected the original contribution by Grossman and Helpman (1991). In fact, Jones (1999, p. 139) writes that, when there is a scale effect, "the growth rate of the economy is proportional to the total amount of research undertaken in the economy. An increase in the size of the population, other things equal, raises the number of researchers and therefore leads to an increase in the growth rate of per capita income. Taken at face value, this prediction is problematic because it means that population growth should lead to accelerating per capita income growth". Thus, we would like to suggest that the introduction of a series of continuous process innovations, along the lines described in the following sections, could add another mechanism through which this scale effect may be tackled or better, as we stated, mitigated to those reviewed by Jones (1999).

The remaining part of the work is organized as follows: Section 2 describes consumers' and firms' behavior, while the innovative sector is more deeply analyzed in Section 3; Section 4 draws the characteristics of the equilibrium outcomes which are "moving" with particular changes in the distribution of workers; Section 5 presents some comments on the results, while Section 6 concludes.

2 Consumers' and firms' behavior

We consider a closed economy in which consumers love variety and their preferences are described by the following intertemporal utility function

$$U = \int_0^\infty e^{-\rho t} \log \left(\sum_{c=1}^{n(t)} D_c(t)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} dt \tag{1}$$

where D_c is the consumption of variety c, ρ is the rate of time preference and $\sigma > 1$ is both the elasticity of substitution between any pair of varieties and the own-price elasticity of demand for any variety. The elasticity of intertemporal substitution in (1) is constant and equal to 1, while n is the total number of produced varieties in t.

Total consumers' expenditure E is defined as

$$E = \sum_{c=1}^{n} p_c D_c$$

where p_c is the price of variety c.

Consumers' demand x_c for any variety c is

$$x_c = \frac{p_c^{-\sigma}}{\sum_{c=1}^{n} p_c^{1-\sigma}} E \tag{2}$$

All varieties are produced by firms which need to buy a patent from the R&D sector to start their activity and which employ γ workers to produce a unit of their output. Given the assumptions of consumers' love for variety and the fact that there are no scope economies, all firms produce different varieties.

Moreover, firms are not all supposed to use the same production process, given that γ is not necessarily equal for all firms. More precisely, we assume that there are i different groups of firms, each of which is characterized by a particular value of γ, that is γ_m, which expresses the specific units of labor required to produce one unit of the output by the n_m firms of the group of type m, with $m = 1, 2, ..., i$.

We assume that the higher the value of m, the lower the value of γ_m is and, therefore, the higher the productivity of workers employed by firms of type m is. Each period is characterized by a particular value of i, which increases when a new group of goods is made available through R&D efforts in the innovative sector. We assume that any time the value of i increases, a new, more productive process is made available and new firms use the more productive production process. More details on how new values of γ are made available will be given in next section which describes the innovative sector. For the moment, we only anticipate that by producing new patents, researchers in the R&D sector exploit the knowledge accumulated by past innovative activities and that the development of a sufficiently large number of varieties allows them to introduce new patents characterized by higher productivity levels (that is, by lower γ values).

Hence, if there are only i groups of variety, each of numerousness n_m, we derive from (2) the demand x_m for any firms characterized by γ_m

$$x_m = \frac{p_m^{-\sigma}}{\sum_{j=1}^{i}\sum_{j=1}^{n_j} p_j^{1-\sigma}} E \tag{3}$$

with $m = 1, 2, ..., i$.

Given that all varieties of type m are symmetric, total expenditure in varieties of the same type is

$$n_m p_m x_m = \frac{n_m p_m^{1-\sigma}}{\sum_{j=1}^{i} n_j p_j^{1-\sigma}} E \tag{4}$$

with $m = 1, 2, ..., i$.

Considering the intertemporal component of consumers' allocation problem, following Grossman and Helpman (1991) we define the index of the manufactured output

$$D \equiv \left(\sum_{m=1}^{i} n_m x_m^{\alpha} \right)^{\frac{1}{\alpha}}$$

where $\alpha = \frac{\sigma-1}{\sigma}$, and the ideal price index of final goods

$$p_D \equiv \left(\sum_{m=1}^{i} n_m p_m^{1-\sigma} \right)^{\frac{1}{1-\sigma}}$$

Given that $D = E/p_D$, the intertemporal utility function (1) becomes

$$U = \int_0^{\infty} e^{-\rho t} \left(\log E - \log p_D \right) dt$$

As Grossman and Helpman (1991, pag. 48) show, the maximization of the previous expression "subject to an intertemporal budget constraint requires that spending evolve according to"

$$\frac{\dot{E}}{E} = r - \rho$$

Then we normalize prices in such a way that total expenditure E is equal to 1.[3] This implies that we have

$$r = \rho \tag{5}$$

Consumption varieties are produced by monopolistically competitive firms, which sustain a fixed cost in order to acquire a patent produced in the innovative sector and a variable cost of production. Since each firm maximizes profits, we know that price p_m is

$$p_m = \frac{1}{\alpha} \gamma_m w \tag{6}$$

where $\alpha = (\sigma - 1)/\sigma$ and w is the nominal wage.

We notice that the ratio between prices of any pair of varieties is proportional to the ratio of labor required to obtain one unit of each type of good with

$$\frac{p_m}{p_j} = \frac{\gamma_m}{\gamma_j} \tag{7}$$

[3] Cfr. Grossman and Helpman (1991)

where $m, j = 1, 2, ..., i$.

Operating profits realized by a single firm of type m are

$$\pi_m = \left(\frac{1-\alpha}{\alpha}\right) w\gamma_m x_m$$

In equilibrium, when supply, x_m, is equal to demand (3), substituting prices from (6), we obtain that profits π_m are

$$\pi_m = (1-\alpha)\frac{\gamma_m^{1-\sigma}}{\sum_{j=1}^{i} n_j \gamma_j^{1-\sigma}} < 1 \tag{8}$$

From the previous expression, we know that profits, π_m, decrease when the total number of firms increases, while they increase as productivity increases. In particular, for the more productive varieties, characterized by $m = i$, we know not only that profits decrease as n_i increases, with $\partial\pi_i/\partial n_i < 0$, but also that they increase as γ decreases, given that $\partial\pi_i/\partial\gamma_i < 0$.

Total labor demand by firms of type m, L_m, is given by

$$L_m = n_m \gamma_m x_m \tag{9}$$

Moreover, considering (4) together with (6), we derive that L_m is equal to

$$L_m = \frac{\alpha n_m p_m^{1-\sigma}}{w \sum_{j=1}^{i} n_j p_j^{1-\sigma}} \tag{10}$$

Given that the total number of units of workers in the economy is L, the labor market clearing condition requires that

$$L = L_R + \sum_{m=1}^{i} L_m \tag{11}$$

where L_R is the total amount of labor employed in the innovative sector and will be described in the following section.[4] Finally, we define L_C as the total amount of labor employed in the production of consumption goods which corresponds to

$$L_C = \sum_{m=1}^{i} L_m$$

[4] We simply assume that the switching technology cost for existing firms consists in a different, too high fixed cost which firms that are already in the market have to sustain to be able to use the process innovation generated within the R&D sector. This enables us to avoid considering the case of old firms switching technology.

3 Product and process innovations

All firms producing consumption goods start their production after buying a patent of price v_i. New patents for new varieties are produced in the R&D sector, and their production is described by the following function

$$\dot{n}_i = \frac{1}{a} n_i L_R \tag{12}$$

where a is an inverse measure of labor productivity in the innovative sector.

Expression (12) shows that the number of new patents produced in the R&D sector is proportional to the units of workers employed in the same sector and to the number of the already existing varieties, whose production process is characterized by the smallest value of γ, that is γ_i. Therefore, we share the assumption in Grossman and Helpman (1991) that nonrivalry of ideas in the innovative sector gives rise to increasing returns. We could have chosen a different functional form for (12) which would have avoided the scale effect in Grossman and Helpman (1991). However, we chose this specification because we would like to show that we can identify a potential way through which the presence of the scale effect can be mitigated, given that, as we show later, its consequences are attenuated when process innovations may take place.

Moreover, we assume that γ values are decreasing in m, with

$$\gamma_1 > \gamma_2 > ... > \gamma_m > .... > \gamma_{i-1} > \gamma_i \tag{13}$$

As a consequence of (12) and (13), when innovations take place, that is when $\dot{n}_i > 0$, new varieties are produced with the most efficient production process, characterized by the smallest available value of γ, that is γ_i.

We know from (12) that labor demand in the innovative sector is

$$L_R = \frac{\dot{n}_i}{n_i} a \tag{14}$$

Following Grossman and Helpman (1991), we assume that stock market value of a patent, v_i, is at any time equal to the present discounted value of the stream of all following profits. Hence, given the interest rate r on a safe asset, we can write that

$$v_i = \int_0^\infty e^{-rt} \pi_i dt \tag{15}$$

The innovative sector is perfectly competitive and the level of employment of workers in the R&D sector is such that it maximizes profits

$$\pi_R = v_i \dot{n}_i - wL_R \tag{16}$$

From the first order condition on the previous maximization problem, we obtain the nominal wage as a function of the price of any new patent v_i and the number of varieties of type i, that is

$$w = \frac{v_i n_i}{a} \tag{17}$$

At this stage, we need to specify how γ evolves over time. We assume that the value of γ evolves along a learning curve and we think that it is more likely that researchers obtain a further improvement in production processes associated to new patents, which reduces the smallest value of γ, when the number of patents associated to the existing more efficient technology, n_i, becomes sufficiently large and when the potential demand dimension is large, that is when L is large. The reason for the first effect is that knowledge accumulates over time and, consequently, it makes further improvements possible. The rationale for the second effect, which will be discussed later, is that when demand is larger, researchers' efforts are increased and productivity improvements take place sooner. Moreover, in this case, researchers know that there is a competition effect generated by the entrance of a higher number of firms due to the fact that, as we will show later, larger values of L, other things equal, tend to increase the rate of growth of varieties, $\dot{n}_i/n_i$, and consequently n_i, lowering expected rewards by researchers. In fact, expression (8) shows that expected profits on varieties of type i are lower when n_i is higher. However, given that (8) also shows that profits for varieties i are higher, the lower γ_i, researchers increase their efforts to find improvements in the available production technology to avoid the larger reduction in profits when L is larger and to try to improve the profitability of new varieties because, once a reduction in γ_i

occurs, there is a gap in the present value of the flow of all future operating profits between old firms and new type firms

$$\int_{t_i}^{\infty} e^{-rt} \left(\pi_i(t) - \pi_m(t)\right) dt = (1-\alpha) \int_{t_i}^{\infty} e^{-rt} \frac{1}{\sum_{j=1}^{i} n_j(t) \left(\gamma_j(t)\right)^{1-\sigma}} \left(\frac{1}{\gamma_i^{\sigma-1}} - \frac{1}{\gamma_m^{\sigma-1}} \right) dt > 0 \tag{18}$$

We observe, in passing, that the gap described in previous expression decreases as t increases.

At this point we notice that workers and firms engaged in the R&D sector have incentives in pursuing process innovations, because these innovations allow them to increase the purchasing power of their wages in terms of new, more productive goods. In fact, as (15) and (18) show, process innovations increase patents' prices paid by firms in the final sector, given that they raise profits realizable by these firms. The increase in patents' prices is, in turn, accompanied by an increase in the wage w of workers employed in the R&D sector, because, in the framework we use, their wage is related to the value of their marginal product of the R&D sector, which depends on patents' prices (17). Then, (6) shows that higher wages results in higher purchasing power in terms of consumption goods, only when process innovations take place, because γ decreases and w/p_i increases. Thus, researchers, who are also consumers, have an incentive to obtain process innovations. For this reason, we think that it is likely to assume that researchers make deliberate efforts not only to produce new patents, but also to have more productive processes and we assume that process innovations take place in the R&D sector provided that a sufficient level of knowledge is accumulated.

Moreover, we can also assume that decreases in γ are more frequent, or larger, with larger population because the larger demand could allow researchers to exploit increasing returns to scale or because workers know, as we show later, that a larger population is associated with a larger number of researchers in equilibrium, and, therefore, with a larger number of goods on which process innovations could increase their purchasing power.

Therefore, following previous reasoning, we may think that the evolution of γ_i depends on n_i

and L with

$$\dot{\gamma}_i = f(n_i, L) \tag{19}$$

To give an example, and we wont need to assume it in the following of the paper, a simple specification for (19), could be the following

$$\dot{\gamma}_i = \begin{cases} c_i < 0 & \text{if } n_i L = \chi_i \\ 0 & \text{if } n_i L < \chi_i \end{cases} \tag{20}$$

where χ_i is a threshold value, which once reached allows us to represent the development of new and more productive varieties. The threshold is not a constant, given that different stages of development may require a different number of patents or different sizes of the population and demand to induce further process improvements. Moreover, c_i expresses the size of the process innovation, whose value is not constant, given that process innovations are certainly not at all equal in their effects and that they may have different impacts on various stages of the growth process.

In more details, once γ decreases, from that point in time onward, index i will represent the new more productive varieties. In particular, to make clear the use of our notation, we note that varieties of type i can also be named with the last integer number of the series for m, which we call h with $m = 1, 2, ..., (h = i)$. Once there is an improvement along the learning curve, described for instance by (19), the series continues in the following way: $m = 1, 2, ..., h, (h + 1 = i)$. Moreover, we notice that in the moment of the change in γ, (12) can be written as follows

$$n_{h+1} = \dot{n}_{h+1} = \frac{1}{a} n_h L_R \tag{21}$$

According to (20), the evolution of γ is related to the "adjusted" size of the market, $n_i L$.

In summary, and more generally with (19), for a given size of the market L, more firms operating in the economy along the frontier (higher n_i) increase the accumulated knowledge which allows researchers in the R&D sector to be able to find the way to introduce further improvements in productivity of firms associated with new patents. But these improvements are more likely to occur when larger dimensions of the market, L, push researchers to increase their efforts in searching for process innovations; firstly, to avoid the decrease in the value of new patents which otherwise would be generated by smaller profits due to the higher competition and, secondly, to increase their purchasing power on a larger number of new more productive varieties.

We know that the no arbitrage condition between patents and a safe asset implies that the following Fisher equation must be satisfied for every value of m

$$\frac{\pi_m}{v_m} + \frac{\dot{v}_m}{v_m} = r \tag{22}$$

We recall that while for $m \neq i$ innovation does not introduce any new varieties, these are developed for the $i - th$ group of firms.

4 Moving equilibrium

In this section we describe the properties of the equilibrium of the model, which will be characterized as a *moving equilibrium*, given that we assume that the number of firms is the slow variable of the economy, while all other variables are the fast variables.[5]

In particular, we know from expression (11) that in equilibrium the labor market is clearing. From (10), (6) and (17) we obtain that employment in the final sector is

$$L_C = \frac{\alpha}{w} = \frac{\alpha a}{v_i n_i} \tag{23}$$

Thus, in any periods between the two subsequent reduction in γ_i, the market clearing condition (11) can be rewritten as

$$L = \frac{\dot{n}_i}{n_i} a + \frac{\alpha a}{v_i n_i}$$

Let us denote with V_i the inverse of the value of the aggregate existing stock of patents of firms of type i, $V_i = \frac{1}{v_i n_i}$. Then, from the previous equation, we derive the growth rate of firms of type

[5] See Schlicht (1985, 1997).

i in any periods between the two subsequent reduction in γ_i, that is

$$g_i = \frac{L}{a} - \alpha V_i \tag{24}$$

where $g_i = \dot{n}_i / n_i$.

For any time before a subsequent reduction in γ_i, we know from expressions (22) and (23) that the rate of change of V_i for firms of type i is

$$\frac{\dot{V}_i}{V_i} = V_i \frac{n_i(1-\alpha)}{\sum_{j=1}^{i} n_j \left(\frac{\gamma_i}{\gamma_j}\right)^{\sigma-1}} - g_i - r \tag{25}$$

From the previous expression, we note that we have Grossman and Helpman's (1991) results only if $\gamma_i = \gamma_j \ \forall j = 1, ..., i-1$, because this would also imply that $n_i = n_j \ \forall j = 1, ..., i-1$. Moreover, in the same particular case, we know that the interception between the two curves (24) and (25) would be unique when $\dot{V}_i = 0$, as is required in equilibrium.

However, when, as it happens in our case, $\gamma_i \neq \gamma_j$, the intersection between the two curves (24) and (25) is not unique and it moves over time as n_i increases in any period between two following values of γ_i are made available. Therefore, more than a fixed steady state equilibrium, as in Grossman and Helpman (1991), our assumptions lead to identify a *moving equilibrium*, which is characterized by continuous changes in the number of firms with different productivities.

Let us define the index b_i as

$$b_i \equiv \frac{n_i \gamma_i^{1-\sigma}}{\sum_{j=1}^{i} n_j \gamma_j^{1-\sigma}} \tag{26}$$

It is readily verifiable that $0 \leq b_i \leq 1$ and that b_i approaches 1 when n_i goes to infinity. b_i gives us some information on the relative weight of firms of type i on the total number of firms, where the weights are given by the productivity measure $\gamma_i^{1-\sigma}$. Hence, given that the value of b_i continuously changes, we have a *moving equilibrium* characterized by continuous changes in the fast variables due to movements in the slow variable b_i. Particularly, we have a moving equilibrium when all variables assume their equilibrium values conditioned to the number of patents already introduced in the R&D sector or, in an equivalent fashion, conditioned to the value of b_i, which depends on the number of patents. Then, changes in the number of available varieties, change b_i and, consequently, other variables, as we show in the rest of this section. Expressions (24), (25) and (5) tell us that for any of those moving equilibria the following condition must be satisfied

$$\dot{V}_i = V_i \left[V_i \left((1-\alpha)\frac{n_i \gamma_i^{1-\sigma}}{\sum_{j=1}^{i} n_j \gamma_j^{1-\sigma}} + \alpha \right) - \frac{L}{a} - \rho \right] \tag{27}$$

Moreover, we use (26) to rewrite profits (8) in the following way

$$\pi_m = \frac{(1-\alpha)}{n_m} b_m < 1 \tag{28}$$

Substituting (26), (6) and (17) into (3), we obtain that the demand for any firm of type m is

$$x_m = \frac{n_m p_m^{1-\sigma}}{n_m p_m \sum_{j=1}^{i} n_j p_j^{1-\sigma}} = \frac{a\alpha b_m}{n_m \gamma_m} V_i \tag{29}$$

Using this expression we derive the total demand $x_m n_m$ for all firms of type m, that is

$$n_m x_m = \frac{a\alpha b_m}{\gamma_m} V_i \tag{30}$$

Expression (30) tells us that when new more productive varieties are made available by the innovative sector, as the innovative process goes on, the total demand $x_m n_m$ for the oldest firms of type m, characterized by the highest values of γ, tends to decrease to zero, because b_m becomes smaller and smaller. On the contrary, the total demand $x_i n_i$ for firms of type i on the technological frontier, tends to increase as b_i increases.

Substituting (26), expression (27) becomes

$$\dot{V}_i = V_i \left\{ V_i \left[(1-\alpha) b_i + \alpha \right] - \frac{L}{a} - \rho \right\} \tag{31}$$

Expression (31) is an upward opening parabola, with $\dot{V}_i = 0$ either when $V_i = 0$ or when $V_i^* = \frac{L/a+\rho}{(1-\alpha)b_i+\alpha} > 0$. The graph is plotted in Figure 1 only for positive values of V_i, because negative values of V_i would have no meaning.

Insert Figure 1 about here

Moreover, in Figure 1 we also plot the actual value of V_i derived from (23), that is

$$V_i = \frac{L_C}{\alpha a} \tag{32}$$

We know from (32) that (11) and (25) are, respectively,

$$L_R = L - L_C \tag{33}$$

and

$$\dot{V}_i = \left(\frac{L_C}{\alpha a}\right)\left[\frac{b_i L_C}{\alpha a}(1-\alpha) - \left(\frac{L_R}{a} + \rho\right)\right] \tag{34}$$

Hence, as both pairs of equations above (31)-(32) and (33)-(34) show, the equilibrium outcomes which we describe within this framework are not stationary, given that b_i changes as the innovating process goes on determining the introduction of new varieties which increase n_i. Thus, between any pair of subsequent process innovations which lead to changes in γ_i, the equilibria we consider are moving equilibria which we need, indeed, to identify.[6]

We notice that we need to know L_C (or L_R) in order to define the exact position of the vertical line (32) in Figure 1, otherwise, we could either have that $L_C/(\alpha a) < V_i^*$ or that $L_C/(\alpha a) > V_i^*$. These two options would imply opposite changes in V_i. In fact, while V_i is increasing when $V_i > V_i^*$, it is decreasing when $V_i < V_i^*$. However, as we show in two steps, V_i must be equal to

[6] In other words, we can consider the economy as described by the following equation system in two vectors of variables x and y:
$\dot{x} = f(x, y)$ and $\dot{y} = g(x, y)$
where the vector of fast variable is $x' = (p_m, x_m, v_m, V_m, w, \pi_R, L_R, L_C, g_i)$ and the vector of slow variable is $y' = (b_i)$. Note that the slow variable b_i is obtained as a transformation of the number of all variables, n_m with $m = 1, 2, ..., i$, which, thus, are considered as slow variables too.
In the paper we assume that the fast vector has already reached its equilibrium for any given and fixed value of the slow variable, b_i, and we prove that the equilibrium is univocally identified for any given value of b_i in the following paragraphs in the text of the paper when we show that $V_i = V_i^* = \frac{L/a+\rho}{(1-\alpha)b_i+\alpha}$.
Particularly, the equilibrium value of the fast vector x is
$x = X(b_i)$, with $f(X(b_i), b_i) = 0$.
Then, given that the slow variable b_i changes over time, then $x = X(b_i)$ gives the corresponding moving equilibrium of x. (See Schlicht (1985, 1997))

V_i^*. In particular, first we recall that this is true in Grossman and Helpman's (1991, ch. 3) case. Then we prove that this is true in our general case.

First of all, we recall that if we were in Grossman and Helpman's (1991, ch. 3) case, γ would assume only one value, that is $\gamma_m = \gamma_i = \gamma_1 \ \forall m$. Moreover, in this case the steady state equilibrium is characterized by $\dot{V}_1 = 0$ and $L_R = L(1-\alpha) - a\alpha\rho$. In fact, we know that in this case the expectations of agents are fulfilled only if the economy jumps immediately to the point in which $\dot{V}_1 = 0$, because if $\dot{V}_1$ were positive we would have V_1 growing to infinity, while if $\dot{V}_1$ were negative, we would end up with $V_1 = 0$. However, Grossman and Helpman (1991) show that both cases are impossible, given that: in the first case we cannot have V_1 growing to infinity because L_R would be drawn to zero, n_1 would stop growing, and v_1 would be different from zero (given that with a finite number of variety, profits are strictly positive); in the second case, we cannot have $V_1 = 0$, because L_R would assume its maximum potential value, L, with $L_C = 0$, and expectations would be contradicted.[7] Finally, we notice that in this case, $b_1 = 1$. If we consider the pair of equations (31)-(32) which describes the equilibrium condition, they would intersect in $V_1 = \frac{L-L_R}{\alpha a} = V_1^* = L/a + \rho$ with $L_R = L(1-\alpha) - a\alpha\rho$ derived from the second pair of equations (33)-(34).

Let us now consider the case in which, in the framework so far described, an innovation process takes place producing new patents characterized by $\gamma_2 < \gamma_1$, which perturbs previous stationary equilibrium.[8] These new patents allow n_2 firms to employ the technology of type 2. We know from (21) that $n_2 = \frac{1}{a} n_1 L_R$ and that $b_2 = \frac{n_2 \gamma_2^{1-\sigma}}{n_1 \gamma_1^{1-\sigma} + n_2 \gamma_2^{1-\sigma}} < 1$.

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[7] In fact, Grossman and Helpman (1991, p. 61) recall that if $L_R = L$, the number of varieties would grow continuously and, at the same time we would have $v_1(t) = \int_t^\infty e^{-r(\tau-t)} \frac{1-\alpha}{n_1(t)} d\tau < \frac{1-\alpha}{r n_1(t)}$.

In fact, $v_1(t) = \int_t^\infty e^{-r(\tau-t)} \frac{1-\alpha}{n_1(t)} d\tau = \left[-\frac{e^{-r(\tau-t)}}{r}\right]_t^\infty \frac{1-\alpha}{n_1(t)} = \left[-\frac{e^{-r\infty}}{r} + \frac{1}{r}\right] \frac{1-\alpha}{n_1(t)} < \frac{1-\alpha}{r n_1(t)}$.

Therefore, we would have that $v_1(t) n_1(t) < \frac{1-\alpha}{r}$ which is equivalent to saying $V_1(t) > \frac{r}{1-\alpha} > 0$ which contradicts the fact that $V_1 = 0$.

[8] In the particular example described by (20) when $n_1 L = \chi_1$. In any case, we recall that we do not need to use this particular specification of the more general expression (19).

After the change in γ, the innovative sector continues to produce new patents of type 2 according to (12). The inverse of the aggregate value of patents of type 1 is equal to $V_1 = \frac{1}{v_1 n_1}$, where n_1 is now a constant. Moreover, from (28), we know that profits of firms of type 1 are from now on $\pi_1 = \frac{(1-\alpha)}{n_1} b_1$. At the same time, there will be continuous increases in n_2, or in other more productive types of firms whenever there are further innovations leading to further reductions in γ. These processes will reduce b_1, reducing profits of firms of type 1 and, therefore, the value of patents of firms of type 1, v_1, thereby, increasing V_1. Therefore, we know that V_1 is increasing in b_1. Moreover, for any given value of b_1, n_1 and v_1 are given and, thus, V_1 is univocally determined. In other words, we are able to rule out bubble paths for the aggregate value of firms which are no longer on the technological frontier, such as firms of type 1, once technology with γ_2 can be used. Furthermore, we may say that this is generally true for any firms of type m different from i, that is, firms at the technological frontier from the production process point of view, because their number n_m does not increase anymore, and because their value v_m must decrease due to the ongoing growth in variety. At the limit, when the weight b_m of firms adopting older technology than the firms at the frontier, γ_i tends to decrease toward zero, and V_m tends to infinity.

Lemma 1 *For any variety which is not at the technological frontier, that is, for any variety produced with $\gamma_m > \gamma_i$, profits decrease and V_m increases as the weight b_m of the group decreases as a consequence of subsequent innovations in the R&D sector which increase the number of patents.*

Moreover, returning to our example, we further observe that once the new patents of type 2 become available at the technological frontier, with $i = 2$, and $b_2 < 1$, then $V_2^* = \frac{L/a+\rho}{(1-\alpha)b_2+\alpha} > V_1^*$.

Then we notice that while firms of type 2 remain at the frontier, for a given value of b_2, if $V_2 = \frac{1}{v_2 n_2}$ does not immediately jump to V_2^*, there could be two other possible cases which we should consider: either we have $V_2 < V_2^*$ (with $\dot{V}_2 < 0$ which would draw V_2 to zero), or $V_2 > V_2^*$ (with $\dot{V}_2 > 0$ and V_2 growing to infinity). We note, in passing, that the following arguments can be generalized to the case in which firms of type i are at the frontier, for given b_i values.

We rule out the first case, that is $V_2 < V_2^*$, because we want to exclude asset bubble paths both in the subcase in which all firms of type 2 will always continue to be on the technological frontier in the future and in the subcase in which, sometime in the future, these firms will no longer be on the technological frontier due to further process innovations, which further reduce γ for future varieties. In the first subcase, V_2 cannot be drawn to zero because this would be possible, for a finite number of firms n_2, only with v_2 increasing to infinity; but with ongoing patent innovations this is impossible. Following the same reasoning, we can exclude asset bubble paths in the present (while firms of type 2 are at the frontier) also in the subcase in which the same firms were no longer supposed to be on the technological frontier at sometime in the future. In fact, if in the future the number of varieties is growing, we can exclude a continuous growth in v_2 and consequent decreases in V_2 in the present, for given value of n_2 and b_2, because profits of firms of type 2 are superiorly limited by variety growth.

We can also rule out the second case, that is $V_2 > V_2^*$, once more both in the subcase in which all firms of type 2 will always continue to be on the technological frontier in the future and in the subcase in which, at sometime in the future, these firms will no longer be on the technological frontier due to further process innovations. In the first subcase, when firms of type 2 remain on the frontier with $i = 2$, this will rapidly lead to employ all workers in the manufacturing sector with no more growth in variety (given that we know from (32) that $V_2 = \frac{L_C}{\alpha a}$) and V_2 increasing to infinity. However, this would be possible only if v_2 were equal to zero and we exclude this case because if the number of varieties stops growing, profits must always be strictly positive. Finally, in the second subcase if n_2 stops growing at sometime in the future because firms of type 2 are displaced from the frontier, in the present (while these firms are at the frontier), we know that V_2 will increase to infinity in the future, because it is in the future that v_2 tends to zero as b_2 (and profits) tends to zero. However, these future increases in V_2 cannot be anticipated in the present, because, otherwise, V_2 would be lead to infinity in the present which would be inconsistent with $L_C < L$. In other words, from (32) we could have it only with $L_C = L$, but at the expense of no innovation at all in the present (because $L_R = 0$) which would exclude the potential future process innovations.

So we have that when firms of type two are at the technological frontier, V_2 immediately jumps to V_2^* and that, for any given value of b_2, $\dot{V}_2 = 0$. We know that b_2 changes in the present with ongoing patent innovations. In particular, it continuously increases, because n_2 increases. As long as firms of type two are at the technological frontier, the continuous increase in b_2 is associated with continuous reductions in $V_2 = V_2^*$, with V_2 having a lower limit. In fact, $V_2 = V_2^*$ tends to V_1^* as n_2 increases.

We already noticed that previous arguments can be generalized to the case in which, instead of firms of type 2 we considers firms of type i at the frontier for given b_i values. For we know now that we have

$$V_i = V_i^* = \frac{L/a + \rho}{(1-\alpha)b_i + \alpha} \tag{35}$$

where $V_i = V_i^*$ is increasing in L, ρ (with $\rho = r$) and decreasing in a, b_i and α.

We may write the following lemma:

Lemma 2 *In general, the inverse of the aggregate value of firms at the frontier, V_i, is $V_i = V_i^* > V_1^*$ with $\dot{V}_i = 0$ when b_i is given. However, given that b_i increases when patent innovations take place for firms at the technological frontier increasing n_i, V_i changes approaching V_1^* as the weight b_i increases.*

The previous lemma is extremely important given that it allows us to identify not a unique steady state equilibrium, but a series of moving equilibria, which can be considered as perturbations of the original steady state equilibrium in Grossman and Helpman (1991, ch. 3), and which continue to change as long as b_i changes and as long as we can have different process innovations which continue to change the technology at the frontier. The implications of our results will be discussed in the following section.

Moreover, from expressions (30) and (35) we obtain the total demand for varieties of type i

$$x_i n_i = \frac{a\alpha}{\gamma_i}\left[\frac{L/a + \rho}{(1-\alpha) + \alpha/b_i}\right]$$

which clearly shows that as b_i increases, total demand for varieties of type i increases. Thus, gradually in our model, while the market share and the demand of previously developed varieties decreases, the market share and the demand of new and more productive varieties made available increases as long as they are on the technological frontier.

5 Structural changes and the scale effect

One of the most striking characteristics of the moving equilibrium we have so far described is that it allows us to represent the effects of ongoing patent innovations which take place together with process innovations. Considering both kinds of innovations gives a more complete picture of the effects of R&D activities and it produces a setup in which the rate of growth of patent innovations varies across time according to workers' distribution between the final and the innovative sectors considered in the model.

In the period in which technology of type i is available, we know from expression (12) that the rate of innovation is proportional to the number of workers employed in the R&D sector, and this number L_R, derived from (33)-(34) when $\dot{V}_i = 0$, depends on the value of b_i, that is

$$L_R = \frac{Lb_i (1-\alpha) - a\rho\alpha}{(1-\alpha) b_i + \alpha} \tag{36}$$

As in Grossman and Helpman (1991), we assume that L is sufficiently large to allow patent innovations to take place: this requires that $L > a\rho\alpha / b_i (1-\alpha)$. Once more, it is readily verifiable that when $b_i = 1$ we obtain the same results as in Grossman and Helpman (1991).

Expression (36) shows that the number of workers employed in the innovative sector is an increasing function of b_i because

$$\frac{\partial L_R}{\partial b_i} = \frac{(1-\alpha)\alpha (L + a\rho)}{((1-\alpha) b_i + \alpha)^2} > 0$$

Therefore, when there are at least two different types of firms producing using different technologies, and the innovative sector intensifies its research in finding new patents for the production of new goods employing the more productive technologies, then any time a new patent is produced and implemented the value of b_i increases. As b_i increases, the final sector in aggregate

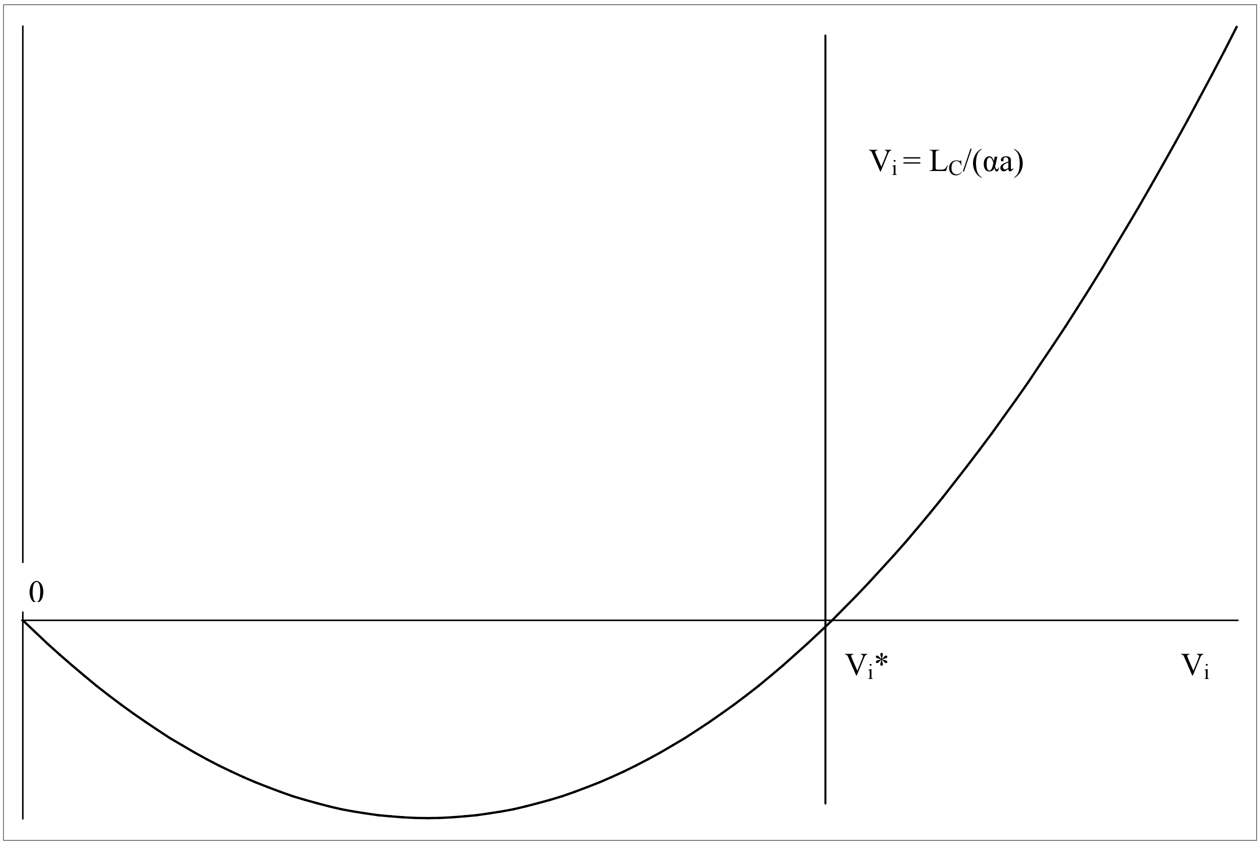

Figure 1

becomes more productive and, therefore, more workers are made available to be employed in the R&D sector. Moreover, the growth rate of new varieties increases as it is shown by the following expression

$$g_i = g = \frac{L_R}{a} = \frac{Lb_i(1-\alpha)/a - \rho\alpha}{(1-\alpha)b_i + \alpha} \tag{37}$$

The growth rate g is superiorly and inferiorly limited because $0 \leq b_i \leq 1$.

In general, the model explains structural changes by means of workers' distribution movements between the two sectors. Indeed, changes in L_R (and L_C) reflect changes in b_i, which are the results of product and process innovations. We know that as long as new patents are produced by means of product innovations, b_i continues to increase over time implying a continuous shift of workers from the sector in which final consumption goods are produced to the innovative sector, with an increasing value of g_i. However, once there is a process innovation which reduces γ_i, changes in b_i are more complex and they explain structural changes of different nature, which may end up also with workers shifted from the innovative sector towards the sector in which consumption goods are produced if b_i for the new type of varieties is larger than it was for previous varieties on the frontier.

Particularly, we may state that there is a redistribution of workers from the innovative (final good) toward the final good (innovative) sector when the value which b_i takes once the process innovation takes place is smaller (larger) than its value for previous varieties on the frontier. In Appendix A we show that when process innovations are relatively not too big, b_i decreases after process innovations take place with workers moving from the innovative sector to the final good sector and, as a consequence, the growth rate of patents decreases.

Once the process innovation has taken place, as long as there are further innovations which increase the number of patents with the same value of γ, workers move from the final to the innovative sector. They are induced to move again to the final sector, once a subsequent process innovation of limited impact takes place.

Regarding the scale effect, we notice that it would still be present in this work if we had not introduced the assumption in (19) that increases in L may produce process innovations. These continuous subsequent process innovations due to increases in L may contribute to continuously lowering the value of b_i and keeping g from increasing.

In particular, this could not happen as long as subsequent patent innovations are related to varieties characterized by the same value of γ. In fact, partially differentiating (36) with respect to L_R, L and b_i, after few steps we obtain

$$\frac{dL_R}{L_R} = \frac{(1-\alpha)\, b_i}{(Lb_i\,(1-\alpha) - a\rho\alpha)} \left(L\frac{dL}{L} + \frac{\alpha\,(L + a\rho)}{((1-\alpha)\, b_i + \alpha)} \frac{db_i}{b_i} \right) \tag{38}$$

From (38) we know that L_R, and consequently g, would be constant only if

$$\frac{db_i}{b_i} = -\frac{L\,((1-\alpha)\, b_i + \alpha)}{\alpha\,(L + a\rho)} \frac{dL}{L} \equiv b^* < 0 \tag{39}$$

This is never the case when varieties of type i remain along the technological frontier given that b_i would continuously increase over time and, therefore, db_i/b_i can only be positive.

However, when L increases, continuous process innovations could continuously lower b_i. If these two effects on b_i balance each other, b_i will be constant implying that L_R, in turn, is constant with no change in the growth rate of the number of varieties. In appendix B we show that this would imply a constant growth of the real gross domestic product (GDP).

Therefore, we may conclude that when process innovations are associated to product innovations, we can obtain equilibrium paths characterized by a stable distribution of workers between the two sectors, which corresponds to a fixed growth rate, provided that b_i continuously decreases over time due to subsequent process innovations.

6 Conclusion

Scholars in the field of international economics and economic growth have devoted great attention to the subject of heterogeneity of firms in the last few years. Productivity differences across firms are, for instance, analyzed in a general equilibrium framework by Melitz (2003) which analyzes the effects of international trade on intra-industry reallocation of firms and on aggregate industry productivity.

Our paper adopts a growth approach to explain how heterogeneous firms producing in a particular period of time are the result of subsequent waves of process innovations which allow more recent firms to produce using more productive techniques. The contemporaneous production of firms characterized by different productivity levels results in a variety of prices set by firms which reflect productivity differences. The latter are also responsible for patents' price differences, given that patent prices of more profitable varieties are higher. Moreover, demand and market shares of older less efficient firms decrease over time as long as new patents, which allow the production of new goods along the technological frontier, are produced in the innovative sector.

In this work we assume that old firms are unable to implement the more productive production processes due to high switching or implementing costs required to adopt the new production processes. However, demand for these firms is still positive given the Dixit and Stiglitz approach, which postulates love for variety in consumption. The assumption of goods which are imperfect substitutes together with that of productivity heterogeneity results in different equilibrium prices for different varieties. Moreover, our results suggest that policy intervention may have a role given that specific policies could be set in order to reduce switching costs when they are particularly high in order to implement a redistribution of the production activity toward the innovative sector with an associated higher rate of innovation and of growth of the overall economy.

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Appendix A

As in the text, we define h in such a way that $m = 1, 2, ..., (h = i)$. Once there is an improvement along the learning curve described by (19), the series continues in the following way: $m = 1, 2, ..., h, (h + 1 = i)$. In this appendix we show when process innovations which increase the value of h as defined above, end up with a smaller (higher) value of b_i. In other words, we show when b_h is higher (lower) than b_{h+1}.

We know from the definition (26) that

$$b_h \equiv \frac{n_h \gamma_h^{1-\sigma}}{\sum_{j=1}^{h} n_j \gamma_j^{1-\sigma}} \quad \text{and} \quad b_{h+1} \equiv \frac{n_{h+1} \gamma_{h+1}^{1-\sigma}}{\sum_{j=1}^{h+1} n_j \gamma_j^{1-\sigma}}$$

Hence, we derive that $b_h > b_{h+1}$ when

$$n_h \gamma_h^{1-\sigma} \sum_{j=1}^{h+1} n_j \gamma_j^{1-\sigma} > n_{h+1} \gamma_{h+1}^{1-\sigma} \sum_{j=1}^{h} n_j \gamma_j^{1-\sigma}$$

or, equivalently, when

$$n_h \gamma_h^{1-\sigma} \sum_{j=1}^{h} n_j \gamma_j^{1-\sigma} + n_h \gamma_h^{1-\sigma} n_{h+1} \gamma_{h+1}^{1-\sigma} - n_{h+1} \gamma_{h+1}^{1-\sigma} \sum_{j=1}^{h} n_j \gamma_j^{1-\sigma} > 0 \tag{40}$$

Expression (40) is true when

$$\frac{\sum_{j=1}^{h} n_j \gamma_j^{1-\sigma}}{\sum_{j=1}^{h-1} n_j \gamma_j^{1-\sigma}} > \frac{n_{h+1} \gamma_{h+1}^{1-\sigma}}{n_h \gamma_h^{1-\sigma}}$$

We substitute n_{h+1} from (21) and we obtain

$$l \equiv \frac{\sum_{j=1}^{h} n_j \gamma_j^{1-\sigma}}{\sum_{j=1}^{h-1} n_j \gamma_j^{1-\sigma}} > \frac{L_R \gamma_{h+1}^{1-\sigma}}{a \gamma_h^{1-\sigma}} = \frac{L_R}{a} \left(\frac{\gamma_h}{\gamma_{h+1}} \right)^{\sigma-1}$$

where the left term in the inequality, l, is always larger than 1. Therefore, given that $\gamma_{h+1} < \gamma_h$, we may at least state that $b_h > b_{h+1}$ is true, when $\frac{L_R}{a} \left(\frac{\gamma_h}{\gamma_{h+1}} \right)^{\sigma-1} < l$. That is when

$$1 < \left(\frac{\gamma_h}{\gamma_{h+1}} \right)^{\sigma-1} < \frac{a}{L_R} l \tag{41}$$

Expression (41) says that when the process innovation produces a reduction in γ which is not relatively high, then b_i decreases.

Appendix B

Following Grossman and Helpman (1991, p. 63) we define the index of the manufactured output

$$D \equiv \left(\sum_{m=1}^{i} n_m x_m^{\alpha} \right)^{\frac{1}{\alpha}}$$

where $\alpha = \frac{\sigma-1}{\sigma}$, while the ideal price index of final goods is p_D.

The gross domestic product (GDP), G, is defined as the sum of the value added in manufacturing and in the R&D sector

$$G \equiv p_D D + v_i \dot{n}_i$$

We know from Grossman and Helpman (1991, p. 63) that the growth of the real GDP is equal to a weigthed average of the growth rates of the manufactured good index, g_D, and of the research output, g_i, with weights given by sector's value shares. In particular, the manufactured goods share is given by $\theta_D \equiv p_D D / (p_D D + v_i \dot{n}_i)$. Thus the growth rate of the real GDP is

$$g_G = \theta_D g_D + (1 - \theta_D) g_i$$

We need to compute g_D for a given value of b_i.

Using (29), (26) and (17), we rewrite D as follows

$$\begin{aligned}
D^{\alpha} &\equiv \sum_{m=1}^{i} n_m \left(\frac{\alpha b_m}{n_m \gamma_m} \right)^{\alpha} w^{-\alpha} = \sum_{m=1}^{i} n_m \left(\frac{\alpha \frac{n_m \gamma_m^{1-\sigma}}{\sum_{j=1}^{i} n_j \gamma_j^{1-\sigma}}}{n_m \gamma_m} \right)^{\alpha} w^{-\alpha} = \\
&= \sum_{m=1}^{i} n_m \left(\alpha \frac{\gamma_m^{-\sigma}}{\sum_{j=1}^{i} n_j \gamma_j^{1-\sigma}} \right)^{\frac{\sigma-1}{\sigma}} w^{-\alpha} = \frac{\alpha^{\frac{\sigma-1}{\sigma}} \sum_{m=1}^{i} n_m \gamma_m^{1-\sigma}}{\left(\sum_{j=1}^{i} n_j \gamma_j^{1-\sigma} \right)^{\frac{\sigma-1}{\sigma}}} w^{-\alpha} = \\
&= \alpha^{\alpha} \left(\sum_{m=1}^{i} n_m \gamma_m^{1-\sigma} \right)^{\frac{1}{\sigma}} w^{-\alpha} = \alpha^{\alpha} \left(\sum_{m=1}^{i} n_m \gamma_m^{1-\sigma} \right)^{1-\alpha} w^{-\alpha}
\end{aligned}$$

Therefore

$$D^{\frac{\alpha}{1-\alpha}} \equiv \alpha^{\frac{\alpha}{1-\alpha}} \left(\sum_{m=1}^{i} n_m \gamma_m^{1-\sigma} \right) w^{-\frac{\alpha}{1-\alpha}}$$

and totally differentiating the previous expression, we obtain

$$\begin{aligned}
\frac{\alpha}{1-\alpha} D^{\frac{\alpha}{1-\alpha}-1} \dot{D} &= -\frac{\alpha}{1-\alpha} \alpha^{\frac{\alpha}{1-\alpha}} w^{-\frac{\alpha}{1-\alpha}-1} \dot{w} \left(\sum_{m=1}^{i} n_m \gamma_m^{1-\sigma} \right) + \alpha^{\frac{\alpha}{1-\alpha}} w^{-\frac{\alpha}{1-\alpha}} \gamma_i^{1-\sigma} \dot{n}_i \\
g_D &= -\hat{w} + \frac{1-\alpha}{\alpha} \frac{n_i \gamma_i^{1-\sigma}}{\left(\sum_{m=1}^{i} n_m \gamma_m^{1-\sigma} \right)} g_i \\
g_D &= -\hat{w} + \frac{(1-\alpha)}{\alpha} b_i g_i
\end{aligned}$$

We know from (17) that

$$\hat{w} = -\hat{V}_i$$

Since for any given value of b_i we know that $\hat{V}_i = 0$, we derive that

$$g_D = \frac{(1-\alpha)}{\alpha} b_i g_i$$

Moreover, given our normalization for manufacturing expenditure, we know that $E = p_D D = 1$ and

$$\theta_D = \frac{1}{1 + \frac{1}{V_i} g_i} \tag{42}$$

Expression (42) tells us that the manufactured goods share, θ_D, is constant if V_i is constant. We know that V_i is constant only if b_i does not change. Consequently, the real GDP growths at the following rate

$$g_G = \left[\theta_D \frac{(1-\alpha)}{\alpha} b_i + (1 - \theta_D)\right] g_i$$

which is constant when b_i is constant, given that we know from (37) that also g_i is constant.