Chapter I

Introduction

The general aim of this work is concerned with quantitative estimates of the convergence of suitable sequences of linear operators to the solution of evolution problems of the form

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = Au(\cdot,t)u(x) \ , \\ u(x,0) = u_0(x) \ , \end{cases} \tag{I.1}$$

where $A : D(A) \longrightarrow X$ is a linear second-order differential operator on some Banach space X, independent of the time variable. Possible boundary conditions of problem (I.1) are included in the domain of A. In most cases we shall have a second-order (possibly degenerate) differential operator

$$Au(x) = \alpha(x)u''(x) + \beta(x)u'(x) \ , \qquad x \in I \ , \tag{I.2}$$

where $I :=]r_1, r_2[$ is a real interval and $\alpha, \beta : I \to \mathbb{R}$ are continuous functions with $\alpha(x) > 0$ for every $x \in I$.

It is well-known that the solution of this problem can be represented using the semigroup theory; namely, if the operator A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on a suitable domain $D(A)$ and $u_0 \in D(A)$ the solution of the preceding problem is given by

$$u(t,x) = (T(t)u_0)(x) \ , \qquad t \geq 0 \ . \tag{I.3}$$

This approach motivated Feller to study the general conditions (that are, the domains) under which the operator A generates a C_0-semigroup and in 1952 he completely characterized this property both on the space of real continuous functions than of integrable functions on a real interval (see [52, 54, 53]). After these papers many questions arose concerning with representation (I.3). In particular, the generation of a C_0-semigroup has been deeply investigated on an assigned domain. Necessary and sufficient conditions in order for A to be the generator of a C_0-semigroup in $C(\overline{I})$

have been given by Clément and Timmermans [44] when Ventcel's boundary conditions are imposed at the endpoints, and by Timmermans in [69] on the maximal domain. More recently, the existence of a C_0-semigroup has been characterized in [29] also in the case of Neumann's type boundary conditions at the endpoints. In the space $L^1(I)$ the characterization of the generation of a C_0-semigroup has been completely achieved in [19] on the adjoint maximal domain, on the adjoint Dirichlet domain and on the adjoint Neumann domain. All the results obtained in [44, 69, 29, 19] are very closely related to the pioneer work by Feller [54]; in the preliminary section we give a brief exposition of them.

Another field of interest which was immediately developed after the work of Feller was the possibility of approximating the semigroup $(T(t))_{t\geq 0}$ generated by $(A, D(A))$. In this respect, starting from usual numeric methods for computing the solution of a partial differential equation [57], in 1958 Trotter [70] provided some important results which have been revealed as the main tools for the investigation and the approximation of the solution of (I.1) until today. More precisely, the idea of Trotter was based on the construction of an implicit Euler scheme for the approximation of the semigroup $(T(t))_{t\geq 0}$. He treated the question of convergence of suitable difference operators in an operator-theoretic fashion, in which semigroup theory give a fundamental contribution. The main idea consist in replacing the differential operator with approximating operators, and taking the solution of the resulting equation as an approximation to the solution of the original equation. Namely, the process is made through a discretization in time and space: to be more precise, for each $n \in \mathbb{N}$, let be h_n the time-size, and S_n a spatial grid; the function $u_n(kh_n, x)$ defined on S_n represents the approximation of the solution of (I.1) at the time $t_k = kh_n$, and can be defined recursively in terms of a linear operator L_n

$$\begin{cases} u_n(t_k, x) = L_n u_n\,(t_{k-1}, x) \\ u_n(0, x) = f_n(x)\ , \end{cases} \tag{I.4}$$

note that $u_n(t_k, x) = L_n^k f(x)$.

In this scheme we approximate the operator A with $A_n = \frac{L_n u - u}{h_n}$, and consequently, if we require that A_n converges to A in some suitable sense, the solution $u(t, x)$ of (I.1) will be approximated by $L_n^{[th_n^{-1}]} f(x)$.

Thus, under suitable assumptions, Trotter obtained a sequence $(L_n)_{n\geq 1}$ of linear operators satisfying

$$\lim_{n\to+\infty} L_n^{k(n)} = T(t)$$

whenever $\lim_{n\to+\infty} k(n)/n = t$ (see [70, Theorem 5.1] and Chapter II).

As an immediate consequence of this result we can represent the solution of problem (I.1) as follows

$$u(t, x) = \lim_{n\to+\infty} L_n^{k(n)}(u_0)(x)\ , \qquad t \geq 0\ .$$

This last representation opened the possibility of approximating the solution of parabolic problems through positive operators.

Among the main applications, the parabolic problems under consideration were concerned with diffusion models in population genetics and models in mathematical finance. Diffusion models in population genetics and in particular Wright-Fisher models were indeed the starting point of the investigation of Feller. The evolution problem (I.1) corresponding to this model in absence of selection, mutation and migration becomes

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \dfrac{x(1-x)}{2}\,\dfrac{\partial^2 u}{\partial x^2}(x,t)\,, & \\ \qquad\qquad\qquad\qquad\qquad\qquad t>0\,, & 0<x<1\,, \\ u(x,0) = u_0(x)\,, & 0 \le x \le 1\,. \end{cases} \tag{I.5}$$

Karlin and Ziegler [56] showed that the solution of the preceding problem can be expressed in terms of iterates of the classical Bernstein operators

$$B_n f(x) = \sum_{k=0}^{n} \binom{n}{k} x^k\,(1-x)^{n-k} f\left(\frac{k}{n}\right)\,, \qquad 0 \le x \le 1\,,$$

evaluated at the initial function u_0.

The connection with the evolution problem is provided by the following Voronovskaja's formula

$$\lim_{n\to+\infty} n(B_n f(x) - f(x)) = \frac{x(1-x)}{2}\,f''(x)\,,$$

which holds uniformly for every $f \in C^2([0,1])$.

Then, applying the Trotter's theorem on the approximation of semigroups, we obtain that the semigroup $(T(t))_{t\geq 0}$ generated by the operator $Au(x) = \frac{x(1-x)}{2}\,u''(x)$ on the Ventcel's domain

$$D(A) = \{u \in C[0,1] \cap C^2]0,1[\mid \lim_{x\to 0,1} Au(x) = 0\}$$

can be represented as

$$T(t)u(x) = \lim_{n\to+\infty} B_n^{[nt]} u(x)$$

and hence the solution of (I.5) can be written as

$$u(x,t) = T(t)u_0(x) = \lim_{n\to+\infty} B_n^{[nt]} u_0(x)\,.$$

After this example, models with more than two alleles and recently even with an infinite numbers of alleles have been considered. Different mathematicians from several schools were involved in this project.

Of particular interest in this context is the generalization introduced of Altomare [2], concerning with positive operators of Bernstein type, namely Bernstein-Schnabl operators, in an infinite dimensional setting by associating them to a positive projection.

This result has motivated many subsequent researches. On one hand there was the possibility of applying the same method to different sequences of positive operators (Stancu, Lototsky and so on; see [9, Chapter VI] and [3, 4, 5, 16, 12, 22, 23]) obtaining in this way the possibility of approximating the solutions of an enlarged class of Cauchy problems.

On the other hand, starting with an assigned degenerate parabolic problem, it has also been considered the inverse problem of approximating its solution by means of suitable sequences of discrete or integral positive operators (see e.g. [6, 7, 19, 11, 13, 14, 15, 18, 27, 24, 25, 26, 28, 30, 35]).

Further, this approach has also been extended to a direct study of the diffusion model (see [8]).

However, the lack of a quantitative estimate of the convergence of the iterates of the positive operators to the limit semigroup has constituted one of the main problems in the applications of Trotter's theorem. Some results in this direction have recently been obtained in [55] and [62]. In these papers the limit semigroup is assumed to have a growth bound equal to 0 and applications to classical approximation processes require restriction to the subspaces of C^4 and C^3-functions respectively.

In Chapter 1, we shall be able to state a general result concerning with the order of convergence to the limit semigroup. Our result holds under the same general assumptions of Trotter's theorem. Moreover we don't make any assumptions on the growth bound of the semigroup and the general result yields better estimates even in the case of growth bound equal to 0.

This main result requires only a quantitative estimate in the Voronovskaja-type formula in terms of suitable seminorms. Then we shall be able to evaluate the norm difference between the $k(n)$-iterates of the linear operator and the semigroup. Consequently a suitable choice of the sequence $(k(n))_{n\in\mathbb{N}}$ will ensure the strong convergence of the iterates to the semigroup.

We also point out that our estimate behaves better even for linear functions and does not require any invariant subspace properties.

A representation by means of iterates of positive operators can be obtained for the resolvent operator of semigroup's generator. In [1] it was given a general description of this kind of representation in terms of approximating processes of the resolvent operator and different formulas of independent interest were obtained. These formulas turn out to be useful for some qualitative properties of the semigroup in different meaningful cases.

Along this line, we shall give a quantitative estimate in the approximation of the resolvent operator starting from a quantitative version of a Voronovskaja-type formula.

Moreover, we have studied the possibility of introducing some general sequences of linear operators obtained from classical approximation processes which turn to be useful in the approximation of the resolvent operators. The aim is the possibility of representing the resolvent operators in terms of classical approximation operators.

The results in this chapter are collected in [36, 37, 38].

In Chapter 2, we shall provide several applications of the general quantitative estimates, by considering some classes of Schnabl-type operators and using a quantitative version of the Voronovskaja-type formula obtained in [9].

As regards the finite dimensional setting we shall provide a general result concerning with a quantitative Voronovskaja-type formula which holds for all functions of class $C^{2,\alpha}(K)$, where K is a compact domain in $\mathbb{R}^d$. So we shall be able to give some estimates of the convergence to the limit semigroup in this space extending considerably the class of functions for which such an estimate have been found.

The order of convergence to the second-order differential operator is strictly related to the convergence of $(L_n(\mathrm{id}-x)^2(x))_{n\in\mathbb{N}}$ to 0. In particular, when we describe the solution of the evolution problem associated with a Fleming-Viot diffusion model in population genetics in terms of Bernstein operators, we can furnish the order of convergence equal to $\alpha/2$ in the space $C^{2,\alpha}(K)$.

We shall show a partial converse result which ensures that this order of convergence is the best possible for the Bernstein operators.

Finally we conclude Chapter 2 with an application to suitable combinations of iterates of Stancu operators. These results can been found in [37] and [39].

In Chapter 3 we consider iterates of generalized Steklov operators and study their convergence to a limit semigroup generated by the second-order differential operator $Af(x) = a(x)f''(x)$, where the coefficient a has a degeneration at least of second order at the endpoints. We also give a quantitative version of Voronovskaja's formula in order to apply the quantitative estimates in Chapter 1.

The choice of Steklov operators is motivated by the possibility of considering the same kind of operators in different setting, such as spaces of continuous functions or weighted spaces of continuous functions both on bounded and unbounded real intervals; hence we can study the convergence to the limit semigroups in all these setting.

The results in this chapter have been obtained in collaboration with I. Rasa (Cluj-Napoca, Romania) and published in [33], [34]. At the end of the chapter we briefly describe a possibile extension to the multivariate case. In this setting we have only some partial results which has been published in [42].

The need of adapting sequences of operators to different settings has lead us to study the possibility of considering a combinations of different sequences of operators in the approximation of the same problem.

In Chapter 4 we introduce a general and simple method which consists in constructing a new approximation process starting with a decomposition of an Hilbert space into the direct sum of orthogonal subspaces and associating to each subspace an assigned approximation process.

In this way are able to obtain new Voronovskaja-type formulas from assigned ones, or conversely we can construct particular approximation processes in order to satisfy a prescribed Voronovskaja-type formula, extending the class of differential problems under consideration. Some similar questions have also been considered in [31] and in [32].

Our method can be applied in different settings. We concentrate our attention on some particular positive approximation processes in spaces of L^2-real functions, namely the Bernstein-Kantorovich and the Bernstein-Durrmeyer operators.

The choice of these operators resides on the fact that both for Bernstein-Kantorovich and Bernstein-Durrmeyer operators the Voronovskaja's formula is related to the evolution process associated with some diffusion models in population genetics. Thus, it may happen that different factors, such as selection, mutation and migration affect only some alleles and therefore involve only a face of the simplex whose vertices represent the totality of alleles; in this case the possibility of using different approximation processes on different orthogonal subspaces allow us to make the appropriate choice of the approximating operators on every subspace.

In connection with this problem we have also studied perturbations of operators obtained by modifying some of its components. In concrete examples, this has been performed for Jackson convolution operators. These results are published in [40] (a further investigation has been performed in [41] for Bernstein-Durrmeyer operators and for the sake of brevity is not included in this work).

Following the same approach of the preceding chapters, in Chapter 5 we study the possibility of approximating the solution of suitable hyperbolic problems using the generation of a cosine function (see [68] and [50] for more details on this approach).

Unlike to the case of semigroup theory there has not been a similar development in the approximation of cosine functions and only in [30, 35] we can find some qualitative results on the convergence of iterates of trigonometric polynomials to suitable cosine functions in the setting of spaces of continuous periodic real functions. We provide a general cosine version of the Trotter's approximation theorem together with a quantitative estimate of the convergence. We also introduce suitable sequences of operators approximating the resolvent operators associated with the generator of the cosine function and also in this case we obtain a quantitative estimate of the convergence. We apply our results to some generalized Rogosinski trigonometric operators.

These results are collected in [43].

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