R&D activity and the dynamic equilibrium. Section 3 contains some concluding remarks.

2 The model

2.1 Preferences

Consider an economy with $\bar{L}$ identical households and differentiated goods produced in N_m varieties, $[x_i]_{i=1}^{N_m}$. The representative household maximizes its lifetime utility:

$$U(t_0) = \int_{t_0}^{\infty} e^{-\rho(t-t_0)} \ln u(t)dt \tag{1}$$

subject to the intertemporal budget constraint that the present discounted value of expenditure cannot be greater than the present discounted value of lifetime labour income, plus initial wealth:

$$\int_{t_0}^{\infty} R(t)Y(t)dt \leq A(t_0) + \int_{t_0}^{\infty} R(t)w(t)dt \tag{2}$$

where $\rho > 0$ is the individual discount rate, $R(t) = e^{-\int_{t_0}^{t} r(s)ds}$ is the cumulative discount factor, Y is nominal per capita expenditure, and A is initial wealth. The typical household takes the path of wages and the interest rate as given. Throughout the analysis, wage is the numéraire.

Preferences are identical for all consumers. We assume that there is a large number of varieties, all of which enter symmetrically into the instantaneous utility function $u(t)$, which we assume to be of the Dixit-Stiglitz type[3]:

$$u = \left(\sum_{i=1}^{N_m} x_i^{\beta} \right)^{\frac{1}{\beta}} \tag{3}$$

where x_i is the consumption of each variety and $0 < \beta < 1$. As is well known, this specification has proved to be the most tractable when product differentiation is the main concern. The love for variety could alternatively be modelled in a slightly different framework, by extending preferences over a continuous product space and assuming that at any given moment in time only a subset of potential varieties are available (Grossman and Helpman, 1989; Krugman, 1980). Over time, innovation can expand this subset, and $N_m(t)$ is the number of varieties at time t. This utility function implies constant elasticity of substitution between any couple of varieties:

$$\sigma = \frac{1}{1-\beta} > 1 \tag{4}$$

[3] In the rest of the paper the time variable, t, is suppressed.

The typical solution of this problem is in two stages. By Euler equation, we first get the optimal dynamic expenditure path:

$$\frac{\dot{Y}}{Y} = r - \rho \tag{5}$$

which also defines optimal saving behavior. Then, by taking the time-path of expenditure as given, we solve the static household maximization problem at any t, i.e. the maximization of u subject to $Y = \sum_{i=1}^{N_m} p_i x_i$.

The h-th household demand for variety i (where $i \in [1, N_m]$) is

$$x_{ih}(p_i) = \frac{Y}{q}\left(\frac{p_i}{q}\right)^{-\sigma} \tag{6}$$

where p_i is the market price of the i-th brand, and q is the 'dual' price index:

$$q = \left[\sum_{i=1}^{N_m} p_i^{1-\sigma}\right]^{\frac{1}{1-\sigma}} \tag{7}$$

Aggregating over the $\bar{L}$ identical consumers, we obtain the demand schedule faced by firms producing the i-th brand:

$$x_i(p_i) = \bar{L}\frac{Y}{q}\left(\frac{p_i}{q}\right)^{-\sigma} \tag{8}$$

Equation (8) is used in the analysis of firms' price-setting behaviour. Since we shall be interested in quantity competition between firms, we consider the corresponding inverse demand function, along the lines suggested by Spence (1976):

$$p_i(x_i) = \bar{L}Y\frac{x_i^{\beta-1}}{Q^{\beta}} \tag{9}$$

where p_i is the market price of the i-th variety, x_i is the aggregate production of the i-th sector, and Q is the economy quantity index given by:

$$Q = \left[\sum_{i=1}^{N_m} x_i^{\beta}\right]^{\frac{1}{\beta}} \tag{10}$$

Notice the immediate interpretation of β in terms of the structure of preferences. The parameter is an indicator of the degree of substitutability between any two differentiated brands: the lower is β, the lower is the inter-variety interdependence. As $\beta \to 0$, the degree of substitution reaches the maximum level (i.e. $\sigma \to 1$) of product differentiation. As $\beta \to 1$, there is no differentiation, the degree of substitutability becomes infinite (i.e. $\sigma \longrightarrow \infty$) and any brand is perfectly substitutable with any other. Clearly, the demand function given in

(8) or (9) encompasses both traditional formulations of oligopoly with a homogeneous good and of monopolistic competition. In order to verify it, it is enough to look at the inverse demand function under the symmetry assumption ($\forall i,j$ $x_i = x_j = \frac{X}{N_m}$ and $p_i = p_j = p$). In this case, for $\beta \to 0$ each sectorial demand is defined independently of the others, and (8) or (9) describes a monopolistic inter-sector competition; for $\beta \to 1$, we obtain a set-up with homogeneous product.

2.2 Technology

On the production side, firms undertake two activities. First, they produce the existing varieties, and second they can divert resources to investment in R&D in order to create new designs. While it is generally assumed that each variety is produced by a single firm, in the sequel we assume that each variety can be manufactured by N competing firms.

This assumption can be justified in different ways. First, the innovative brand may be not patentable; because its inventor has difficulties in applying the patents' law in order to prevent unauthorized use; or, furthermore, because the product innovation is not a real invention but, simply, a combination of existing varieties. In this latter interpretation the new product may indeed look new to the consumers, but being not really original, is not patentable. In other words, inventors might be unable to exclude others from making free use of their ideas, because of difficulties in the definition of property rights in the innovation. Second, especially when international trade is allowed for, similar varieties could indeed exhibit many overlapping characteristics (the case of the automobile sector provides clear examples in this respect), and this creates an environment where a (nearly) identical product is offered by many firms. Finally, we recall that Grossman and Helpman (1991) exclude any incentive to imitation on the basis that an intra-sector Bertrand price competition would immediately lead profits to zero, so that the copier would not be able to recoup the positive cost of imitation. Their argument is clearly based on the idea that firms compete in prices. But if we imagine a different types of competition, the scope for imitation may indeed arise. If the intra-sector competition is consistent with a positive mark-up over marginal costs, the imitation costs can be covered and different firms can find it convenient to produce the same homogeneous good.

Since there are N_m varieties, each of them produced by N firms, each firm simultaneously faces two different competitive environments. Horizontally, at the inter-sector level each firm competes with other firms producing an imperfect substitute of its own product. Also, it competes with other firms producing a homogeneous product at the intra-sector level.[4] Therefore, there is an inter-sector competition (i.e. between different varieties) of the standard monopolistic

[4] Notice that also in Grossman and Helpman (1991) there is a schematic discussion of possible forms of intra-sector competition. In particular they suggest that the research labs could be involved in quality improvements of existing varieties, so that intra-sector competition may turn to vertical product differentiation.

type, and an intra-sector competition (within the same variety). As suggested above, we assume that the latter is in quantities, so that the market for each variety can be thought of as a traditional Cournot oligopoly.

The j-th firm ($j \in [1, N]$) operating in the i-th sector, is single-product. Each good can be produced through labour according to the linear technology:

$$z_{ij}(L_{ij}) = L_{ij} \tag{11}$$

where L_{ij} is the amount of labour employed in the i-th sector by the j-th firm, and z_{ij} is the firm's output. Hence for the j-th firm, the cost function is $C(z_{ij}) = z_{ij}$ (remember that the wage, w, is the numéraire). Obviously, the aggregate production for the i-th sector is:

$$x_i = \left(\sum_{j=1}^{N} z_{ij} \right) \tag{12}$$

while the economy aggregate production is:

$$X = \left(\sum_{i=1}^{N_n} x_i \right) \tag{13}$$

Therefore, the number of workers employed is:

$$L_X = \left[\sum_{i=1}^{N_n} \left(\sum_{j=1}^{N} L_{ij} \right) \right] = X \tag{14}$$

Each firm chooses its production level in order to maximize profits:

$$\pi_{ij} = p_i(x_i) z_{ij} - z_{ij} \tag{15}$$

Notice that, given the large number of existing varieties, each firm in each sector perceives the economy quantity index as given. In turn, this implies that the negligibility assumption holds: each firm perceives its own level of production, z_{ij}, as irrelevant with respect to the aggregate production of the other sectors, and of the economy as a whole, Q. It is the large number of available varieties which allows for the inter-sector monopolistic competition. On the contrary, in the intra-sector perspective each firm takes into account the effect of its own production on the aggregate sectorial production level. This latter competition is based upon Cournot conjectures.

Substituting (9) into (15), and using (12), we can rewrite profits in terms of individual quantity as

$$\pi_{ij} = \bar{L}Y \frac{\left[\sum_{j=1} z_{ij} \right]^{\beta - 1} z_{ij}}{Q^{\beta}} - z_{ij} \tag{16}$$

The first order condition under Cournot conjectures for any given levels of z_{hk}, $h \neq i$ and $k \neq j$, is

$$\frac{\partial \pi_{ij}}{\partial z_{ij}} = 0 \iff \bar{L}\frac{Y}{Q^{\beta}}\left[(\beta - 1)\left(\sum_{j=1} z_{ij}\right)^{\beta - 2} z_{ij} + \left(\sum_{j=1} z_{ij}\right)^{\beta - 1}\right] - 1 = 0 \quad (17)$$

Under symmetry $z_{ij} = z \ \forall \ i, j$, and the Nash equilibrium level of individual output is:

$$z^{*} = \bar{L}Y\frac{(\beta - 1 + N)}{N^{2}N_{m}} \quad (18)$$

Sectorial aggregate production is, from (12):

$$x^{*} = \bar{L}Y\frac{(\beta - 1 + N)}{NN_{m}}. \quad (19)$$

The related market price is given by (9):

$$p^{*} = \frac{N}{\beta - 1 + N}. \quad (20)$$

Therefore the equilibrium level of profits is:

$$\pi^{*} = \bar{L}Y\frac{1 - \beta}{N^{2}N_{m}} \quad (21)$$

Notice that the optimal quantity produced by any firm is inversely proportional to the number of existing varieties, N_m. The same holds for profits and sectorial aggregate production, while the price level is independent of N_m. Notice, also, the (expected) result about the influence of the degree of substitutability. For a low level of β, inter-sector competition is less fierce because of the low interdependence level among sectors. Table 1 synthetizes the equilibrium outcome under these two extreme configurations on inter-sector competition.

Table 1 about here

At the intra-sectorial level, a simple indicator of the degree of competition is the firm's sectorial market share. Profits and prices are directly related with the sectorial market power in the admitted range, $s = \frac{1}{N} \in \,]0, 1]$. In this respect, a large number of firms in the sector (i.e. $N \to \infty$) means that no limits to imitation exist. Free entry implies a negligible share (i.e. $s \to 0$) and the intra-sector competition resembles perfect competition: prices equal marginal costs and profits are driven to zero. On the contrary, when a strict patent system prevents imitation and unauthorized entry into the sector, a single firm supplies the entire sector ($s = 1$), i.e. each variety is produced by only one firm ($N = 1$), which behaves like a monopolist. This situation collapses to that described by the Grossman and Helpman model, where the intra-sector competition is absent: prices and profits are higher, while aggregate production level is lower. Table 2 synthetizes the extreme configuration of intra-sector competition.

Table 2 about here

It must be stressed that the market share s can be interpreted in two different ways. It is an index of the degree of competition of market structure, but it can also be seen as an indicator of the degree of enforcement both of patent law and deterrence of imitation . In this respect, the extreme values, $s \to 0$ and $s = 1$ arise under perfect competition (absence of patents) and monopoly power (perfect patents). For intermediate values, in the range $]0, 1[$, we have some strategic interaction: the higher the value of s, the lower the degree of competition and the higher the level of protection provided by.

Finally, we recall that if the intra-sector competition were à la Bertrand, we would have the competitive price (because of homogeneity), independently of the properties of the inter-sectorial competition.

2.3 Research & Development

Following Grossman and Helpman (1991) and Lucas (1998) among others, we assume that the production of new varieties takes place according to the innovation function:

$$\frac{\partial N_m}{\partial t} = \dot{N}_m = \frac{1}{a} L_R k(t) \tag{22}$$

where a is a positive parameter, L_R is the number of workers employed in R&D and $k(t)$ is the stock of knowledge at time t. Equation (22) is the most common formulation of R&D technology in the endogenous growth literature: it shows a positive relation between the development of new varieties and the stock of available knowledge at each moment in time.

Since the number of varieties changes over time, the stock of knowledge is not constant; rather, it depends, in a proportional way, on the number of existing varieties. This can be justified in terms of learning by doing: each innovation, by increasing the level of knowledge, makes R&D more productive.

The simplest function linking the stock of knowledge to the number of varieties is the linear one:

$$k(t) = N_m. \tag{23}$$

On the basis of the R&D technology, the cost of the creation of a new variety is:

$$I_v(t) = \frac{a}{k(t)} = \frac{a}{N_m}. \tag{24}$$

Making use of (23), equation (22) determines the endogenous growth rate:

$$g = \frac{\dot{N}_m}{N_m} = \hat{N}_m = \frac{1}{a} L_R. \tag{25}$$

Assuming free entry in R&D activity, the present value of profits for any variety discovered at time t must be equal to its cost of creation:

$$V(t_0) = \int_{t_0}^{\infty} R(t)\pi(t)dt = \frac{a}{N_m}. \tag{26}$$

At each moment in time the Fisher equation must hold: the current profit plus the rate of capital gain must be equal to the value of profitable capital investment:

$$\pi_t + \dot{V}_t = rV_t. \tag{27}$$

Suppressing the time notation, and expressing (27) in proportional terms, we have:

$$\frac{\pi}{V} + \frac{\dot{V}}{V} = r. \tag{28}$$

The rate of profit is given by $\frac{\pi}{V} = \bar{L}Y\frac{(1-\beta)}{aN^2}$, while the percentage change in the present value of profits is $\hat{V} = \frac{\dot{V}}{V} = -\hat{N}_m = -g$, so that the Fisher equation gives:

$$r = \bar{L}Y\frac{(1-\beta)}{aN^2} - g, \tag{29}$$

which establishes, as usual, the equality between the interest rate and the dividend rate, plus the capital gain rate.

By using the labour market clearing condition, the amount of available labour is allocated between the two activities: L_X for production and L_R for R&D. If the supply of labour is fixed at the level $\bar{L}$, we have:

$$\bar{L} = L_R + L_X \tag{30}$$

Equation (30) can now be rewritten, by making use of (25), together with (13), (14) and (19); by performing these substitutions, the growth rate is expressed by:

$$g = \frac{\bar{L}}{a}\left(1 - Y\frac{(\beta - 1 + N)}{N}\right) \tag{31}$$

Since new varieties are produced through the residual workers not employed in production, the number of varieties increases over time at the same rate g at which production in each sector decreases. Assuming full employment, the constraint on labour resources given by equation (31) must be always satisfied. The higher is the rate of innovation, the greater is the employment in R&D and the lower the number of workers left for manufacturing.

2.4 Dynamics

The general equilibrium is described by the equations (5), (29) and (31). By substituting (31) into (29) for g we get

$$r = \frac{\bar{L}}{a}\left[Y^{-1}\left(s^2(1-\beta)-s(1-\beta)+1\right)-1\right] \tag{32}$$

which in turn can be substituted for r into (5), in order to obtain the following dynamic equation in Y (where we have used the definition of s)

$$\dot{Y} = Y^2\left(\frac{\bar{L}}{a}\left[s^2(1-\beta)-s(1-\beta)+1\right]\right)-Y\left(\frac{\bar{L}+a\rho}{a}\right). \tag{33}$$

This Bernoullian equation has two steady state solutions. The graph of $\dot{Y}$ (figure 1) cuts the horizontal axis twice at the origin, and also at Y^{SS}. The first solution, stable, is $Y=0$. The second, unstable, is

$$Y^{SS} = \frac{a\rho+\bar{L}}{\bar{L}}\frac{1}{s^2(1-\beta)-s(1-\beta)+1}. \tag{34}$$

The qualitative properties can be described by a phase line graph.

Figure 1 about here

For all values of Y within the interval $\left[0, Y^{SS}\right]$, expenditure must be decreasing, indicating that $\dot{Y}<0$; this is why the arrows in this interval. For values of $Y > Y^{ss}$ the opposite holds, $\dot{Y}>0$ and expenditure increases

While stable, the first solution ($Y=0$) is economically meaningless. If in the long run the aggregate expenditure approaches zero, the rate of innovation reaches its maximum value; in this situation the entire supply of labour is employed in R&D and there is no production activity. However in this case, the number of products would be growing continuously at the positive rate $g=\frac{\bar{L}}{a}$, expenditure and profits would approach to zero, but the arbitrage condition would be violated: the present value of profit would be lower than the positive entry cost. The second solution, though economically meaningful, is unstable. Therefore we must impose stability, by assuming that starting from any initial value Y, being a non-predetermined variable, jumps instantaneously to Y^{SS}[5]. In the steady state, $r=\rho$: constant household's expenditure must involve a constant interest rate, equal to the subjective discount rate.[6]

[5] Even though it would seem questionable in Industrial Organization literature, it is common in the endogenous growth models

[6] The assumption that Y jumps to its steady state equilibrium (other than $Y=0$) is the equivalent, in this set-up, of Grossman and Helpman hypothesis that nominal expenditure is normalized and constant to one.

We now substitute the steady state equilibrium value Y^{SS} and r into the Fisher equation, in order to obtain the steady state solution for the growth rate[7]:

$$g^{SS} = \frac{\rho a\left[s\left(1-\beta\right)-1\right] + \bar{L}(1-\beta)s^2}{s^2(1-\beta) - s(1-\beta)+1}\frac{1}{a}. \quad (35)$$

In order to analyze the properties of the steady state solution it is useful to see it explicitly as the simultaneous solution (for Y and g) of the labour market clearing condition (31) and the free entry condition in R&D given by the Fisher equation (29), the latter evaluated at $r = \rho$[8]

$$g = \bar{L}Y\frac{(1-\beta)}{a}s^2 - \rho \quad (36)$$

$$g = \frac{1}{a}\bar{L}\left(1 + Y\left[s\left(1-\beta\right)-1\right]\right) \quad (37)$$

These two linear equations are represented in figure 2.

Figure 2 about here

Equation (36) is positively sloped in the (Y, g) plane while equation (37) is negatively sloped since $0 < s \leq 1$ and $\beta < 1$. The intersection between the two equations gives us the same steady state equilibrium values (34) and (35) for Y and g. This intersection point is one where the division of labour resources, between the two activities, production and R&D, remains constant over time. In this respect, the rate of product development exactly matches the decline of entry cost and the innovation would continue at a fixed rate g^{SS}.

Now we are in a position to evaluate the relationship between the degree of competition of market structure, captured by the market share, and the growth rate. Considering the position of two equations, the higher s is (i.e. the lower the degree of competition is), the greater is the (positive) slope of the first equation (36), and the smaller is the (negative) slope in (37). The total effect is an higher growth rate. On the contrary, when the degree of competition is high (i.e. the level for s is low), the resulting growth rate is lower.

[7] Notice that for the admitted range of parameter s and β, the denominator of (35) is positive, while for the numerator we consider values for the resource base ($\bar{L}$), the productivity in reserch lab (a) and the households' patient (ρ) in order to have a meaningful (positive) growth rate. Violating these specifications, economy immediately jumps a stationary state without innovation. In this case may be that R&D will not be profitable or productivity of researchers is inefficient; the resource base are insufficient or households do not value wide enough variety in consumption. (also see: Grossman and Helpman (1991); and Romer (1990) for insufficient endowment of the resources needed for industrial research.)

[8] Obviously, for $r = \rho$ expenditure is constant over time; in this respect this level of the interest rate is the sole compatible with constant expenditure.

The interpretation of this relationship is straightforward. Suppose the lowest level for s, i.e. $s \to 0$, the firms market share is negligible and the intra-sector market tends to be highly competitive. The equality between price and marginal cost implies the highest production level: more workers are employed in production, leaving few resources for R&D activity and, obviously, a lower growth rate.

On the contrary, when the market share is maximum, no firm has direct competitors: setting the monopoly price, more workers are available for R&D activities, and so, the growth rate raises. Furthermore, a lower degree of interdependence among firms leads to a higher level of profits, giving more incentives to innovation activities.

The relationship between the growth rate and market share can also be analyzed by evaluating the derivative of g^{SS} in (35) with respect to s:

$$\frac{\partial g^{SS}}{\partial s} = \frac{[2 - s(1-\beta)][s(1-\beta)]\left[\rho + \frac{\bar{L}}{a}\right]}{[s^2(1-\beta) - s(1-\beta) + 1]^2} > 0.$$

Sustained innovations should be possible for $s \neq 0$. This means that a positive growth rate results if, and only if, some intellectual property rights prevent the free use of innovations.

Note that when $s = 1$, there is only one firm per sector and this implies the traditional Grossman-Helpman result:

$$g^{SS}_{G-H} = \frac{\bar{L}}{a}(1-\beta) - \rho\beta. \tag{38}$$

In their formulation the parameter β plays a fundamental role with respect to the product differentiation degree. The lower it is, the lower is the level of substitutability, σ, between goods, and the higher is the profits level; thus the grows rate rises. On the other hand, when the product differentiation is minimum (i.e. $\beta \to 1$), the varieties are perfectly substitutable among themselves and profits are driven to zero. Now, there is no incentive for innovation because zero profits do not permit the recovery of the positive cost of R&D. The same situation occurs when no patents exist ($s \to 0$).

3 Conclusions

The contribution of this paper can be found in its analysis of two market structures, usually considered as alternative types of competition.

In recent attempts to involve the strategic interaction in endogenous growth models an ambiguous relationship emerges between the growth rate and the degree of competition of market structure.

Here, denoting the degree of competition by the intra-sector market share, the main result of the paper concerns its unambiguous effect on the growth rate. When the degree of competition is high, prices go down, the aggregate quantity raises and the available labor force for R&D activity are reduced, so the growth