Chapter 2

Kernel estimates for Markov semigroups

This chapter is devoted to the study of kernels of elliptic operators. As we have seen in Chapter 1, even if the coefficients of the operators are unbounded, the semigroup generated in the space of continuous and bounded functions admits an integral representation through a kernel p. We are interested in finding pointwise upper bounds for such kernels. However we will not consider the whole operator, our attention will be first turned toward Kolmogorov operators not containing a zero order derivative term. In a second moment we will analyse also Schrödinger operators not containing a drift term.
In both cases we use Lyapunov function techniques.

2.1 Kernel estimates for a class of Kolmogorov semigroups

We consider the second order elliptic operator

$$A = \sum_{i,j=1}^{N} a_{ij} D_{ij} + \sum_{i=1}^{N} F_i D_i = A_0 + F \cdot D$$

where $A_0 = \sum_{i,j=1}^{N} a_{ij} D_{ij}$ and the associated parabolic problem

$$\begin{cases} u_t(x,t) = Au(x,t), & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = f(x) & x \in \mathbb{R}^N \end{cases} \tag{2.1}$$

with initial datum $f \in C_b(\mathbb{R}^N)$.
The operator A is endowed with the maximal domain in $C_b(\mathbb{R}^N)$ given by

$$D_{max}(A) = \{u \in C_b(\mathbb{R}^N) \cap W^{2,p}_{loc}(\mathbb{R}^N) \quad \text{for all} \quad p < \infty : \ Au \in C_b(\mathbb{R}^N)\}.$$

As proved in Chapter 1, assuming that (a_{ij}) is a symmetric matrix, $a_{ij} \in C^{\alpha}_{loc}(\mathbb{R}^N)$, $F_i \in C^{\alpha}_{loc}(\mathbb{R}^N)$ for some $0 < \alpha < 1$ and the ellipticity condition

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2$$

for every $x,\ \xi \in \mathbb{R}^N$ and suitable $0 < \lambda \leq \Lambda$, it is possible to prove the existence of a bounded classical solution of such problem, i.e. a function $u \in C(\mathbb{R}^N \times [0, +\infty)) \cap C^{1,2}(\mathbb{R}^N \times (0, +\infty))$ which is bounded in $\mathbb{R}^N \times [0, T]$ for any $T > 0$ and satisfies $\partial_t u,\ D^2u \in C^{\alpha}(\mathbb{R}^N \times (0, +\infty))$ and (2.1). In their work, Metafune, Pallara and Rhandi (see [27]), using Lyapunov functions independent of t, prove estimates of the form

$$p(x, y, t) \leq c(t)\omega(y).$$

For instance, if the drift term is given by $F(x) = -|x|^r \dfrac{x}{|x|}$ and the second order part is the Laplacian, they prove that, for any $\gamma < 1/(r+1)$ and for some positive constants c_1 and c_2, $p(x, y, t) \leq c_1 \exp\left(c_2 t^{-\frac{r+1}{r-1}}\right) \exp(-\gamma|y|^{r+1})$ for small times t and for all $x,\ y \in \mathbb{R}^N$.
Following their idea, but considering Lyapunov functions depending also on the time variable for the operator $\partial_t + A$, we deduce estimates of the form

$$p(x, y, t) \leq c(t)\omega(y, t).$$

In particular, in the special case mentioned above, for small times, we obtain

$$p(x, y, t) \leq c_1 t^{-\delta} \exp(-t^{\alpha}\gamma|y|^{r+1}).$$

We remark that, although for $0 < t \leq 1$ $\exp\{-c|y|^{r+1}\} \leq \exp\{-ct^{\alpha}|y|^{r+1}\}$, the function $c(t)$ blows up polynomially in our estimates and exponentially in [27]. Therefore, using Lyapunov functions for the parabolic operator depending also on the time variable t, we gain a better behaviour for the function $c(t)$.
We start by proving the integrability of certain Lyapunov functions with respect to the measure $p(x, \cdot, t)dy$. Moreover an estimate of the L^1-norm of the Lyapunov functions with respect to the measure above is obtained. Assuming suitable assumptions on the radial component of the drift F, examples of Lyapunov functions for the parabolic operator are given.
Following [27, Section 3], it is proved how, underthe hypothesis of integrability of some power k of the drift with respect to the measure p, the kernel is in some Lebesgue spaces L^r or in some other spaces embedded in L^{∞} for k large enough. Then the main result is proved, we apply an estimate for the L^{∞}-norm of solutions of certain parabolic problems to deduce the claimed result. An useful tool employed here is a result of Sobolev regularity for transition probabilities.
In some recent papers, Bogachev, Krylov, Röckner and Shaposhnikov (see [6], [7] and [8]) have proved existence and regularity properties for parabolic problems

having measures as initial data. The authors assume also integrability properties of the drift term, comparables to ours, and deduce the uniform boundedness of the solutions in $\mathbb{R}^N \times [0,T]$ whenever $T < 1$. Their results do not apply to our situation since the fundamental solution p is singular for $t = 0$.

All over the section we will assume the existence of a Lyapunov function for the operator A, that is a function $0 \le V \in C^2(\mathbb{R}^N)$ such that $\lim_{|x|\to\infty} V(x) = +\infty$ and $AV(x) \le \lambda V(x)$ for some positive λ. We recall that this assumption insures that the domain of the weak generator $\hat{D}$ coincides with the maximal domain $D_{max}(A)$ (see Theorem 1.2.5). We will see later that Lyapunov functions exist for the operators we are interested in.
Moreover, since we will deal with differential quotients and we have to apply the integration by parts formula, we suppose that the coefficients a_{ij} of the operator are of class $C_b^1(\mathbb{R}^N)$.

2.1.1 L^1- estimates of some Lyapunov functions

In this section we show how to obtain the integrability of certain unbounded functions with respect to the kernel p. Later pointwise estimates will be deduced from L^1-bounds.
Our technique rests on the following definition, where $L = \partial_t + A$.

We say that a continuous function $W : [0,T] \times \mathbb{R}^N \to [0,+\infty)$ is a Lyapunov function for the operator L if it belongs to $C^{2,1}(Q_T)$, $\lim_{|x|\to\infty} W(x,t) = +\infty$ uniformly with respect to t in compact sets of $(0,T]$ and there exists $h : (0,T] \to [0,\infty)$ integrable in a neighborhood of 0 such that $LW(x,t) \le h(t)W(x,t)$ for all $(x,t) \in Q_T$. Note that we do not require that $W(x,0)$ tends to ∞ as $|x| \to \infty$.

We refer the reader to [30] for results similar to the next proposition, when the Lyapunov function is independent of t.

Proposition 2.1.1. *For each $t \in [0,T]$, a Lyapunov function $W(\cdot,t)$ is integrable with respect to the measure $p(x,\cdot,t)$. Moreover, setting*

$$\xi_W(x,t) = \int_{\mathbb{R}^N} p(x,y,t)W(y,t)dy, \tag{2.2}$$

the inequality

$$\xi_W(x,t) \le e^{\int_0^t h(s)ds} W(x,0) \tag{2.3}$$

holds.

Proof. Let us consider, for every $\alpha \ge 0$, $\psi_\alpha \in C_b^\infty(\mathbb{R})$ such that $\psi_\alpha(s) = s$ for $s \le \alpha$, ψ_α is constant in $[\alpha+1,\infty)$, $\psi'_\alpha \ge 0$ and $\psi''_\alpha \le 0$. From the concavity of ψ_α it follows that

$$s\psi'_\alpha(s) \le \psi_\alpha(s) \qquad \forall\ s \ge 0. \tag{2.4}$$

Obviously $\psi_\alpha \circ W \in BUC(Q_T)$ and, moreover, it belongs to $BUC^{2,1}(Q(\varepsilon,T))$ for every $\varepsilon > 0$, since is constant for $t \geq \varepsilon > 0$ and large $|x|$. We set $\xi_\alpha(x,t) = \int_{\mathbb{R}^N} p(x,y,t)\psi_\alpha(W(y,t))dy$. For every fixed $t \geq \varepsilon$, the function $(\psi_\alpha \circ W)(\cdot,t)$ belongs to $D_{max}(A)$, which coincides with the domain of the generator by the assumption of the existence of Lyapunov functions for A. It follows that

$$\partial_t \xi_\alpha(\cdot,t) = e^{tA} A(\psi_\alpha \circ W)(\cdot,t) + e^{tA}\partial_t(\psi_\alpha \circ W)(\cdot,t)$$

and then

$$\partial_t \xi_\alpha(x,t) = \int_{\mathbb{R}^N} p(x,y,t) L(\psi_\alpha \circ W)(y,t)dy.$$

By a straightforward computation we obtain

$$\begin{aligned} L(\psi_\alpha \circ W)(x,t) =& \psi'_\alpha(W(x,t))LW(x,t) \\ &+ \psi''_\alpha(W(x,t)) \sum_{i,j=1}^N a_{ij} D_j W(x,t) D_i W(x,t) \\ \leq& \psi'_\alpha(W(x,t))LW(x,t). \end{aligned}$$

Thus, for $t \geq \varepsilon$,

$$\partial_t \xi_\alpha(x,t) \quad \leq \quad \int_{\mathbb{R}^N} p(x,y,t)\psi'_\alpha(W(y,t))LW(y,t)dy.$$

Using the property of W, the positivity of ψ' and (2.4) we get

$$\partial_t \xi_\alpha(x,t) \leq h(t) \int_{\mathbb{R}^N} p(x,y,t)\psi_\alpha(W(y,t))dy = h(t)\xi_\alpha(x,t).$$

Therefore for $t \geq \varepsilon$

$$\xi_\alpha(x,t) \leq e^{\int_\varepsilon^t h(s)ds} \xi_\alpha(x,\varepsilon). \tag{2.5}$$

Now we prove that $\xi_\alpha(x,\varepsilon) \to \psi_\alpha(W(x,0))$ as $\varepsilon \to 0$. We have

$$\begin{aligned} |\xi_\alpha(x,\varepsilon) - \psi_\alpha(W(x,0))| \quad &= \quad \left| \int_{\mathbb{R}^N} p(x,y,\varepsilon)\psi_\alpha(W(y,\varepsilon))dy - \psi_\alpha(W(x,0))\, dy \right| \\ &\leq \quad \int_{\mathbb{R}^N} p(x,y,\varepsilon)|\psi_\alpha(W(y,\varepsilon)) - \psi_\alpha(W(y,0))|\, dy \\ &+ \quad |T(\varepsilon)\psi_\alpha(W(x,0)) - \psi_\alpha(W(x,0))|. \end{aligned}$$

The second term in the right member obviously goes to 0 as $\varepsilon \to 0$ since $\psi_\alpha \circ W \in C_b(\mathbb{R}^N)$ and $T(t)f \to f$ as $t \to 0$ uniformly on compact sets of $\mathbb{R}^N$ for $f \in C_b(\mathbb{R}^N)$ (see Theorem 1.1.7). Concerning the first addend, we fixe $R > |x| + 1$ and we split it in the integral over B_R and the integral over the complementary of B_R. We have

$$\begin{aligned}\int_{\mathbb{R}^N} & p(x,y,\varepsilon)|\psi_\alpha(W(y,\varepsilon)) - \psi_\alpha(W(y,0))|\,dy \\ &= \int_{B_R} p(x,y,\varepsilon)|\psi_\alpha(W(y,\varepsilon)) - \psi_\alpha(W(y,0))|\,dy \\ &+ \int_{\mathbb{R}^N\setminus B_R} p(x,y,\varepsilon)|\psi_\alpha(W(y,\varepsilon)) - \psi_\alpha(W(y,0))|\,dy.\end{aligned}$$

The integral on B_R tends to 0 as $\varepsilon \to 0$ since $\psi_\alpha(W(y,\varepsilon)) \to \psi_\alpha(W(y,0))$ uniformly on B_R. Consider the integral on the complementary of B_R. Let h_R be a smooth function on $\mathbb{R}^N$ such that $\chi_{\mathbb{R}^N\setminus B_R} \le h_R \le \chi_{\mathbb{R}^N\setminus B_{R-1}}$. Observe that $h_R \in D_{max}(A)$ and hence $T(\varepsilon)h_R \to h_R$ uniformly in $\mathbb{R}^N$ since

$$T(\varepsilon)h_R(x) - h_R(x) = \int_0^\varepsilon T(s)Ah_R(x)\,ds$$

for all $x \in \mathbb{R}^N$ (see [38, Proposition 3.2]). Therefore, given $\delta > 0$, there exists $\varepsilon_0 > 0$ such that, for $\varepsilon \le \varepsilon_0$, $T(\varepsilon)h_R \le \delta + h_R$. By means of the previous remarks, since $|x| < R-1$, we deduce

$$\begin{aligned}\int_{\mathbb{R}^N\setminus B_R} p(x,y,\varepsilon)|\psi_\alpha(W(y,\varepsilon)) - \psi_\alpha(W(y,0))| &\le 2(\alpha+1)\int_{\mathbb{R}^N\setminus B_R} p(x,y,\varepsilon) \\ &= 2(\alpha+1)T(\varepsilon)\chi_{\mathbb{R}^N\setminus B_R}(x) \\ &\le 2(\alpha+1)T(\varepsilon)h_R(x) \\ &\le 2(\alpha+1)[\delta + h_R(x)] \\ &= 2(\alpha+1)\delta\end{aligned}$$

for $\varepsilon \le \varepsilon_0$. Letting $\varepsilon \to 0$ in (2.5) we obtain

$$\xi_\alpha(x,t) \le e^{\int_0^t h(s)ds}\psi_\alpha(W(x,0)).$$

Letting $\alpha \to \infty$ in the previous inequality and using Fatou's Lemma we get

$$\int_{\mathbb{R}^N} p(x,y,t)W(y,t)dy \le \liminf_{\alpha\to\infty}\xi_\alpha(x,t) \le e^{\int_0^t h(s)ds}W(x,0).$$

□

In the next proposition we prove that suitable exponential functions in x and t are of Lyapunov for a class of Kolmogorov operators.

Proposition 2.1.2. *Let $L = \partial_t + A_0 + F\cdot D$ such that*

$$\limsup_{|x|\to\infty} |x|^{-r}F(x)\cdot\frac{x}{|x|} < -c \tag{2.6}$$

for some positive c and $r > 1$. Then, if $\alpha > \frac{r+1}{r-1}$, $\delta < \frac{c}{\Lambda(r+1)}$ and $0 < t \le 1$, $W(x,t) = \exp\{\delta t^\alpha |x|^{r+1}\}$ is a Lyapunov function for L. Moreover $\xi_W(x,t) \le CW(x,0) = C$ for some positive constant C and for all $x \in \mathbb{R}^N$ and $0 < t \le 1$.

Proof. An easy computation gives

$$\begin{aligned} LW(x,t) &= \delta(r+1)t^\alpha W(x,t)\Big[\alpha\frac{|x|^{r+1}}{t(r+1)} + (r-1)|x|^{r-3}\sum_{i,j=1}^N a_{ij}x_ix_j \\ &+ |x|^{r-1}\sum_{i=1}^N a_{ii} + \delta(r+1)t^\alpha|x|^{2r-2}\sum_{i,j=1}^N a_{ij}x_ix_j + |x|^r F\cdot\frac{x}{|x|}\Big] \\ &\le \delta(r+1)t^\alpha W(x,t)\Big[\alpha\frac{|x|^{r+1}}{t(r+1)} + \big[\Lambda(r-1)+\sum_{i=1}^N a_{ii}\big]|x|^{r-1} \\ &+ \Lambda\delta(r+1)t^\alpha|x|^{2r} + |x|^r F\cdot\frac{x}{|x|}\Big]. \end{aligned}$$

Considering suitable space-time regions it is possible to estimate the right hand side in the previous inequality.
Let $\gamma > \dfrac{1}{r-1}$. If $|x| > \dfrac{1}{t^\gamma}$, $0<t<1$,

$$\begin{aligned} LW(x,t) &\le \delta(r+1)t^\alpha W(x,t)\Big[\alpha\frac{|x|^{r+1}}{t(r+1)} + \big[\Lambda(r-1)+\sum_{i=1}^N a_{ii}\big]|x|^{r-1} \\ &+ \Lambda\delta(r+1)t^\alpha|x|^{2r} + |x|^r F\cdot\frac{x}{|x|}\Big] \\ &\le \delta(r+1)t^\alpha W(x,t)\Big[\frac{\alpha}{r+1}|x|^{r+1+\frac{1}{\gamma}} + \big[\Lambda(r-1)+\sum_{i=1}^N a_{ii}\big]|x|^{r-1} \\ &+ \Lambda\delta(r+1)|x|^{2r} + |x|^r F\cdot\frac{x}{|x|}\Big] \\ &\le \delta(r+1)t^\alpha|x|^{2r} W(x,t)\Big[\frac{\alpha}{r+1}|x|^{r+1+\frac{1}{\gamma}-2r} \\ &+ \big[\Lambda(r-1)+\sum_{i=1}^N a_{ii}\big]|x|^{-r-1} + \Lambda\delta(r+1) + |x|^{-r} F\cdot\frac{x}{|x|}\Big]. \end{aligned}$$

By assumption (2.6), if $|x|$ is large enough,

$$\begin{aligned} LW(x,t) \le \delta(r+1)t^\alpha|x|^{2r}W(x,t)\Big[&\frac{\alpha}{r+1}|x|^{-r+1+\frac{1}{\gamma}} \\ &+ \big[\Lambda(r-1)+\sum_{i=1}^N a_{ii}\big]|x|^{-r-1} + \Lambda\delta(r+1) - c\Big]. \end{aligned}$$

Since $\delta < \dfrac{c}{\Lambda(r+1)}$ and $\gamma > \dfrac{1}{r-1}$, for $|x|$ large enough and belonging to the considered region $LW \le 0$. For the remaining small values of x in this region $LW(x,t) \le C \le CW(x,t)$.

If $|x| \le \dfrac{1}{t^\gamma}$ and is large enough in order that the term containing the drift is negative,

$$\begin{aligned} LW(x,t) &\le \\ &W(x,t)\left[\frac{\delta\alpha}{t^{\gamma(r+1)+1-\alpha}} + \delta(r+1)\left(\Lambda(r-1)+\sum_{i=1}^N a_{ii}\right)\frac{1}{t^{\gamma(r-1)-\alpha}}\right. \\ &\left. + \Lambda\delta^2(r+1)^2\frac{1}{t^{2\gamma r-2\alpha}}\right]. \end{aligned}$$

If we choose $\gamma < \frac{\alpha}{r+1}$, we have $\gamma(r+1)-\alpha+1<1$ and $2r\gamma-2\alpha<0$. If $|x|$ is small we obtain the estimate as in the other region. In any case

$$LW(x,t) \le h(t)W(x,t)$$

with h integrable near 0. Observe moreover that the conditions on γ are compatible since $\alpha > \frac{r+1}{r-1}$. The existence of Lyapunov functions for the elliptic operator is guaranteed under the assumption (2.6) (see [27, Prop. 2.6]). Then by Proposition 2.1.1 the estimate of $\xi_W(x,t)$ follows. □

Example 2.1.3. In particular, Proposition 2.1.2 applies if

$$L = \partial_t + \Delta - |x|^r \frac{x}{|x|}\cdot D$$

with $r>1$. Then, for $\alpha > \dfrac{r+1}{r-1}$, $\delta < \frac{1}{r+1}$ and $0<t\le 1$, $W(x,t) = \exp\{\delta t^\alpha |x|^{r+1}\}$ is a Lyapunov function for L and $\xi_W(x,t) \le CW(x,0) = C$ for some positive constant C, for all $x \in \mathbb{R}^N$ and $0<t\le 1$.

2.1.2 Integrability and regularity results for the kernel

Following [27, Section 3 and Appendix A], in this subsection we collect some useful and of independent interest results. We prove embedding theorems for the spaces $\mathcal{H}^{k,1}$ due to Krylov (see [21]) and, using the same methods, we deduce also embedding theorems for the spaces Θ^k (see definitions below).
Then we fix $T>0$, $0<a_0<a<b<b_0\le T$, assume $b_0-b\ge a-a_0$ and consider p as a function depending on $(y,t)\in\mathbb{R}^N\times(0,T)$ for arbitrary, but fixed, $x\in\mathbb{R}^N$.
Setting

$$\Gamma(k,x,a_0,b_0) = \left(\int_{Q(a_0,b_0)} |F(y)|^k p(x,y,t)\,dy\,dt\right)^{\frac{1}{k}}$$

and making use of the embeddings above, we show global regularity result for p with respect to the variables (y,t) assuming $\Gamma(k,x,a_0,b_0)<\infty$ for suitable $k\ge 1$.

Definition 2.1.4. *Given $k \geq 1$, $\mathcal{H}^{k,1}(Q_T)$ denotes the space of all functions $u \in W_k^{1,0}(Q_T)$ with $\partial_t u \in (W_{k'}^{1,0}(Q_T))'$, the dual space of $W_{k'}^{1,0}(Q_T)$, endowed with the norm*

$$\|u\|_{\mathcal{H}^{k,1}(Q_T)} := \|\partial_t u\|_{(W_{k'}^{1,0}(Q_T))'} + \|u\|_{W_k^{1,0}(Q_T)},$$

with $\frac{1}{k} + \frac{1}{k'} = 1$.

Definition 2.1.5. *For $k > 2$, $\Theta^k(Q_T)$ is the space of all functions u belonging to $W_k^{1,0}(Q_T)$ such that there exists $C > 0$ for which*

$$\left| \int_{Q_T} u \partial_t \phi \, dx \, dt \right| \leq C \left(\|\phi\|_{L^{\frac{k}{k-2}}(Q_T)} + \|D\phi\|_{L^{\frac{k}{k-1}}(Q_T)} \right)$$

for every ϕ such that the right hand side above is finite. Observe that $\frac{k}{k-1} = k'$ and $\frac{k}{k-2} = \left(\frac{k}{2}\right)'$. $\Theta^k(Q_T)$ is a Banach space endowed with the norm

$$\|u\|_{\Theta^k(Q_T)} = \|u\|_{W_k^{1,0}(Q_T)} + \|\partial_t u\|_{\frac{k}{2},k;Q_T},$$

where $\|\partial_t u\|_{\frac{k}{2},k;Q_T}$ is the best constant such that the above estimate holds.

By using a reflection argument and standard approximation by smooth functions methods one can prove the following extension and density results.

Lemma 2.1.6. *There exists linear, continuous extension operators*

$$E_1 : \mathcal{H}^{k,1}(Q_T) \to \mathcal{H}^{k,1}(\mathbb{R}^{N+1})$$

and

$$E_2 : \Theta^k(Q_T) \to \Theta^k(\mathbb{R}^{N+1}).$$

Lemma 2.1.7. *The restrictions of functions in $C_c^\infty(\mathbb{R}^{N+1})$ to Q_T are dense in $\mathcal{H}^{k,1}(Q_T)$ and in $\Theta^k(Q_T)$.*

Theorem 2.1.8. *The following embeddings of $\mathcal{H}^{k,1}$ in L^r spaces hold.*

(i) If $1 < k < N+2$, then $\mathcal{H}^{k,1}(Q_T)$ is continuously embedded in $L^r(Q_T)$ for $\frac{1}{r} = \frac{1}{k} - \frac{1}{N+2}$.

(ii) If $k = N+2$, then $\mathcal{H}^{k,1}(Q_T)$ is continuously embedded in $L^r(Q_T)$ for $N + 2 \leq r < \infty$.

(iii) If $k > N+2$, then $\mathcal{H}^{k,1}(Q_T)$ is continuously embedded in $L^\infty(Q_T)$.

Proof. Since the restrictions of functions in the space $C_c^\infty(\mathbb{R}^{N+1})$ are dense in $\mathcal{H}^{k,1}(Q_T)$, in any case we will prove the estimate

$$\|u\|_{L^r(Q_T)} \leq \|u\|_{\mathcal{H}^{k,1}(Q_T)} \tag{2.7}$$

for every function $u \in C_c^\infty(\mathbb{R}^{N+1})$ and some positive constant C independent of u. Let G be the fundamental solution of the operator $\partial_t - \Delta$ in $\mathbb{R}^{N+1}$ given by

$$G(x,t) = \begin{cases} \frac{1}{(4\pi t)^{\frac{N}{2}}} \exp\left(-\frac{1}{4t}|x|^2\right) & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t \le 0. \end{cases} \tag{2.8}$$

Let $u \in C_c^\infty(\mathbb{R}^{N+1})$, $\psi \in C_c^\infty(Q_T)$ and set $\phi = G * \psi$. Then $\phi \in C^2(\mathbb{R}^{N+1})$ and, by [20, Theorem 8.4.2], it satisfies $\partial_t\phi - \Delta\phi = \psi$. Moreover, since ψ has support in $\mathbb{R}^N \times [0,T]$, then $G * \psi = G_T * \psi$ where $G_T = G\chi_{[0,T]}$. By simple computations it immediately follows that $G_T \in L^s(\mathbb{R}^{N+1})$ for $1 \le s < \frac{N+2}{N}$ and $DG_T \in L^s(\mathbb{R}^{N+1})$ for $1 \le s < \frac{N+2}{N+1}$ where the gradient is understood with respect to the space variable. Young's inequality yields $\|\phi\|_{W_s^{1,0}(Q_T)} \le C\|\psi\|_{L^1(Q_T)}$.
We have

$$\begin{aligned} \left|\int_{Q_T} u\psi \, dx \, dt\right| &= \left|\int_{Q_T} u(\partial_t\phi - \Delta\phi) \, dx \, dt\right| \\ &= \left|\int_{Q_T} u\partial_t\phi + Du \cdot D\phi) \, dx \, dt\right| \\ &\le C\|u\|_{\mathcal{H}^{k,1}(Q_T)}\|\phi\|_{W_{k'}^{1,0}(Q_T)}. \end{aligned} \tag{2.9}$$

Let us prove (i). Let $1 < k < N+2$, r such that $\frac{1}{r} = \frac{1}{k} - \frac{1}{N+2}$. By Theorem A.0.8, $\|\phi\|_{W_{r'}^{2,1}(Q_T)} \le c\|\psi\|_{L^{r'}(Q_T)}$, by the embedding $W_{r'}^{2,1}(Q_T) \subset W_{k'}^{1,0}(Q_T)$ (see Theorem A.0.9) and the previous inequality (2.9), we obtain

$$\begin{aligned} \left|\int_{Q_T} u\psi \, dx \, dt\right| &\le C\|u\|_{\mathcal{H}^{k,1}(Q_T)}\|\phi\|_{W_{k'}^{1,0}(Q_T)} \le C\|u\|_{\mathcal{H}^{k,1}(Q_T)}\|\phi\|_{W_{r'}^{2,1}(Q_T)} \\ &\le C\|u\|_{\mathcal{H}^{k,1}(Q_T)}\|\psi\|_{L^{r'}(Q_T)}. \end{aligned}$$

This implies (2.7).
Let now $k = N+2$, $N+2 \le r < \infty$ and choose $1 < s < \frac{N+2}{N+1}$ such that

$$\frac{1}{k'} = \frac{1}{s} + \frac{1}{r'} - 1.$$

Young's inequality yields $\|\phi\|_{W_{k'}^{1,0}(Q_T)} \le C\|\psi\|_{L^{r'}(Q_T)}$ and then by 2.9 we deduce (ii). Finally, if $k > N+2$, then $k' < \frac{N+2}{N+1}$ and by Young's inequality we get $\|\phi\|_{W_{k'}^{1,0}(Q_T)} \le C\|\psi\|_{L^1(Q_T)}$. By (2.9),

$$\left|\int_{Q_T} u\psi \, dx \, dt\right| \le C\|u\|_{\mathcal{H}^{k,1}(Q_T)}\|\phi\|_{W_{k'}^{1,0}(Q_T)} \le C\|u\|_{\mathcal{H}^{k,1}(Q_T)}\|\psi\|_{L^1(Q_T)}.$$

□

Theorem 2.1.9. *If $k > N+2$, then $\Theta^k(Q_T)$ is continuously embedded in $L^\infty(Q_T)$. Moreover the following estimate holds*

$$\|u\|_{L^\infty(Q_T)} \le C(\|Du\|_{L^k(Q_T)} + \|\partial_t u\|_{\frac{k}{2},k;Q_T}).$$

Proof. Let $u \in C_c^\infty(\mathbb{R}^{N+1})$ and ϕ, ψ as in the proof of the previous theorem. As before we have

$$\left|\int_{Q_T} u\psi\,dx\,dt\right| = \left|\int_{Q_T} u(\partial_t\phi - \Delta\phi)\,dx\,dt\right| = \left|\int_{Q_T} (u\partial_t\phi + Du\cdot D\phi)\,dx\,dt\right| \tag{2.10}$$
$$\leq (\|Du\|_{L^k(Q_T)} + \|\partial_t u\|_{\frac{k}{2},k;Q_T})\left(\|D\phi\|_{L^{\frac{k}{k-1}}(Q_T)} + \|\phi\|_{L^{\frac{k}{k-2}}(Q_T)}\right).$$

Now, since $k > N+2$, $k' < \frac{N+2}{N+1}$ and $\frac{k}{k-2} < \frac{N+2}{N}$. By Young's inequality we get $\|\phi\|_{W_{k'}^{1,0}(Q_T)} \leq C\|\psi\|_{L^1(Q_T)}$ and $\|\phi\|_{L^{\frac{k}{k-2}}(Q_T)} \leq C\|\psi\|_{L^1(Q_T)}$. Therefore

$$\left|\int_{Q_T} u\psi\,dx\,dt\right| \leq (\|Du\|_{L^k(Q_T)} + \|\partial_t u\|_{\frac{k}{2},k;Q_T})\|\psi\|_{L^1(Q_T)}.$$

and the claim follows. □

The embedding theorems above allow us to prove some integrability and regularity properties for the kernel p. A preliminary lemma is needed.

Lemma 2.1.10. *Let $0 \leq t_1 < t_2$ and $\phi \in C^{2,1}(Q(t_1,t_2))$ such that $\phi(\cdot,t)$ has compact support for every $t \in [t_1,t_2]$. Then*

$$\begin{aligned}&\int_{Q(t_1,t_2)} (\partial_t\phi(y,t) + A\phi(y,t))p(x,y,t)\,dy\,dt\\ =\ &\int_{\mathbb{R}^N} (p(x,y,t_2)\phi(y,t_2) - p(x,y,t_1)\phi(y,t_1))dy.\end{aligned}$$

Proof. Note that if $\psi \in C_c^2(\mathbb{R}^N)$ then by Proposition 1.2.2 and by Proposition 1.2.3

$$\partial_t T(t)\psi = T(t)A\psi.$$

Let $\phi(y,t)$ be as in the statement. We have

$$\partial_t(T(t)\phi(\cdot,t)) = T(t)\partial_t\phi(\cdot,t) + T(t)A\phi(\cdot,t).$$

Integrating this identity over the interval $[t_1,t_2]$ and writing $T(t)$ in terms of the kernel we obtain the claim. □

Recall that, for every $k \geq 1$, $\Gamma(k,x,a_0,b_0) = (\int_{Q(a_0,b_0)} |F(y)|^k p(x,y,t)\,dy)^{\frac{1}{k}}$.

Proposition 2.1.11. *If $\Gamma(1,x,a_0,b_0) < \infty$, then $p \in L^r(Q(a_0,b_0))$ for all $r \in [1, \frac{N+2}{N+1})$ and*

$$\|p\|_{L^r(Q(a_0,b_0))} \leq C(1 + \Gamma(1,x,a_0,b_0))$$

for some constant $C > 0$.

Proof. Consider $\phi \in C^{2,1}(Q_T)$ such that $\phi(\cdot,T)=0$ and such that $\phi(\cdot,t)$ has compact support for all t. By Lemma 2.1.10, we deduce

$$\int_{Q(a_0,b_0)} p(\partial_t\phi + A_0\phi)dy\,dt = -\int_{Q(a_0,b_0)} pF\cdot D\phi dy\,dt + \int_{\mathbb{R}^N}(p(x,y,b_0)\phi(y,b_0) - p(x,y,a_0)\phi(y,a_0))dy$$

where $A_0 = \sum_{i,j=1}^N a_{ij}D_{ij}$. Since $\int_{\mathbb{R}^N} p(x,y,t) \le 1$ for all $t\ge 0$, $x\in\mathbb{R}^N$, it follows that

$$\left|\int_{Q(a_0,b_0)} p(\partial_t\phi + A_0\phi)dy\,dt\right| \le \Gamma(1,x,a_0,b_0)\|\phi\|_{W^{1,0}_\infty(Q(a_0,b_0))} + 2\|\phi\|_\infty \le (2+\Gamma(1,x,a_0,b_0))\|\phi\|_{W^{1,0}_\infty(Q(a_0,b_0))}. \tag{2.11}$$

Fix $\psi \in C_c^\infty(Q(a_0,b_0))$ and consider the parabolic problem

$$\begin{cases} \partial_t\phi + A_0\phi = \psi & \text{in } \ Q_T, \\ \phi(y,T) = 0 & y\in\mathbb{R}^N. \end{cases} \tag{2.12}$$

By the Schauder theory (see Theorem A.0.10), there exists a solution $\phi \in C^{2+\alpha,1+\frac{\alpha}{2}}(Q_T)$. Fixing $r_1' > N+2$, by Theorem A.0.8, we have that $\phi \in W^{2,1}_{r_1'}(Q_T)$ and satisfies

$$\|\phi\|_{W^{2,1}_{r_1'}(Q_T)} \le C\|\psi\|_{L^{r_1'}(Q(a_0,b_0))}$$

and, by the Sobolev embedding Theorems (see Theorem A.0.9) and the previous inequality, we deduce that

$$\|\phi\|_{W^{1,0}_\infty(Q(a_0,b_0))} \le \|\phi\|_{W^{1,0}_\infty(Q_T)} \le C\|\phi\|_{W^{2,1}_{r_1'}(Q_T)} \le C\|\psi\|_{L^{r_1'}(Q(a_0,b_0))}. \tag{2.13}$$

Observe that the solution of the parabolic problem just found cannot be immediately inserted in (2.11) since in general it is not with compact support with respect to the space variable. Anyway we can approximate the solution ϕ with functions which satisfy (2.11) as follows. Let $\theta\in C_c^\infty(\mathbb{R}^N)$ such that $\theta(y)=1$ for $|y|\le 1$ and, for each $n\in\mathbb{N}$, consider $\phi_n(y,t)=\theta(\frac{y}{n})\phi(y,t)$. Then ϕ_n satisfies (2.11) and, letting $n\to\infty$ by dominated convergence, by (2.13) we obtain

$$\left|\int_{Q(a_0,b_0)} p\psi dy\,dt\right| \le C(1+\Gamma(1,x,a_0,b_0))\|\psi\|_{L^{r_1'}(Q(a_0,b_0))}.$$

This proves that $p\in L^{r_1}(Q(a_0,b_0))$ where $\frac{1}{r_1}+\frac{1}{r_1'}=1$. By the arbitrarity of $r_1' > N+2$, it follows that $p\in L^r(Q(a_0,b_0))$ for all $1\le r<\frac{N+2}{N+1}$ with

$$\|p\|_{L^r(Q(a_0,b_0))} \le C(1+\Gamma(1,x,a_0,b_0)).$$

□

Lemma 2.1.12. *If $\Gamma(k,x,a_0,b_0)<\infty$ for some $k>1$ and $p\in L^r(Q(a_0,b_0))$ for some $1<r\le\infty$, then $p\in\mathcal{H}^{s,1}(Q(a,b))$ for $s:=\frac{rk}{r+k-1}$ if $r<\infty$ and $s=k$ if $r=\infty$.*

Proof. Let η be a smooth function such that $0\le\eta\le 1$, $\eta(t)=1$ for $a\le t\le b$, $\eta(t)=0$ for $t\le a_0$ and $t\ge b_0$ and $|\eta'|\le\frac{2}{a-a_0}$. Let $\phi\in C^{2,1}(Q_T)$ such that $\phi(\cdot,t)$ has compact support for all t. Then also $\eta\phi$ has compact support for all t and by Lemma 2.1.10, setting $q=\eta p$, we obtain

$$\int_{Q_T} q(\partial_t\phi+A_0\phi)dy,\ dt=-\int_{Q_T}(qF\cdot D\phi+p\phi\partial_t\eta)dy\,dt.$$

Now we estimate the right hand side of the previous equality by using the Hölder inequality and the integrability assumption on p. We have

$$\begin{aligned}\int_{Q(a_0,b_0)}|F|^s p^s dy\,dt &= \int_{Q(a_0,b_0)}|F|^s p^{\frac{s}{k}}p^{s(1-\frac{1}{k})}dy\,dt\\ &\le\left(\int_{Q(a_0,b_0)}|F|^k p\,dy\,dt\right)^{\frac{s}{k}}\left(\int_{Q(a_0,b_0)}p^{\frac{s(k-1)}{k-s}}dy\,dt\right)^{1-\frac{s}{k}}\\ &=\left(\int_{Q(a_0,b_0)}|F|^k p\,dy\,dt\right)^{\frac{s}{k}}\left(\int_{Q(a_0,b_0)}p^r dy\,dt\right)^{1-\frac{s}{k}}\\ &\le\Gamma(k,x,a_0,b_0)^s\left(\int_{Q(a_0,b_0)}p^r dy\,dt\right)^{1-\frac{s}{k}},\end{aligned}$$

hence we have

$$\|Fp\|_{L^s(Q(a_0,b_0))}\le C\|p\|^{\frac{k-1}{k}}_{L^r(Q(a_0,b_0))}$$

where C is a generic constant depending on k, x, a_0, b_0. Therefore

$$\left|\int_{Q_T} q(\partial_t\phi+A_0\phi)dy,\ dt\right|\le C\|p\|^{\frac{k-1}{k}}_{L^r(Q(a_0,b_0))}\|\phi\|_{W^{1,0}_{s'}(Q_T)}$$

with $\frac{1}{s}+\frac{1}{s'}=1$. Observe that we can replace ϕ by its difference quotients with respect to the variable y given by

$$\tau_{-h}\phi(y,t):=\frac{1}{|h|}(\phi(y-he_j,t)-\phi(y,t)),\quad (y,t)\in Q_T,\ 0\neq h\in\mathbb{R}.$$

In this way and recalling that $a_{ij}\in C^1_b(\mathbb{R}^N)$, we obtain

$$\left|\int_{Q_T}\tau_h q(\partial_t\phi+A_0\phi)dy,\ dt\right|\le C\|p\|^{\frac{k-1}{k}}_{L^r(Q(a_0,b_0))}\|\phi\|_{W^{1,0}_{s'}(Q_T)}\tag{2.14}$$

where C depends on k, x, a_0, b_0 and the $C^1_b(\mathbb{R}^N)$ norm of the coefficients a_{ij}. Observe that, since $q\in L^s(Q_T)$, by approximation, as in the proof of Lemma

2.1.11, the inequality (2.14) remains true for functions $\phi \in W^{2,1}_{s'}(Q_T)$. Moreover, since $q \in L^s(Q_T)$, then $|\tau_h q|^{s-2}\tau_h q \in L^{s'}(Q_T)$. By Theorem A.0.8, there exists $\phi \in W^{2,1}_{s'}(Q_T)$ such that

$$\begin{cases} \partial_t \phi + A_0\phi = |\tau_h q|^{s-2}\tau_h q & \text{in} \quad Q_T, \\ \phi(y,T) = 0 & y \in \mathbb{R}^N \end{cases} \tag{2.15}$$

and

$$\|\phi\|_{W^{2,1}_{s'}(Q_T)} \leq C\||\tau_h q|^{s-1}\|_{L^{s'}(Q_T)}.$$

By (2.14), we get

$$\int_{Q_T} |\tau_h q|^s dy\, dt \leq C\|p\|^{\frac{k-1}{k}}_{L^r(Q(a_0,b_0))}\|\tau_h q\|^{s-1}_{L^s(Q_T)},$$

By means of the properties of the differential quotients we deduce

$$\|Dq\|_{L^s(Q_T)} \leq C\|p\|^{\frac{k-1}{k}}_{L^r(Q_T)}.$$

This implies $Dq \in L^s(Q_T)$ and so $q \in W^{1,0}_s(Q_T)$ and $p \in W^{1,0}_s(Q(a,b))$. Concerning the first order time derivative, by the estimate above, integrating by parts and recalling that $a_{ij} \in C^1_b(\mathbb{R}^N)$, we have

$$\begin{aligned} \left|\int_{Q_T} q\partial_t\phi\, dy\, dt\right| &\leq \left|\int_{Q_T} qA_0\phi\, dy\, dt\right| + C\|p\|^{\frac{k-1}{k}}_{L^r(Q(a_0,b_0))}\|\phi\|_{W^{1,0}_{s'}(Q_T)} \\ &\leq \left|\int_{Q_T}\sum_{i,j=1}^N a_{ij}D_i\phi D_j q\, dy\, dt\right| + C\|p\|^{\frac{k-1}{k}}_{L^r(Q(a_0,b_0))}\|\phi\|_{W^{1,0}_{s'}(Q_T)} \\ &\leq C\|Dq\|_{L^s(Q_T)}\|\phi\|_{W^{1,0}_{s'}(Q_T)} + C\|p\|^{\frac{k-1}{k}}_{L^r(Q(a_0,b_0))}\|\phi\|_{W^{1,0}_{s'}(Q_T)} \\ &\leq C\|p\|^{\frac{k-1}{k}}_{L^r(Q(a_0,b_0))}\|\phi\|_{W^{1,0}_{s'}(Q_T)} \end{aligned}$$

and the claim follows. □

Proposition 2.1.13. *If $\Gamma(k,x,a_0,b_0) < \infty$ for some $1 < k \leq N+2$, then $p \in L^r(Q(a,b))$ for all $r \in [1, \frac{N+2}{N+2-k})$ and $p \in \mathcal{H}^{s,1}(Q(a,b))$ for all $s \in (1, \frac{N+2}{N+3-k})$.*

Proof. The result follows by applying iteratively Lemma 2.1.12 and Proposition 2.1.11.

Let $r_1 < \frac{N+2}{N+1}$. Observe that $\Gamma(h,x,a_0,b_0) \leq C\Gamma(k,x,a_0,b_0)$ for $h \leq k$ and for some positive constant C. Therefore we can apply Proposition 2.1.11 and deduce $p \in L^{r_1}(Q(a_0,b_0))$. Fix a parameter m (to be chosen later) depending on k and r. Set $a_n = a_0 + \frac{n(a-a_0)}{m}$, $b_n = b_0 - \frac{n(b_0-b)}{m}$ for $n = 1, ..., m$. Suppose that $p \in L^{r_n}(Q(a_0,b_0))$ and take $s_n := \frac{kr_n}{k+r_n-1}$. Then $1 < s_n < r_n$, $s_n < k$ and $r_n = \frac{s_n(k-1)}{k-s_n}$. As in the previous proof, we consider $q = \eta p$ with $\eta(t) = 1$ for

$a_{n+1} \le t \le b_{n+1}$ and $\eta(t) = 0$ for $t \le a_n$, $t \ge b_n$, $|\eta'| \le \frac{2m}{a-a_0}$. As in the proof of Lemma 2.1.12, we get

$$\left| \int_{Q_T} q \partial_t \phi \, dy \, dt \right| \le C \|p\|^{\frac{k-1}{k}}_{L^{r_n}(Q(a_n,b_n))} \|\phi\|_{W^{1,0}_{s'_n}(Q_T)}$$

and

$$\|Dq\|_{L^{s_n}(Q_T)} \le C \|p\|^{\frac{k-1}{k}}_{L^{r_n}(Q(a_n,b_n))}.$$

with C depending on k, x, a_0, b_0. Therefore $p \in \mathcal{H}^{s_n,1}(Q(a_{n+1}, b_{n+1}))$. By the embedding Theorem for the $\mathcal{H}^{s,1}$ spaces (see Theorem 2.1.8), we have that $p \in L^{r_{n+1}}(Q(a_{n+1}, b_{n+1}))$ where

$$\frac{1}{r_{n+1}} = \frac{1}{s_n} - \frac{1}{N+2} = \frac{k + r_n - 1}{k r_n} - \frac{1}{N+2} = \frac{1}{r_n}\left(1 - \frac{1}{k}\right) + \frac{1}{k} - \frac{1}{N+2}.$$

Since $\frac{1}{r_1} > \frac{N+1}{N+2}$, it follows that

$$\frac{1}{r_2} - \frac{1}{r_1} < -\frac{1}{k}\left(1 - \frac{1}{N+2}\right) + \frac{1}{k} - \frac{1}{N+2} = \frac{1}{N+2}\left(\frac{1}{k} - 1\right) < 0.$$

By induction, since $\dfrac{1}{r_{n+1}} = g\left(\dfrac{1}{r_n}\right)$ with g increasing function, $\left(\dfrac{1}{r_n}\right)$ is a positive and decreasing sequence which converges to $\frac{N+2-k}{N+2}$. This implies that, for any $r < \frac{N+2}{N+2-k}$, after a finite number of steps m, we get $r_n > r$ and $p \in L^r(Q(a,b))$. Finally, by Lemma 2.1.12, we handle $p \in \mathcal{H}^{s,1}(Q(a,b))$ for all $s \in (1, \frac{N+2}{N+3-k})$. □

Corollary 2.1.14. *If $\Gamma(k, x, a_0, b_0) < \infty$ for some $k > N + 2$, then $p \in L^\infty(Q(a,b))$.*

Proof. By assumption, $\Gamma(k, x, a_0, b_0) < \infty$ for some $k > N + 2$, therefore $\Gamma(N + 2, x, a_0, b_0) \le C\Gamma(k, x, a_0, b_0) < \infty$ and, by Proposition 2.1.13, $p \in L^r(Q(a,b))$ for all $r \in [1, \infty)$. By Proposition 2.1.12, $p \in \mathcal{H}^{s,1}(Q(a,b))$ for all $1 < s < k$ and then, choosing $s > N+2$, by Theorem 2.1.8, $p \in L^\infty(Q(a,b))$. □

2.1.3 Pointwise estimates of kernels

We recall that T is a fixed positive number and a_0, a, b, b_0 are such that $0 < a_0 < a < b < b_0 \le T$. Assume that W_1, W_2 are Lyapunov functions for L, $W_1 \le W_2$ and there exists $1 \le \omega \in C^2(\mathbb{R}^N \times (0, \infty))$ such that for some positive constants $c_1(a_0, b_0)$, $c_2(a_0, b_0)$, $c_3(a_0, b_0)$, $c_4(a_0, b_0)$, $c_5(a_0, b_0)$ and $k > N + 2$

$$\begin{aligned} &\omega \le c_1 W_1; \quad |D\omega| \le c_2 \omega^{\frac{k-1}{k}} W_1^{\frac{1}{k}}; \\ &|D^2\omega| \le c_3 \omega^{\frac{k-2}{k}} W_1^{\frac{2}{k}}; \quad |\partial_t \omega| \le c_4 \omega^{\frac{k-2}{k}} W_1^{\frac{2}{k}}; \end{aligned} \tag{2.16}$$

$$\omega |F|^k \le c_5 W_2 \tag{2.17}$$

pointwise almost everywhere in $Q(a_0, b_0)$. Using the notation of the previous section, we write $\xi_1(x,t)$ to denote $\int_{\mathbb{R}^N} p(x,y,t)W_1(y,t)dy$ and ξ_2 for the analogous integral with W_2. Under these assumptions the following main theorem can be stated.

Theorem 2.1.15. *There exists a positive constant C such that*

$$0 < \omega(y,t)p(x,y,t) \le C\Bigg[(c_2^k + c_5 + c_3^{\frac{k}{2}} + c_2^{\frac{k}{2}}c_5^{\frac{1}{2}}) \int_{a_0}^{b_0} \xi_2 \tag{2.18}$$

$$+ \left(\frac{c_1}{(a-a_0)^{\frac{k}{2}}} + c_4^{\frac{k}{2}}\right) \int_{a_0}^{b_0} \xi_1\Bigg] \tag{2.19}$$

for all $x,\ y \in \mathbb{R}^N$ and $a \le t \le b$.

As preliminary result we prove an estimate of the L^∞ norm of solutions of certain parabolic problems.

Theorem 2.1.16. *Let $k > N+2$, $v \in L^k(Q_T)$, $w \in L^{\frac{k}{2}}(Q_T)$ and assume that $u \in L^k(Q_T)$ satisfies*

$$\int_{Q_T} u(\partial_t \phi + A_0 \phi)\, dx\, dt = \int_{Q_T} (v \cdot D\phi + w\phi)\, dx\, dt \tag{2.20}$$

for every $\phi \in C^{2,1}(Q_T)$ such that $\phi(\cdot,t)$ has compact support for every t. Then $u \in \Theta^k(Q_T)$ and

$$\|u\|_{L^\infty(Q_T)} \le C\|u\|_{\Theta^k(Q_T)} \le C(\|v\|_{L^k(Q_T)} + \|w\|_{L^{\frac{k}{2}}(Q_T)})$$

where C is a positive constant depending on N, T, k and the C_b^1-norm of the coefficients a_{ij}.

Proof. First we prove that

$$\|u\|_{L^k(Q_T)} \le C(\|v\|_{L^k(Q_T)} + \|w\|_{L^{\frac{k}{2}}(Q_T)}). \tag{2.21}$$

As in other proofs, we observe that, since $u \in L^k(Q_T)$, by approximation, (2.20) holds for functions $\phi \in W^{2,1}_{k'}(Q_T)$. Let $\psi \in C_c^\infty(Q_T)$. By Theorem A.0.8 there exists $\phi \in W^{2,1}_{k'}(Q_T)$ such that

$$\begin{cases} \partial_t \phi + A_0 \phi = \psi & \text{in} \quad Q_T, \\ \phi(x,T) = 0, & x \in \mathbb{R}^N \end{cases}$$

and the estimate

$$\|\phi\|_{W^{2,1}_{k'}(Q_T)} \le C\|\psi\|_{L^{k'}(Q_T)}$$

holds with a constant C depending on k, T and the coefficients a_{ij}. Moreover by the Sobolev embedding theorems (see Theorem A.0.9)

$$\|\phi\|_{L^{\frac{k}{k-2}}(Q_T)} \le C\|\phi\|_{W^{2,1}_{k'}(Q_T)}.$$

By assumption (2.20), we deduce

$$\left|\int_{Q_T} u\psi\right| \le C(\|v\|_{L^k(Q_T)}\|D\phi\|_{L^{k'}(Q_T)} + \|w\|_{L^{\frac{k}{2}}(Q_T)}\|\phi\|_{L^{\frac{k}{k-2}}(Q_T)})$$
$$\le C(\|v\|_{L^k(Q_T)} + \|w\|_{L^{\frac{k}{2}}(Q_T)})\|\psi\|_{L^{k'}(Q_T)}$$

and so the estimate for the $\|u\|_{L^k(Q_T)}$ follows.

Now let us prove the claim. As proved above, we have

$$\left|\int_{Q_T} u(\partial_t\phi + A_0\phi)\right| \le C\left(\|v\|_{L^k(Q_T)}\|D\phi\|_{L^{k'}(Q_T)} + \|w\|_{L^{\frac{k}{2}}(Q_T)}\|\phi\|_{L^{\frac{k}{k-2}}(Q_T)}\right)$$

for all $\phi \in W^{2,1}_{k'}(Q_T)$. Replacing ϕ by its differential quotients with respect to the space variable, we obtain

$$\left|\int_{Q_T} \tau_h u(\partial_t\phi + A_0\phi)\right| \le C\big[\big(\|u\|_{L^k(Q_T)} + \|v\|_{L^k(Q_T)}\big)\|\phi\|_{W^{2,1}_{k'}(Q_T)}$$
$$+ \|w\|_{L^{\frac{k}{2}}(Q_T)}\|D\phi\|_{L^{\frac{k}{k-2}}(Q_T)}\big].$$

By Sobolev embedding Theorem (see Theorem A.0.9),

$$\|D\phi\|_{L^s(Q_T)} \le C\|\phi\|_{W^{2,1}_{\frac{k}{k-1}}(Q_T)}$$

if $\dfrac{1}{s} = 1 - \dfrac{1}{k} - \dfrac{1}{N+2}$. Since $\dfrac{k}{k-1} < \dfrac{k}{k-2} < s$ by the assumption $k > N+2$, we have

$$\|D\phi\|_{L^{\frac{k}{k-2}}(Q_T)} \le C\|\phi\|_{W^{2,1}_{\frac{k}{k-1}}(Q_T)}$$

and so

$$\left|\int_{Q_T} \tau_h u(\partial_t\phi + A_0\phi)\right| \le C\big(\|u\|_{L^k(Q_T)} + \|v\|_{L^k(Q_T)} + \|w\|_{L^{\frac{k}{2}}(Q_T)}\big)\|\phi\|_{W^{2,1}_{k'}(Q_T)}. \tag{2.22}$$

Let now $\phi \in W^{2,1}_{k'}(Q_T)$ such that

$$\begin{cases} \partial_t\phi + A_0\phi = |\tau_h u|^{k-2}\tau_h u, & \text{in } \quad Q_T \\ \phi(x,T) = 0, & x \in \mathbb{R}^N \end{cases}$$

and

$$\|\phi\|_{W^{2,1}_{k'}(Q_T)} \le \||\tau_h u|^{k-1}\|_{L^{k'}(Q_T)} = \|\tau_h u\|^{k-1}_{L^k(Q_T)}.$$

For a ϕ so done, by (2.22), we deduce $u \in W^{1,0}_k(Q_T)$ and

$$\|Du\|_{L^k(Q_T)} \le C\big(\|u\|_{L^k(Q_T)} + \|v\|_{L^k(Q_T)} + \|w\|_{L^{\frac{k}{2}}(Q_T)}\big). \tag{2.23}$$

Consider the time derivative. By assumption we have

$$\int_{Q_T} u\partial_t\phi\,dx\,dt = \int_{Q_T}\left(-\sum_{i,j=1}^{N} a_{ij}(D_{ij}\phi)u + v\cdot D\phi + w\phi\right)dx\,dt$$
$$= \int_{Q_T}\left(\sum_{i,j=1}^{N} a_{ij}D_iu\,D_j\phi + \sum_{i,j=1}^{N}(D_ia_{ij})u\,D_j\phi + v\cdot D\phi + w\phi\right)dx\,dt$$

and, as above,

$$\left|\int_{Q_T} u\partial_t\phi\,dx\,dt\right| \le C\big[(\|Du\|_{L^k(Q_T)} + \|u\|_{L^k(Q_T)} + \|v\|_{L^k(Q_T)})\|D\phi\|_{L^{k'}(Q_T)}$$
$$+ \|w\|_{L^{\frac{k}{2}}(Q_T)}\|\phi\|_{L^{\frac{k}{k-2}}(Q_T)}\big].$$

By (2.23) we obtain

$$\left|\int_{Q_T} u\partial_t\phi\,dx\,dt\right| \le C\big[(\|u\|_{L^k(Q_T)} + \|v\|_{L^k(Q_T)} + \|w\|_{L^{\frac{k}{2}}(Q_T)})\|D\phi\|_{L^{k'}(Q_T)}$$
$$+ \|w\|_{L^{\frac{k}{2}}(Q_T)}\|\phi\|_{L^{\frac{k}{k-2}}(Q_T)}\big]$$

and, by (2.21),

$$\left|\int_{Q_T} u\partial_t\phi\,dx\,dt\right| \le C\big[(\|v\|_{L^k(Q_T)} + \|w\|_{L^{\frac{k}{2}}(Q_T)})\|D\phi\|_{L^{k'}(Q_T)}$$
$$+ \|w\|_{L^{\frac{k}{2}}(Q_T)}\|\phi\|_{L^{\frac{k}{k-2}}(Q_T)}\big].$$

(2.21), (2.23) and the last inequality imply that $u \in \Theta^k(Q_T)$ with

$$\|u\|_{\Theta^k(Q_T)} = \|u\|_{W_k^{1,0}(Q_T)} + \|\partial_t u\|_{\frac{k}{2},k;Q_T} \le C(\|v\|_{L^k(Q_T)} + \|w\|_{L^{\frac{k}{2}}(Q_T)}).$$

Finally, Theorem 2.1.9 implies

$$\|u\|_{L^\infty(Q_T)} \le C_1\|u\|_{\Theta^k(Q_T)} \le C_2(\|v\|_{L^k(Q_T)} + \|w\|_{L^{\frac{k}{2}}(Q_T)}).$$

□

We can prove the main theorem.

PROOF. (Theorem 2.1.15) In the first part of the proof we assume that ω is bounded.

Let $\Gamma(k,x,a_0,b_0) = (\int_{Q(a_0,b_0)}|F(y)|^k p(x,y,t)dy\,dt)^{\frac{1}{k}}$. Then, by (2.17) and Proposition 2.1.1,

$$\begin{aligned}\Gamma(k,x,a_0,b_0) &\le \int_{Q(a_0,b_0)}\omega|F(y)|^k p(x,y,t)dy\,dt\\ &\le c_5\int_{Q(a_0,b_0)} p(x,y,t)W_2(y,t) \le c_5\int_{a_0}^{b_0}\xi_2(x,t) < \infty.\end{aligned}$$

From Corollary 2.1.14, $p \in L^\infty(Q(a,b))$. Let η be a smooth function such that $\eta(t) = 1$ for $a \le t \le b$, $\eta(t) = 0$ for $t \le a_0$, $t \ge b_0$, $|\eta'| \le \frac{2}{a-a_0}$ and let $\psi \in C^{2,1}(Q_T)$ be such that $\psi(\cdot,t)$ has compact support for every t. We set $q = \eta^{\frac{k}{2}} p$ and $\phi(y,t) = \eta^{\frac{k}{2}}(t)\omega(y,t)\psi(y,t)$. By Lemma 2.1.10, we obtain

$$\int_{Q_T} (\partial_t \phi(y,t) + A\phi(y,t))p(x,y,t)dy\,dt = 0$$

and then, after some computations,

$$\begin{aligned}\int_{Q_T} \omega q(-\partial_t\psi - A_0\psi)dy\,dt = \int_{Q_T}\Bigg[q\Bigg(\psi A_0\omega + 2\sum_{i,j=1}^N a_{ij}D_i\omega D_j\psi \\ + \omega F\cdot D\psi + \psi F\cdot D\omega + \psi\partial_t\omega\Bigg) + \frac{k}{2}p\omega\psi\eta^{\frac{k-2}{2}}\partial_t\eta\Bigg]dy\,dt.\end{aligned}$$

Since ω is bounded, $\omega q \in L^1(Q_T)\cap L^\infty(Q_T)$. By Theorem (2.1.16),

$$\begin{aligned}\|\omega q\|_{L^\infty(Q_T)} \le C\big(\|qD\omega\|_{L^k(Q_T)} + \|\omega q F\|_{L^k(Q_T)} + \|qD^2\omega\|_{L^{\frac{k}{2}}(Q_T)} \\ + \|qF\cdot D\omega\|_{L^{\frac{k}{2}}(Q_T)} + \|q\partial_t\omega\|_{L^{\frac{k}{2}}(Q_T)} + \frac{1}{a-a_0}\|p\omega\eta^{\frac{k-2}{2}}\|_{L^{\frac{k}{2}}(Q_T)}\big)\end{aligned} \tag{2.24}$$

where C depends on N, k, T and the C^1_b-norm of a_{ij}. Now we estimate the right hand side in (2.24) by using (2.16) and (2.17).

$$\begin{aligned}\|\omega q F\|_{L^k(Q_T)} &= \left(\int_{Q_T}|\omega q F|^k\right)^{\frac{1}{k}} \le \left(\int_{Q_T}(q\omega)^{k-1}\omega q|F|^k\right)^{\frac{1}{k}} \\ &\le c_5(a_0,b_0)^{\frac{1}{k}}\left(\int_{Q_T}(q\omega)^{k-1}qW_2\right)^{\frac{1}{k}} \le c_5(a_0,b_0)^{\frac{1}{k}}\|\omega q\|^{\frac{k-1}{k}}_{L^\infty(Q_T)}\left(\int_{a_0}^{b_0}\xi_2 dt\right)^{\frac{1}{k}}.\end{aligned}$$

In a similar way

$$\|p\omega\eta^{\frac{k-2}{2}}\|_{L^{\frac{k}{2}}(Q_T)} \le c_1(a_0,b_0)^{\frac{2}{k}}\|\omega q\|^{\frac{k-2}{k}}_{L^\infty}\left(\int_{a_0}^{b_0}\xi_1 dt\right)^{\frac{2}{k}};$$

$$\|qD\omega\|_{L^k(Q_T)} \le c_2(a_0,b_0)\|\omega q\|^{\frac{k-1}{k}}_{L^\infty(Q_T)}\left(\int_{a_0}^{b_0}\xi_1 dt\right)^{\frac{1}{k}};$$

$$\|qD^2\omega\|_{L^{\frac{k}{2}}(Q_T)} \le c_3(a_0,b_0)\|\omega q\|^{\frac{k-2}{k}}_{L^\infty(Q_T)}\left(\int_{a_0}^{b_0}\xi_1 dt\right)^{\frac{2}{k}};$$

$$\|q\partial_t\omega\|_{L^{\frac{k}{2}}(Q_T)} \le c_4(a_0,b_0)\|\omega q\|^{\frac{k-2}{k}}_{L^\infty(Q_T)}\left(\int_{a_0}^{b_0}\xi_1 dt\right)^{\frac{2}{k}}$$

and

$$\|qF \cdot D\omega\|_{L^{\frac{k}{2}}(Q_T)} \leq c_2(a_0,b_0)c_5(a_0,b_0)^{\frac{1}{k}}\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}} \left(\int_{a_0}^{b_0} \xi_2 dt\right)^{\frac{2}{k}}.$$

Therefore, by (2.24) and the bounds above,

$$\begin{aligned}
\|\omega q\|_{L^\infty(Q_T)} \quad &\leq \quad C\Bigg[(c_2(a_0,b_0) + c_5(a_0,b_0)^{\frac{1}{k}})\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-1}{k}}\left(\int_{a_0}^{b_0} \xi_2\right)^{\frac{1}{k}} \\
&+ \quad (c_3(a_0,b_0) + c_2(a_0,b_0)c_5(a_0,b_0)^{\frac{1}{k}})\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}}\left(\int_{a_0}^{b_0} \xi_2\right)^{\frac{2}{k}} \\
&+ \quad \left(\frac{c_1(a_0,b_0)^{\frac{2}{k}}}{a-a_0} + c_4(a_0,b_0)\right)\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}}\left(\int_{a_0}^{b_0} \xi_1\right)^{\frac{2}{k}}\Bigg]
\end{aligned}$$

and then

$$\begin{aligned}
\|\omega q\|_{L^\infty(Q_T)}^{\frac{2}{k}} \quad &\leq \quad C\Bigg[(c_2(a_0,b_0) + c_5(a_0,b_0)^{\frac{1}{k}})\|\omega q\|_{L^\infty(Q_T)}^{\frac{1}{k}}\left(\int_{a_0}^{b_0} \xi_2\right)^{\frac{1}{k}} \\
&+ \quad (c_3(a_0,b_0) + c_2(a_0,b_0)c_5(a_0,b_0)^{\frac{1}{k}})\left(\int_{a_0}^{b_0} \xi_2\right)^{\frac{2}{k}} \\
&+ \quad \left(\frac{c_1(a_0,b_0)^{\frac{2}{k}}}{a-a_0} + c_4(a_0,b_0)\right)\left(\int_{a_0}^{b_0} \xi_1\right)^{\frac{2}{k}}\Bigg].
\end{aligned}$$

Setting

$$A = (c_2(a_0,b_0) + c_5(a_0,b_0)^{\frac{1}{k}})\left(\int_{a_0}^{b_0} \xi_2\right)^{\frac{1}{k}},$$

$$\begin{aligned}
B = &(c_3(a_0,b_0) + c_2(a_0,b_0)c_5(a_0,b_0)^{\frac{1}{k}})\left(\int_{a_0}^{b_0} \xi_2\right)^{\frac{2}{k}} \\
&+ \left(\frac{c_1(a_0,b_0)^{\frac{2}{k}}}{a-a_0} + c_4(a_0,b_0)\right)\left(\int_{a_0}^{b_0} \xi_1\right)^{\frac{2}{k}}
\end{aligned}$$

and $X = \|\omega q\|_{L^\infty(Q_T)}^{\frac{1}{k}}$, the inequality above can be written as $X^2 \leq AX + B$ and so $X \leq \frac{A+\sqrt{A^2+4B}}{2}$. It easily follows that

$$\begin{aligned}
&0 < \omega(y,t)p(x,y,t) \\
&\qquad \leq C\left[(c_2^k + c_5 + c_3^{\frac{k}{2}} + c_2^{\frac{k}{2}}c_5^{\frac{1}{2}})\int_{a_0}^{b_0} \xi_2 + \left(\frac{c_1}{(a-a_0)^{\frac{k}{2}}} + c_4^{\frac{k}{2}}\right)\int_{a_0}^{b_0} \xi_1\right].
\end{aligned}$$

If ω is not bounded, we set $\omega_\varepsilon = \dfrac{\omega}{1+\varepsilon\omega}$. Obviously ω_ε is bounded. It is easy to see that ω_ε satisfies (2.16) and (2.17) with constants $c_1,\ c_2,\ c_3,\ c_4,\ c_5$

independent of ε. Then the estimate of $\|\omega_\varepsilon q\|_{L^\infty(Q_T)}$ holds with constants in the right hand side of the previuos inequality which do not depend on ε. Letting $\varepsilon \to 0$ we deduce the claim. □

Remark 2.1.17. If W is a Lyapunov function for the operator A, in particular it is a Lyapunov function for L indeed it does not depend on the time variable and so it satisfies $\partial_t W = 0$ and $LW = AW \leq \lambda W$. We can therefore apply Theorem 2.1.15 to deduce upper bounds on the kernels as in [27, Theorem 4.1].

Proposition 2.1.18. *Suppose that the drift satisfies*

$$\limsup_{|x|\to\infty} |x|^{-r} F(x) \cdot \frac{x}{|x|} < -c \tag{2.25}$$

for some $r > 1$ and $c > 0$. Fix $T = 1$, then if $\alpha > \dfrac{r+1}{r-1}$, $\delta < \dfrac{c}{\Lambda(r+1)}$, $k > N+2$

$$p(x,y,t) \leq \frac{C}{t^{\frac{\alpha k r}{r+1}-1}} \exp\{-\delta t^\alpha |y|^{r+1}\}$$

for all $x,\ y \in \mathbb{R}^N$, $0 < t \leq 1$ and for a suitable constant C.

Proof. Let us verify assumptions (2.16) and (2.17).
Let

$$W_1(x,t) = W_2(x,t) = \exp\{t^\alpha \delta_1 |x|^{r+1}\}, \quad \omega = \exp\{t^\alpha \delta |x|^{r+1}\}$$

with $\delta < \delta_1 < \dfrac{c}{\Lambda(r+1)}$. By Proposition 2.1.2 we know that W_1 is a Lyapunov function for L. Obviously $\omega \geq 1$ and $\omega \leq W_1$ with constant $c_1 = 1$. We have to find $c_2(a_0,b_0)$ such that

$$|D\omega| \leq c_2(a_0,b_0)\omega^{\frac{k-1}{k}} W_1^{\frac{1}{k}}$$

that is

$$\begin{aligned}&\delta t^\alpha (r+1)|x|^r \exp\left\{t^\alpha \delta |x|^{r+1}\right\}\\ &\qquad \leq c_2(a_0,b_0)\exp\left\{\frac{k-1}{k}\delta t^\alpha |x|^{r+1}\right\}\exp\left\{\frac{1}{k}\delta_1 t^\alpha |x|^{r+1}\right\}\end{aligned}$$

or, equivalently,

$$\begin{aligned}\delta t^\alpha (r+1)|x|^r &\leq c_2(a_0,b_0)\exp\left\{\left(\delta\frac{k-1}{k} + \frac{\delta_1}{k} - \delta\right) t^\alpha |x|^{r+1}\right\}\\ &= c_2(a_0,b_0)\exp\left\{\frac{\delta_1 - \delta}{k} t^\alpha |x|^{r+1}\right\}.\end{aligned}$$

Observing that

$$\begin{aligned}\delta t^\alpha (r+1)|x|^r &= \frac{1}{|x|}\delta(r+1)\frac{k}{\delta_1-\delta}\frac{\delta_1-\delta}{k} t^\alpha |x|^{r+1}\\ &\leq \delta(r+1)\frac{k}{\delta_1-\delta}\exp\left\{\frac{\delta_1-\delta}{k}t^\alpha |x|^{r+1}\right\}\end{aligned}$$

for $|x| \geq 1$ and

$$\delta t^{\alpha}(r+1)|x|^{r} \leq \delta(r+1)$$

for $|x| < 1$, we obtain that the desired inequality is true with

$$c_2 = \delta(r+1)\max\left\{1, \frac{k}{\delta_1 - \delta}\right\},$$

independent of a_0 and b_0.
Similarly we obtain that

$$\begin{aligned}|D^2\omega| &\leq C(\delta^2 t^{2\alpha}(r+1)^2|x|^{2r} + \delta t^{\alpha}(r+1)(r-1+N)|x|^{r-1}) \\ &\leq c_3 \exp\left\{\frac{2(\delta_1-\delta)}{k}t^{\alpha}|x|^{r+1}\right\}\end{aligned}$$

with c_3 not depending on a_0 and b_0.
Concerning $c_4(a_0, b_0)$, we have

$$\begin{aligned}|\partial_t\omega| &= \delta\alpha t^{\alpha-1}|x|^{r+1}\exp\{t^{\alpha}\delta|x|^{r+1}\} \\ &\leq c_4(a_0,b_0)\exp\left\{\frac{k-2}{k}t^{\alpha}\delta|x|^{r+1}\right\}\exp\left\{\delta_1\frac{2}{k}|x|^{r+1}\right\}\end{aligned}$$

or equivalently

$$\begin{aligned}\delta\alpha t^{\alpha-1}|x|^{r+1} &= \frac{\alpha}{t}\frac{k}{2(\delta_1-\delta)}\delta\frac{2(\delta_1-\delta)}{k}t^{\alpha}|x|^{r+1} \\ &\leq c_4(a_0,b_0)\exp\left\{\frac{2(\delta_1-\delta)}{k}t^{\alpha}\delta|x|^{r+1}\right\}\end{aligned}$$

with $c_4(a_0, b_0) = \dfrac{\alpha\delta k}{2(\delta_1-\delta)a_0}$.
Finally, we have to find $c_5(a_0, b_0)$ such that

$$\exp\{\delta t^{\alpha}|x|^{r+1}\}|x|^{kr} \leq c_5(a_0,b_0)\exp\{\delta_1 t^{\alpha}|x|^{r+1}\}.$$

The function

$$f(s) = \frac{s^{kr}}{\exp\{(\delta_2-\delta)t^{\alpha}s^{r+1}\}}$$

attaints its maximum for $s = \dfrac{c(k,r,\delta,\delta_1)}{t^{\frac{\alpha}{r+1}}}$. Therefore $f(s) \leq \dfrac{c}{t^{\frac{\alpha kr}{r+1}}}$ and we can set

$$c_5(a_0,b_0) = \frac{c(k,r,\delta,\delta_1)}{a_0^{\frac{\alpha kr}{r+1}}}.$$

From (2.18), choosing $a_0 = \frac{1}{2}t,\ a = t,\ b = \frac{3}{2}t,\ b_0 = 2t$ and estimating ξ_1 as in Proposition 2.1.2, we deduce

$$\begin{aligned}p(x,y,t) &\leq C\left(\frac{1}{t^{\frac{\alpha kr}{r+1}-1}} + \frac{1}{t^{\frac{\alpha kr}{2(r+1)}-1}} + \frac{1}{t^{\frac{k}{2}-1}}\right)\exp\{-\delta t^{\alpha}|y|^{r+1}\} \\ &\leq \frac{C}{t^{\frac{\alpha kr}{r+1}-1}}\exp\{-\delta t^{\alpha}|y|^{r+1}\}\end{aligned}$$

for all $x,\ y \in \mathbb{R}^N$ and $t \leq 1$. $\square$

Remark 2.1.19. The estimate of the kernel proved in Proposition 2.1.18 in particular holds when A is given by $\Delta - |x|^r \frac{x}{|x|} \cdot D$. In the unidimensional case, consider for example the operator $A = D^2 - x^3 D$. We deduce the following bound for the kernel. If $\alpha > 2$, $\delta < \frac{1}{4}$, $k > 3$

$$p(x,y,t) \leq \frac{C}{t^{\frac{3\alpha k}{4}-1}} \exp\{-\delta t^\alpha y^4\}$$

for some positive C and for all $x,\ y \in \mathbb{R}$, $0 < t \leq 1$.

2.2 Heat kernel bounds for Schrödinger operators

A method similar to the one applied in the first section works also for Schrödinger operators. In this section, using Lyapunov functions techniques and parabolic regularity, we prove pointwise upper bounds on the kernel p.
We will deal with the problem of finding upper bounds for the kernels of Schrödinger operators in the next chapter too. The approach will be different and sometimes will give more refined estimates. Anyway, it is interesting to complete the study started in the previous section and to prove some estimates for Schrödinger operators making use of suitable Lyapunov functions.
We consider the operator $A = -\Delta + V$ with a nonnegative potential $V \in C^\alpha_{\text{loc}}(\mathbb{R}^N)$, $0 < \alpha < 1$. According to the results previously obtained, the semigroup e^{-tA} generated by the operator $-A$ can be represented in the form

$$e^{-tA} f(x) = \int_{\mathbb{R}^N} p(x,y,t) f(y) dy, \quad t > 0,\ x \in \mathbb{R}^N,$$

where p is a positive $C^{2+\alpha, 2+\alpha, 1+\frac{\alpha}{2}}_{\text{loc}}$ function, symmetric with respect to x and y which is pointwise dominated by the heat kernel of the Laplacian in $\mathbb{R}^N$, see Remark 1.3.21. More refined bounds are known when the potential V tends to ∞ at infinity in a polynomial way, see [13, Corollary 4.5.5] or [45] where also lower bounds are proved. In the case of $V(x) = |x|^\alpha$ we obtain estimates similar to those in [45]. However our method does not allow us to prove Davies-Simon estimate. On the other hand, it is not confined to special polynomial potentials but applies also to logarithmic or exponential growths.
As in the case of Kolmogorov operators, given a Lyapunov function ω we estimate the integral of ω against the kernel p, that is the function

$$\xi_\omega(x,t) = \int_{\mathbb{R}^N} p(x,y,t) \omega(y,t)\, dy.$$

Then we use parabolic regularity for Schrödinger operators with unbounded coefficients to deduce L^∞- bounds for ωp from the L^1-bounds. The same arguments have been applied in [28] but with Lyapunov functions independent of t,

yielding estimates in the form of Davies and Simon.
To shorten the notation we use $L = \partial_t - A = \partial_t + \Delta - V$. Observe however that the parabolic operator associated with A is $\partial_t + A$ and not L.

2.2.1 Integrability of Lyapunov functions

Since p admits Gaussian estimates, it is clear that any function with, say, an exponential growth is integrable with respect to p. Taking into account the growth of the potential V it is possible to integrate functions diverging very fast at infinity.

We say that $\omega : Q_T \to [0, +\infty)$ is a Lyapunov function for the operator L if it belongs to $C^{2,1}(Q_T)$, $\lim_{|x|\to\infty}\omega(x,t) = +\infty$ uniformly with respect to t in compact sets of $(0,T]$ and there exists $h : (0,\infty) \to [0,\infty)$ integrable in a neighborhood of 0 such that $L\omega(x,t) \le h(t)\omega(x,t)$ for all $(x,t) \in Q_T$. Note that we do not require that $\omega(x,0)$ tends to ∞ as $|x| \to \infty$.

In the proof of the proposition below we need to approximate e^{-tA} with the semigroups generated by some Schrödinger operators with bounded potentials. To this purpose we fix $0 \le \eta \in C_c^\infty(\mathbb{R})$ decreasing such that $\eta(s) = 1$ for $|s| \le 1$, $\eta(s) = 0$ for $|s| \ge 2$ and define $V_n(x) = \eta\left(|\frac{x}{n}|\right) V(x)$. Let moreover e^{-tA_n} be the semigroup generated by $-A_n = \Delta - V_n$ and $p_n(x,y,t)$ its kernel. By the maximum principle one easily obtains that $p_n \ge p_{n+1}$ and that $p_n \to p$ pointwise. Note that a Lyapunov function for A always exists since $V \ge 0$ (take for example $V(x) = 1 + |x|^2$, $x \in \mathbb{R}^N$) and therefore the maximum principle holds for bounded $C^{2,1}$ solutions of the Cauchy problem associated with the Schrödinger operator.

Lemma 2.2.1. *Consider the analytic semigroup generated by $-A_n$ in $C_b(\mathbb{R}^N)$. Then, for every $f \in C_b^{2+\alpha}(\mathbb{R}^N)$ the function $e^{-tA_n}f(x)$ converges to $e^{-tA}f(x)$ in $C^{2,1}(\mathbb{R}^N \times [0,T])$.*

Proof. Let $f \in C_b^{2+\alpha}(\mathbb{R}^N)$. Set $u_n(x,t) = e^{-tA_n}f(x)$, $u(x,t) = e^{-tA}f(x)$. Let us fix a radius $\rho > 0$. If $n > \rho + 1$, by the Schauder estimates for the operator A (see [20, Theorem 8.1.1]) we obtain

$$\|u_n\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(B_\rho\times[0,T])} \le C(\|u_n\|_{L^\infty(\mathbb{R}^N\times[0,\infty))} + \|f\|_{C^{2+\alpha}(\mathbb{R}^N)}).$$

By Ascoli's Theorem the sequence (u_n) converges to a function v in $C^{2,1}(\mathbb{R}^N \times [0,\infty))$. Since $\partial_t u_n + A_n u_n = 0$ in $B_\rho \times (0,T]$ for $n > \rho$ we have $\partial_t v + Av = 0$ in $\mathbb{R}^N \times (0,T]$. Moreover $v(x,0) = f(x)$ and $|v(x,t)| \le \|f\|_\infty$. Consider now the difference $w = u - v$. Obviously $w \in C^{2,1}(\mathbb{R}^N \times [0,T])$, is bounded and satisfies

$$\begin{cases} \partial_t w + Aw = 0 & \text{in} \quad \mathbb{R}^N \times (0,T] \\ w(x,0) = 0 & \text{in} \quad \mathbb{R}^N. \end{cases}$$

By the maximum principle it follows $w = 0$ and then u_n converges to u in $C^{2,1}(\mathbb{R}^N \times [0,\infty))$.
Observe that if f is only a $C_b(\mathbb{R}^N)$ function u_n converges pointwise to u. □

We also need the following lemma.

Lemma 2.2.2. *Assume that $V \in L^\infty(\mathbb{R}^N)$ and let $f \in BUC(Q_T)$. Then the function*

$$F(x,t) = \int_{\mathbb{R}^N} p(x,y,t) f(y,t)\, dy$$

is continuous in Q_T. Moreover, if $f \in BUC^{2,1}(Q_T)$, then

$$\partial_t F(x,t) = \int_{\mathbb{R}^N} p(x,y,t) L f(y,t)\, dt$$

with $L = \partial_t - A$.

Proof. Since V is bounded, the semigroup $(e^{-tA})_{t\geq 0}$ is strongly continuous in $BUC(\mathbb{R}^N)$ (the space of bounded and uniformly continuous functions on $\mathbb{R}^N$). Writing $F(\cdot,t) = e^{-tA} f(\cdot,t)$ its continuity easily follows. If $f \in BUC^{2,1}(Q_T)$, then, for every fixed t, the function $f(\cdot,t)$ belongs to the domain of the generator of $(e^{-tA})_{t\geq 0}$ in $BUC(\mathbb{R}^N)$. It follows that

$$\partial_t F(\cdot,t) = -e^{-tA} A f(\cdot,t) + e^{-tA}\partial_t f(\cdot,t)$$

and the proof follows. □

We refer the reader to [28, Proposition 2.5] and to [5, Lemma 2.32] for results similar to the next proposition, when the Lyapunov function is independent of t.

Proposition 2.2.3. *For each $t \in [0,T]$, the Lyapunov function $\omega(\cdot,t)$ is integrable with respect to the measure $p(x,\cdot,t)$. Moreover, setting*

$$\xi_\omega(x,t) = \int_{\mathbb{R}^N} p(x,y,t)\omega(y,t) dy, \tag{2.26}$$

the inequality

$$\xi_\omega(x,t) \leq e^{\int_0^t h(s)ds}\omega(x,0) \tag{2.27}$$

holds.

Proof. Let us consider, for every $\alpha \geq 0$, $\psi_\alpha \in C_b^\infty(\mathbb{R})$ such that $\psi_\alpha(s) = s$ for $s \leq \alpha$, ψ_α is constant in $[\alpha+1,\infty)$, $\psi'_\alpha \geq 0$ and $\psi''_\alpha \leq 0$. From the concavity of ψ_α it follows that

$$s\psi'_\alpha(s) \leq \psi_\alpha(s) \qquad \forall\ s \geq 0. \tag{2.28}$$

Obviously $\psi_\alpha \circ \omega \in BUC(Q_T)$ and, moreover, it belongs to $BUC^{2,1}(Q(\varepsilon,T))$ for every $\varepsilon > 0$, since is constant for $t \geq \varepsilon > 0$ and large $|x|$. According with the previous notation we set $\xi_\alpha^n(x,t) = \int_{\mathbb{R}^N} p_n(x,y,t)\psi_\alpha(\omega(y,t))dy$. Lemma 2.2.2 yields for $t \geq \varepsilon$

$$\partial_t \xi_\alpha^n(x,t) = \int_{\mathbb{R}^N} p_n(x,y,t) L_n(\psi_\alpha \circ \omega)(y,t) dy$$

where $L_n = \partial_t - A_n$. By (2.28) we obtain

$$\begin{aligned} L_n(\psi_\alpha \circ \omega)(x,t) &= \psi'_\alpha(\omega(x,t))L_n\omega(x,t) + V_n(x)[\psi'_\alpha(\omega(x,t))\omega(x,t) \\ &- \psi_\alpha(\omega(x,t))] - \psi''_\alpha(\omega(x,t))|D\omega(x,t)|^2 \\ &\le \psi'_\alpha(\omega(x,t))L_n\omega(x,t). \end{aligned}$$

Thus, for $t \ge \varepsilon$,

$$\begin{aligned} \partial_t \xi^n_\alpha(x,t) &\le \int_{\mathbb{R}^N} p_n(x,y,t)\psi'_\alpha(\omega(y,t))L_n\omega(y,t)dy \\ &\le \int_{\mathbb{R}^N} p_n(x,y,t)\psi'_\alpha(\omega(y,t))L\omega(y,t)dy \end{aligned}$$

if n is sufficiently large since, for fixed α, the function $\psi'_\alpha(\omega(y,t))$ has compact support. Using the property of ω, the positivity of ψ' and (2.28) again we get

$$\partial_t \xi^n_\alpha(x,t) \le h(t)\int_{\mathbb{R}^N} p_n(x,y,t)\psi_\alpha(\omega(y,t))dy = h(t)\xi^n_\alpha(x,t).$$

Therefore, by Gronwall's Lemma, for $t \ge \varepsilon$.

$$\xi^n_\alpha(x,t) \le e^{\int_\varepsilon^t h(s)ds}\xi_\alpha(x,\varepsilon).$$

Since $\xi_\alpha(x,\varepsilon) \to \psi_\alpha(\omega(x,0))$ as $\varepsilon \to 0$, by Lemma 2.2.2, letting $\varepsilon \to 0$ we obtain

$$\xi^n_\alpha(x,t) \le e^{\int_\varepsilon^t h(s)ds}\psi_\alpha(\omega(x,0)).$$

Letting $\alpha \to \infty$ in the previous inequality and using Fatou's Lemma we get

$$\int_{\mathbb{R}^N} p_n(x,y,t)\omega(y,t)dy \le \liminf_{\alpha\to\infty}\xi^n_\alpha(x,t) \le e^{\int_0^t h(s)ds}\omega(x,0).$$

Letting $n \to \infty$, the first member in the previous inequality tends to $\xi_\omega(x,t)$ by monotone convergence so the claim follows. □

2.2.2 Regularity for parabolic problems and some interpolative estimates

We prove a parabolic regularity result needed in the following subsection to deduce pointwise estimates for the kernels.

Theorem 2.2.4. *Let $1 < k < \infty$ and suppose that for every $\gamma > 0$ there exists $C_\gamma > 0$ such that $|DV| \le \gamma V^{\frac{3}{2}} + C_\gamma$. If $u \in L^k(Q_T) \cap W^{2,1}_k(B_R \times [0,T])$ for every $R > 0$ solves*

$$\begin{cases} \partial_t u - \Delta u + Vu = g & \text{in} \quad Q_T \\ u(y,0) = 0 & y \in \mathbb{R}^N \end{cases}$$

with $g \in L^k(Q_T)$, then

$$\|u\|_{W^{2,1}_k(Q_T)} + \|Vu\|_{L^k(Q_T)} \le C_0\|g\|_{L^k(Q_T)}$$

where C_0 depends on N, k, T and C_γ.

Proof. By [31, Proposition 6.5], there exists a function $z \in W^{2,1}_k(Q_T)$ with $Vz \in L^k(Q_T)$ which solves the problem above and satisfies the estimate

$$\|z\|_{W^{2,1}_k(Q_T)} + \|Vz\|_{L^k(Q_T)} \le C\|g\|_{L^k(Q_T)}.$$

Then we have to prove that $u = z$. The difference $w = u - z \in L^k(Q_T) \cap W^{2,1}_k(B_R \times [0,T])$ for every $R > 0$ and satisfies

$$\int_{Q_T} w(-\partial_t \phi - \Delta\phi + V\phi) = 0 \tag{2.29}$$

for every $\phi \in C^{2,1}(Q_T)$ vanishing at the time T and with support in $B_R \times [0,T]$ for some $R > 0$. By density (2.29) holds for every $\phi \in W^{2,1}_{k'}(Q_T)$ such that ϕ vanishes at the time T and $V\phi \in L^{k'}(Q_T)$. By using [31, Proposition 6.5] again, we obtain that, given $\psi \in L^{k'}(Q_T)$, there exists $\phi \in W^{2,1}_{k'}(Q_T)$ with $\phi(\cdot, T) = 0$ and $V\phi \in L^{k'}(Q_T)$ such that $-\partial_t\phi - \Delta\phi + V\phi = \psi$. Therefore

$$\int_{Q_T} w\psi = 0$$

for every $\psi \in L^{k'}(Q_T)$ and then $w = 0$ and $u = v$. $\square$

The following interpolative estimate for the sup norm of u will be crucial in the next section.

Proposition 2.2.5. *Assume that $k > \frac{N+2}{2}$. Then there exists $C > 0$ such that for every $u \in W^{2,1}_k(Q_T)$ the estimate*

$$\|u\|_{L^\infty(Q_T)} \le C\|u\|^{1-\theta}_{L^1(Q_T)}\|u\|^{\theta}_{W^{2,1}_k(Q_T)}$$

holds with

$$\theta = \frac{N+2}{(N+2)\left(1 - \dfrac{1}{k}\right) + 2}.$$

Proof. Since there exists a linear extension operator from $W^{2,1}_k(Q_T)$ to $W^{2,1}_k(\mathbb{R}^{N+1})$ which is also continuous from $L^r(Q_T)$ to $L^r(\mathbb{R}^{N+1})$ for $1 \le r \le \infty$ we prove the claimed estimate for functions in $W^{2,1}_k(\mathbb{R}^{N+1})$. Let R be an unitary cube of $\mathbb{R}^{N+1}$. We start by proving that there exists a positive constant C such that

$$\|u\|_{L^\infty(R)} \le C(\|u\|_{L^1(R)} + \|\partial_t u\|_{L^k(R)} + \|D^2 u\|_{L^k(R)})$$

for every $u \in W^{2,1}_k(R)$. Suppose that this is not true, then for every $n \in \mathbb{N}$ there exists $u_n \in W^{2,1}_k(R)$ such that

$$\|u_n\|_{L^\infty(R)} \ge n(\|u_n\|_{L^1(R)} + \|\partial_t u_n\|_{L^k(R)} + \|D^2 u_n\|_{L^k(R)}). \tag{2.30}$$

We can also suppose $\|u_n\|_{L^\infty(R)} = 1$. Obviously we have $\|u_n\|_{L^k(R)} \le 1$ and, by (2.30), we deduce that $(u_n)_{n\in\mathbb{N}}$ is bounded in $W^{2,1}_k(R)$. Since the embedding of

$W_k^{2,1}(R)$ into $C(\overline{R})$ is compact (see Theorem A.0.9), there exists a subsequence (u_{n_k}) converging in $L^\infty(R)$ to some function $v \in C(\overline{R})$. In particular (u_{n_k}) converges to v in $L^1(R)$, but, by (2.30), $\|u_n\|_{L^1(R)} \leq \frac{1}{n}$ and then $v = 0$. This is a contraddiction since $\|u_n\|_{L^\infty(R)} = 1$. It immediately follows that there exists a positive constant C such that

$$\|u\|_{L^\infty(\mathbb{R}^{N+1})} \leq C(\|u\|_{L^1(\mathbb{R}^{N+1})} + \|\partial_t u\|_{L^k(\mathbb{R}^{N+1})} + \|D^2 u\|_{L^k(\mathbb{R}^{N+1})})$$

for every $u \in W_k^{2,1}(\mathbb{R}^{N+1})$. Let $\lambda > 0$. Choosing $v(x,t) = u(\lambda x, \lambda^2 t)$, we get

$$\begin{aligned}\|u\|_{L^\infty(\mathbb{R}^{N+1})} \leq\ & C(\lambda^{-(N+2)}\|u\|_{L^1(\mathbb{R}^{N+1})} \\ & + \lambda^{(2-\frac{N+2}{k})}(\|\partial_t u\|_{L^k(\mathbb{R}^{N+1})} + \|D^2 u\|_{L^k(\mathbb{R}^{N+1})}))\end{aligned}$$

for all $\lambda > 0$ and $u \in W_k^{2,1}(\mathbb{R}^{N+1})$. It follows that the function

$$\begin{aligned}g(\lambda) = \|u\|_{L^\infty(\mathbb{R}^{N+1})} - C(\lambda^{-(N+2)}\|u\|_{L^1(\mathbb{R}^{N+1})} \\ + \lambda^{(2-\frac{N+2}{k})}(\|\partial_t u\|_{L^k(\mathbb{R}^{N+1})} + \|D^2 u\|_{L^k(\mathbb{R}^{N+1})})) \leq 0\end{aligned}$$

for all $\lambda > 0$ and, in particular, minimising over λ, in correspondence of

$$\lambda = \left[\frac{N+2}{2-\frac{N+2}{k}} \frac{\|u\|_{L^1(\mathbb{R}^{N+1})}}{\|\partial_t u\|_{L^k(\mathbb{R}^{N+1})} + \|D^2 u\|_{L^k(\mathbb{R}^{N+1})}}\right]^{\frac{k}{4k+Nk-N-2}},$$

we obtain then claimed inequality. $\square$

Finally, we state an interpolative inequality.

Proposition 2.2.6. *Let $1 \leq k \leq \infty$ and suppose that for every $\gamma > 0$ there exists $C_\gamma > 0$ such that $|DV| \leq \gamma V^{\frac{3}{2}} + C_\gamma$. Then there exists two constants $m,\ \mu_0$ such that for every $u \in W_k^{2,1}(Q_T)$ with $Vu \in L^k(Q_T)$ the following estimate holds for $0 < \mu \leq \mu_0$*

$$\|V^{\frac{1}{2}} Du\|_{L^k(Q_T)} \leq \mu \|u\|_{W_k^{2,1}(Q_T)} + \frac{m}{\mu}\|Vu\|_{L^k(Q_T)}.$$

Proof. Let u be a smooth function with compact support contained in $B_R \times [0,T]$ for some $R > 0$. By [31, Proposition 2.3] there exist two positive constants $m,\ \mu_0$ such that for $0 < \mu \leq \mu_0$

$$\int_{\mathbb{R}^N} V(x)^{\frac{k}{2}} |Du(x,t)|^k\, dx \leq \mu^k \int_{\mathbb{R}^N} |\Delta u(x,t)|^k\, dx + \frac{m^k}{\mu^k}\int_{\mathbb{R}^N} V(x)^k |u(x,t)|^k\, dx.$$

Integrating over $[0,T]$ with respect to t, the estimate follows for smooth and with compact support functions. By density we deduce the claim. $\square$

2.2.3 Pointwise estimates on kernels

To prove the main result of this paper we need the following assumptions on the potential V and on the Lyapunov function ω.

(A1) $0 \le V \in C^1(\mathbb{R}^N)$ and $\forall\, \gamma > 0$ there exists $C_\gamma > 0:\ |DV| \le \gamma V^{\frac{3}{2}} + C_\gamma$;

(A2) $0 < \omega \in C^{2,1}(\mathbb{R}^N \times ([0,\infty))$ is a Lyapunov function satisfying

$$\frac{|\partial_t \omega|}{\omega} + \frac{|D\omega|^2}{\omega^2} + \frac{|\Delta \omega|}{\omega} \le \gamma V + C \tag{2.31}$$

where γ, C are suitable positive constants. We denote by ξ_ω the function introduced in 2.26 and relative to ω and fix $0 < a_0 < a < b < b_0 < T$ with the property $b_0 - b \ge a - a_0$.

Theorem 2.2.7. *There exists $\gamma_0 > 0$ such that if assumptions (A1) and (A2) are satisfied with $\gamma < \gamma_0$, then*

$$\omega(y,t)p(x,y,t) \le \frac{C}{(a-a_0)^{\frac{N+2}{2}}} \int_{a_0}^{b_0} \xi_\omega(x,t)dt$$

for $a \le t \le b$ and $x,\ y \in \mathbb{R}^N$.

Proof. In the whole proof x will be considered as a parameter and we regard the kernel as a function of the variables (y,t). Similarly, all the differential operators with respect to the space variables will act on the y variable. Observe that p satisfies $p_t = \Delta p - Vp$ for $y \in \mathbb{R}^N, t > 0$. Moreover it belongs to $L^k(Q(a,b))$ for every $1 \le k \le \infty$ since it admits Gaussian estimates. Let η be a smooth function such that $0 \le \eta \le 1$, $\eta(t) = 1$ for $a \le t \le b$, $\eta(t) = 0$ for $t \le a_0$ and $t \ge b_0$, $0 \le |\eta_t| \le \frac{2}{a-a_0}$ and set $q = \eta^k p$. Then $q \in L^k(Q_T) \cap W^{2,1}_k(B_R \times [0,T])$ for all $R > 0$ and satisfies the parabolic problem

$$\begin{cases} \partial_t q - \Delta q + Vq = k\eta^{k-1} p \eta_t & \text{in } Q_T \\ q(y,0) = 0 & y \in \mathbb{R}^N. \end{cases}$$

From Theorem 2.2.4 it follows that, for all $1 < k < \infty$, $q \in W^{2,1}_k(Q_T)$ and $Vq \in L^k(Q_T)$. In particular, from Proposition 2.2.6, $V^{\frac{1}{2}} Dq \in L^k(Q_T)$. Let $\omega_\varepsilon = \omega/(1+\varepsilon\omega)$ for $0 < \varepsilon < 1$. We have

$$\begin{aligned} \frac{D\omega_\varepsilon}{\omega_\varepsilon} &= \frac{D\omega}{\omega(1+\varepsilon\omega)}; \quad \frac{\partial_t \omega_\varepsilon}{\omega_\varepsilon} = \frac{\partial_t \omega}{\omega(1+\varepsilon\omega)}; \\ \frac{\Delta\omega_\varepsilon}{\omega_\varepsilon} &= \frac{\Delta\omega}{\omega(1+\varepsilon\omega)} - \frac{2\varepsilon}{(1+\varepsilon\omega)^2}\frac{|D\omega|^2}{\omega}. \end{aligned}$$

Using the last equations we obtain estimates like (2.31) for ω_ε, namely

$$\frac{|\partial_t \omega_\varepsilon|}{\omega_\varepsilon} + \frac{|D\omega_\varepsilon|^2}{\omega_\varepsilon^2} + \frac{|\Delta \omega_\varepsilon|}{\omega_\varepsilon} \le 3(\gamma V + C). \tag{2.32}$$

The function $\omega_\varepsilon q$ satisfies the parabolic equation

$$\begin{cases} \partial_t(\omega_\varepsilon q) - \Delta(\omega_\varepsilon q) + V\omega_\varepsilon q = (\partial_t \omega_\varepsilon) q + k\eta^{k-1} p \omega_\varepsilon \eta_t & \\ \qquad\qquad -q\Delta\omega_\varepsilon - 2D\omega_\varepsilon \cdot Dq & \text{in } Q_T \\ \omega_\varepsilon(y,0) q(y,0) = 0 & y \in \mathbb{R}^N. \end{cases}$$

Observe that $V\omega_\varepsilon q \in L^k(Q_T)$ since ω_ε is bounded and $Vq \in L^k(Q_T)$. In a similar way we obtain that $k\eta^{k-1}p\omega_\varepsilon\eta_t \in L^k(Q_T)$. Using (2.32) we see that the other terms in the right hand side of the previous equality are in $L^k(Q_T)$. In fact we have

$$|\partial_t\omega_\varepsilon| q \leq \gamma V\omega_\varepsilon q + C\omega_\varepsilon q \in L^k(Q_T).$$

Similarly for the remaining terms. This implies that $\omega_\varepsilon q \in W^{2,1}_k(Q_T)$. We rewrite the previous equation in the form

$$\begin{cases} \partial_t(\omega_\varepsilon q) - \Delta(\omega_\varepsilon q) + V\omega_\varepsilon q = (\partial_t\omega_\varepsilon)q + k\eta^{k-1}p\omega_\varepsilon\eta_t & \\ \qquad\qquad -2\dfrac{D\omega_\varepsilon}{\omega_\varepsilon}D(\omega_\varepsilon q) - q\Delta\omega_\varepsilon + 2\dfrac{|D\omega_\varepsilon|^2}{\omega_\varepsilon}q & \text{in } Q_T \\ \omega_\varepsilon(y,0)q(y,0) = 0 & y \in \mathbb{R}^N \end{cases}$$

and estimate the L^k-norm of the right hand side choosing k greater then $\frac{N+2}{2}$. We have

$$\begin{aligned} \|(\partial_t\omega_\varepsilon)q\|_{L^k(Q_T)} &\leq \gamma\|\omega_\varepsilon qV\|_{L^k(Q_T)} + C\|\omega_\varepsilon q\|_{L^k(Q_T)} &(2.33)\\ &\leq \gamma\|\omega_\varepsilon qV\|_{L^k(Q_T)} + C\|\omega_\varepsilon q\|^{\frac{k-1}{k}}_{L^\infty(Q_T)}\left(\int_{Q(a_0,b_0)}\omega p\right)^{\frac{1}{k}} \end{aligned}$$

$$\|k\eta^{k-1}p\omega_\varepsilon\eta_t\|_{L^k(Q_T)} \leq \frac{2k}{a-a_0}\|\omega_\varepsilon q\|^{\frac{k-1}{k}}_{L^\infty(Q_T)}\left(\int_{Q(a_0,b_0)}\omega p\right)^{\frac{1}{k}} \tag{2.34}$$

$$\begin{aligned} \left\|q\left(\Delta\omega_\varepsilon - 2\frac{|D\omega_\varepsilon|^2}{\omega_\varepsilon}\right)\right\|_{L^k(Q_T)} &\leq 6\Big[\gamma\|V\omega_\varepsilon q\|_{L^k(Q_T)} &(2.35)\\ &\qquad + C\|\omega_\varepsilon q\|^{\frac{k-1}{k}}_{L^\infty(Q_T)}\left(\int_{Q_T}\omega q\right)^{\frac{1}{k}}\Big] &(2.36) \end{aligned}$$

and finally, using Proposition 2.2.6 and the interpolative inequality

$$\|D(\omega_\varepsilon q)\|_{L^k(Q_T)} \leq \delta\|\omega_\varepsilon q\|_{W^{2,1}_k(Q_T)} + \frac{K}{\delta}\|\omega_\varepsilon q\|_{L^k(Q_T)},$$

for all $\delta > 0$ we obtain

$$\begin{aligned} \left\|\frac{D\omega_\varepsilon}{\omega_\varepsilon}D(\omega_\varepsilon q)\right\|_{L^k(Q_T)} &\leq \sqrt{3}\big\{\gamma^{\frac{1}{2}}\|V^{\frac{1}{2}}D(\omega_\varepsilon q)\|_{L^k(Q_T)} &(2.37)\\ &+ C^{\frac{1}{2}}\|D(\omega_\varepsilon q)\|_{L^k(Q_T)}\big\} &(2.38)\\ &\leq \sqrt{3}\Big\{\gamma^{\frac{1}{2}}\left(\mu\|\omega_\varepsilon q\|_{W^{2,1}_k(Q_T)} + \frac{m}{\mu}\|V\omega_\varepsilon q\|_{L^k(Q_T)}\right)\\ &+ C^{\frac{1}{2}}\left(\delta\|\omega_\varepsilon q\|_{W^{2,1}_k(Q_T)} + \frac{K}{\delta}\|\omega_\varepsilon q\|_{L^k(Q_T)}\right)\Big\} \end{aligned}$$

for all $\delta > 0$ and $\mu \le \mu_0$. Setting

$$\Lambda = \frac{2}{a-a_0}\left(\int_{Q(a_0,b_0)} \omega p\right)^{\frac{1}{k}} = \frac{2}{a-a_0}\left(\int_{a_0}^{b_0} \xi_\omega(x,t)dt\right)^{\frac{1}{k}},$$

from (2.33), (2.34), (2.35) and (2.37) and Theorem 2.2.4, we obtain

$$\begin{aligned} & \|\omega_\varepsilon q\|_{W_k^{2,1}(Q_T)} + \|V\omega_\varepsilon q\|_{L^k(Q_T)} \le C_0\Bigg\{\Bigg(k + 7C\frac{a-a_0}{2} \\ + \quad & \frac{a-a_0}{2}\sqrt{3}C^{\frac{1}{2}}\frac{K}{\delta}\Bigg)\|\omega_\varepsilon q\|_{L^\infty(Q_T)}^{\frac{k-1}{k}}\Lambda + \left(\sqrt{3}\gamma^{\frac{1}{2}}\mu + \sqrt{3}C^{\frac{1}{2}}\delta\right)\|\omega_\varepsilon q\|_{W_k^{2,1}(Q_T)} \\ + \quad & \left(7\gamma + \sqrt{3}\gamma^{\frac{1}{2}}\frac{m}{\mu}\right)\|V\omega_\varepsilon q\|_{L^k(Q_T)}\Bigg\} \end{aligned}$$

for all $\delta > 0$ and $\mu \le \mu_0$. Choosing γ, δ small enough so that $\sqrt{3}C_0(\gamma^{\frac{1}{2}}\mu_0 + C^{\frac{1}{2}}\delta) < 1$ and $C_0(7\gamma + \sqrt{3}\gamma^{\frac{1}{2}}m/\mu_0) < 1$ we deduce

$$\|\omega_\varepsilon q\|_{W_k^{2,1}(Q_T)} + \|V\omega_\varepsilon q\|_{L^k(Q_T)} \le C\|\omega_\varepsilon q\|_{L^\infty(Q_T)}^{\frac{k-1}{k}}\Lambda,$$

with C independent of ε. By Proposition 2.2.5 we have

$$\|\omega_\varepsilon q\|_{L^\infty(Q_T)} \le C\|\omega_\varepsilon q\|_{L^1(Q_T)}^{1-\theta}\|\omega_\varepsilon q\|_{W_k^{2,1}(Q_T)}^{\theta}$$

with $\theta = \frac{N+2}{(N+2)(1-\frac{1}{k})+2}$ and therefore

$$\|\omega_\varepsilon q\|_{W_k^{2,1}(Q_T)} \le C\Lambda\|\omega_\varepsilon q\|_{L^1(Q_T)}^{(1-\theta)\frac{k-1}{k}}\|\omega_\varepsilon q\|_{W_k^{2,1}(Q_T)}^{\theta\frac{k-1}{k}}.$$

This yields

$$\begin{aligned} \|\omega_\varepsilon q\|_{W_k^{2,1}(Q_T)} & \le C\Lambda\|\omega_\varepsilon q\|_{L^1(Q_T)}^{\left(1-\frac{N+2}{2k}\right)(1-\frac{1}{k})} \\ & \le C\Lambda\|\omega q\|_{L^1(Q_T)}^{\left(1-\frac{N+2}{2k}\right)(1-\frac{1}{k})}. \end{aligned}$$

Using again the interpolative estimate of Proposition 2.2.5 we obtain

$$\|\omega_\varepsilon q\|_{L^\infty(Q_T)} \le C\|\omega_\varepsilon q\|_{L^1(Q_T)}^{1-\theta}\|\omega_\varepsilon q\|_{W_k^{2,1}(Q_T)}^{\theta} \le C\Lambda\|\omega q\|_{L^1(Q_T)}^{\left(1-\frac{N+2}{2k}\right)}$$

and, finally, estimating the integrals of $\omega_\varepsilon q$ trough ξ_ω,

$$\omega_\varepsilon(y,t)p(x,y,t) \le C\frac{1}{(a-a_0)^{\frac{N+2}{2}}}\int_{a_0}^{b_0}\xi_\omega(x,t)dt$$

for $a \le t \le b$ and $x,\ y \in \mathbb{R}^N$. Observing that the constant in the right hand side does not depend on ε and letting $\varepsilon \to 0$ we conclude the proof. $\square$

2.2.4 Small time estimates

In this section we apply Theorem 2.2.7 to get explicit bounds, for small times, of the heat kernels of some Schrödinger operators with unbounded potentials.

Proposition 2.2.8. *Assume that $V(x) \geq M|x|^\alpha$ for some $\alpha > 2,\ M > 0$. Then there exist $0 < c < \frac{2\sqrt{M}}{2+\alpha},\ C > 0$ such that*

$$p(x,y,t) \leq \frac{C}{t^{\frac{N}{2}}} \exp\left\{-ct(|x|^{1+\frac{\alpha}{2}} + |y|^{1+\frac{\alpha}{2}})\right\}$$

for all $x,\ y \in \mathbb{R}^N$ and $0 < t \leq 1$.

Proof. By Remark 1.3.21 we may assume that $V(x) = M|x|^\alpha$. We define $\omega(x,t) = \exp\{ct|x|^{1+\frac{\alpha}{2}}\}$. By an easy computation we get

$$\begin{aligned}L\omega(x,t) &= \omega(x,t)\big[c|x|^{1+\frac{\alpha}{2}} + c^2(1+\frac{\alpha}{2})^2t^2|x|^\alpha \\ &\quad + c(1+\frac{\alpha}{2})(\frac{\alpha}{2}-1+N)t|x|^{\frac{\alpha}{2}-1} - V(x)\big] \\ &\leq \omega(x,t)|x|^\alpha\big[c|x|^{1-\frac{\alpha}{2}} + c^2(1+\frac{\alpha}{2})^2t^2 \\ &\quad + c(1+\frac{\alpha}{2})(\frac{\alpha}{2}-1+N)t|x|^{-\frac{\alpha}{2}-1} - M\big].\end{aligned}$$

Recalling that $t \leq 1,\ \alpha \geq 2$ and $c < \frac{2\sqrt{M}}{2+\alpha}$, we see that the last member in the previous inequality is negative for $|x|$ large. If $|x|$ is small clearly there exists a positive constant λ such that $L\omega \leq \lambda \leq \lambda\omega$. This proves that ω is a Lyapunov function with $h(t) = \lambda$ and for $0 < t \leq 1$, so, from the Proposition 2.2.3, it follows that

$$\xi_\omega(x,t) \leq e^{\lambda t}\omega(x,0) = e^{\lambda t} \leq C$$

for t small. Now we verify the hypotheses of Theorem 2.2.7. Obviously the potential V is positive, smooth and it is easy to see that V satisfies (A1). Moreover

$$\begin{aligned}\frac{|D\omega|^2}{\omega^2} + \frac{|\Delta\omega|}{\omega} &\leq c^2t^2\left(1+\frac{\alpha}{2}\right)^2|x|^\alpha + c\left(1+\frac{\alpha}{2}\right)\left(\frac{\alpha}{2}-1+N\right)t|x|^{\frac{\alpha}{2}-1} \\ &\leq \left[c^2\left(1+\frac{\alpha}{2}\right)^2 + c\left(1+\frac{\alpha}{2}\right)\left(\frac{\alpha}{2}-1+N\right)\right]|x|^\alpha.\end{aligned}$$

and

$$\frac{|\partial_t\omega|}{|\omega|} = c|x|^{1+\frac{\alpha}{2}} \leq c|x|^\alpha.$$

Choosing c small enough the hypotheses of Theorem 2.2.7 are fulfilled and there exists $C > 0$ such that

$$\omega(y,t)p(x,y,t) \leq \frac{C}{(a-a_0)^{\frac{N+2}{2}}}\int_{a_0}^{b_0}\xi_\omega(x,t)dt$$

for $0 < a \le t \le b \le 1$ and $x,\ y$ in $\mathbb{R}^N$. Setting $a_0 = \frac{t}{2},\ a = t,\ b = \frac{3}{2}t,\ b_0 = 2t$ we obtain

$$p(x,y,t) \le \frac{C}{t^{\frac{N+2}{2}}}\omega(y,t)^{-1}\int_{\frac{t}{2}}^{\frac{3}{2}t} e^{\lambda s}ds \le \frac{C}{t^{\frac{N}{2}}}\omega(y,t)^{-1} = \frac{C}{t^{\frac{N}{2}}}\exp\{-ct|y|^{1+\frac{\alpha}{2}}\}.$$

Using the symmetry of p in x and y one has also

$$p(x,y,t) \le \frac{C}{t^{\frac{N}{2}}}\exp\{-ct|x|^{1+\frac{\alpha}{2}}\}.$$

Multiplying the right and the left hand side in the inequalities obtained above, we deduce

$$p(x,y,t) \le \frac{C}{t^{\frac{N}{2}}}\exp\{-\frac{c}{2}t(|x|^{1+\frac{\alpha}{2}} + |y|^{1+\frac{\alpha}{2}})\}.$$

$\square$

Proposition 2.2.9. *Assume that* $V(x) \ge M|x|^\alpha$ *for some* $0 < \alpha \le 2,\ M > 0$. *Then there exist* $0 < c < M,\ C > 0$ *such that*

$$p(x,y,t) \le \frac{C}{t^{\frac{N}{2}}}\exp\left\{-ct[(|x|^2+1)^{\frac{\alpha}{2}} + (|y|^2+1)^{\frac{\alpha}{2}}]\right\}$$

for all $x, y \in \mathbb{R}^N$ *and* $0 < t \le 1$.

Proof. As before we assume that $V(x) = M|x|^\alpha$. Let $\omega(x,t) = \exp\{ct(|x|^2 + 1)^{\frac{\alpha}{2}}\}$. By an easy computation we get

$$\begin{aligned} L\omega(x,t) &= \omega(x,t)\big[c(|x|^2+1)^{\frac{\alpha}{2}} + c^2\alpha^2t^2|x|^2(|x|^2+1)^{\alpha-2} \\ &+ c\alpha(\alpha-2)t|x|^2(|x|^2+1)^{\frac{\alpha}{2}-2} + ct\alpha N(|x|^2+1)^{\frac{\alpha}{2}-1} - V(x)\big]. \end{aligned}$$

Proceeding as in the proof of the Proposition 2.2.8 we conclude the proof. $\square$

Proposition 2.2.10. *Assume that* $V(x) \ge M\exp\{c|x|^\alpha\}$ *for some* $\alpha > 0,\ c,\ M > 0$. *Then there exist* $c_1,\ c_2,\ C > 0$ *such that*

$$p(x,y,t) \le \frac{C}{t^{\frac{N}{2}}}\exp\{-tc_1(\exp\{c_2|x|^\alpha\} + \exp\{c_2|y|^\alpha\})\}$$

for all $x, y \in \mathbb{R}^N$ *and* $0 < t \le 1$.

Proof. As before we assume that $V(x) = M\exp\{c|x|^\alpha\}$.
Let $\omega(x,t) = \exp\{c_1 t\exp\{c_2|x|^\alpha\}\}$. By an easy computation we get

$$\begin{aligned} L\omega(x,t) &= \omega(x,t)\big[c_1\exp\{c_2|x|^\alpha\} + t^2c_1^2c_2^2\alpha^2|x|^{2\alpha-2}\exp\{2c_2|x|^\alpha\} \\ &+ tc_1c_2\alpha^2\exp\{c_2|x|^\alpha\}|x|^{2\alpha-2} + tc_1c_2\alpha(\alpha-2+N)\exp\{c_2|x|^\alpha\}|x|^{\alpha-2} \\ &- V(x)\big] = \omega(x,t)\exp\{c|x|^\alpha\}\big[c_1\exp\{(c_2-c)|x|^\alpha\} \\ &+ t^2c_1^2c_2^2\alpha^2|x|^{2\alpha-2}\exp\{(2c_2-c)|x|^\alpha\} + tc_1c_2\alpha^2\exp\{(c_2-c)|x|^\alpha\}|x|^{2\alpha-2} \\ &+ tc_1c_2\alpha(\alpha-2+N)\exp\{(c_2-c)|x|^\alpha\}|x|^{\alpha-2} - M\big]. \end{aligned}$$

Recalling that $t \leq 1$, estimating the polynomial factors with exponentials and choosing c_2 small enough, we obtain that, for $|x|$ large, the last member in the previous inequality is negative. If $|x|$ is small, by continuity there exists a positive constant λ such that $A\omega \leq \lambda \leq \lambda\omega$. This proves that ω is a Lyapunov function with $h(t) = \lambda$ and for $0 < t \leq 1$ and then Proposition 2.2.3 gives $\xi_\omega(x,t) \leq C$ for t small. The potential V satisfies assumption (A1). Moreover

$$\begin{aligned}\frac{|D\omega|^2}{\omega^2} + \frac{|\Delta\omega|}{\omega} &= 2t^2c_1^2c_2^2\alpha^2 \exp\{2c_2|x|^\alpha\}|x|^{2\alpha-2} \\ &+ tc_1c_2\alpha^2 \exp\{c_2|x|^\alpha\}|x|^{2\alpha-2} \\ &+ tc_1c_2\alpha(\alpha-2+N)\exp\{c_2|x|^\alpha\}|x|^{\alpha-2}\end{aligned}$$

and

$$\frac{|\partial_t\omega|}{|\omega|} = c_1 \exp\{c_2|x|^\alpha\}.$$

Therefore (A2) is satisfied choosing c_1 and c_2 small enough and Theorem 2.2.7 yields

$$\omega(y,t)p(x,y,t) \leq \frac{C}{(a-a_0)^{\frac{N+2}{2}}}\int_{a_0}^{b_0} \xi_\omega(x,t)dt$$

for $0 < a \leq t \leq b \leq 1$ and $x,\ y$ in $\mathbb{R}^N$. As in Proposition 2.2.8 one concludes the proof. $\square$

Proposition 2.2.11. *Assume $V(x) \geq M\log(1+|x|^2)$. Then there exists $C > 0$ and $\alpha < M$ such that*

$$p(x,y,t) \leq \frac{C}{t^{\frac{N}{2}}}(1+|x|^2)^{-\frac{\alpha}{2}t}(1+|y|^2)^{-\frac{\alpha}{2}t}$$

for all $x,\ y \in \mathbb{R}^N$ and $0 < t \leq 1$.

Proof. Let $\omega(x,t) = (1+|x|^2)^{\alpha t}$. Then

$$\begin{aligned}L\omega(x,t) &= \omega(x,t)\Bigg[\alpha\log(1+|x|^2) + \frac{\alpha t(\alpha t-1)4|x|^2}{(1+|x|^2)^2} + \frac{2\alpha tN}{1+|x|^2} \\ &- M\log(1+|x|^2)\Bigg] \leq 0\end{aligned}$$

for $|x|$ large since $t \leq 1$ and $\alpha < M$. Hence ω is a Lyapunov function. Moreover V satisfies (A1) and

$$\frac{|\partial_t\omega|}{\omega} = \alpha\log(1+|x|^2),$$

$$\frac{|D\omega|^2}{\omega^2} + \frac{|\Delta\omega|}{\omega} \leq 4\alpha^2\frac{|x|^2}{(1+|x|^2)^2} + 4\alpha(\alpha+1)\frac{|x|^2}{(1+|x|^2)^2} + \frac{2\alpha N}{1+|x|^2}.$$

Choosing α small enough we can apply Theorem 2.2.7 and obtain

$$\omega(y,t)p(x,y,t) \leq \frac{C}{(a-a_0)^{\frac{N+2}{2}}}\int_{a_0}^{b_0} \xi_\omega(x,t)dt.$$

for $0 < a \le t \le b \le 1$ and $x,\ y$ in $\mathbb{R}^N$. Arguing as in the examples before, one concludes the proof. □

Remark 2.2.12. We can easily add a Gaussian term in our estimates as follows. For example, multiplying the left and the right hand side in Proposition 2.2.8 respectively with the left and right hand side of the Gaussian bound

$$p(x,y,t) \le \frac{C}{t^{\frac{N}{2}}} \exp\left\{-c\frac{|x-y|^2}{t}\right\},$$

we find

$$p(x,y,t) \le \frac{C}{t^{\frac{N}{2}}} \exp\left\{-c_1 t(|x|^{1+\frac{\alpha}{2}} + |y|^{1+\frac{\alpha}{2}})\right\} \exp\left\{-c_2\frac{|x-y|^2}{t}\right\}$$

for suitable $c_1,\ c_2\ , C > 0$. The other cases are similar.

Remark 2.2.13. Finally we discuss the sharpness of the estimate proving lower bounds similar to the upper bounds obtained in the examples above with the method of [13, Theorem 4.5.10].

We start with the potential $V(x) = |x|^\alpha$, $0 < \alpha \le 2$, considered in Proposition 2.2.9. We consider the ball $B_1(x)$ of center x and radius 1 and the Schrödinger operator A_D in $B_1(x)$ with Dirichlet boundary conditions. The maximum principle yields $e^{-tA} \ge e^{-tA_D}$ in $B_1(x)$. Since $V \le (1+|x|)^\alpha$ in $B_1(x)$ we have $e^{-tA} \ge e^{-tA_D} \ge e^{-t(1+|x|^\alpha)}e^{-t\Delta_D}$ in $B_1(x)$, where Δ_D is the Laplacian with Dirichlet boundary conditions. Taking the inequality for the corresponding kernels and using the estimate

$$p_{\Delta_D}(x,x,t) \ge ct^{-N/2},$$

see [13, Lemma 3.3.3], we obtain

$$p_A(x,x,t) \ge e^{-t(1+|x|^\alpha)} p_{\Delta_D}(x,x,t) \ge \frac{C}{t^{\frac{N}{2}}} e^{-t(1+|x|^\alpha)}$$

for some positive constant C. This shows that Proposition 2.2.9 is sharp, concerning the exponent α appearing in the exponential. Our method does not give a precise estimate of the constant c which, however, turns out to be $1+\varepsilon$, see [45] and the next chapter.

In a similar way we obtain that, if $V(x) = \exp\{c|x|^\alpha\}$ for some $\alpha,\ c > 0$, then, as above,

$$p(x,x,t) \ge \frac{C}{t^{\frac{N}{2}}} \exp\{-t\exp\{c(1+|x|)^\alpha\}\}.$$

Therefore in the case of exponential potentials the estimate in 2.2.10 is sharp, with the exception of constants c_1, c_2.

For a logarithmic potentials $V = M\log(1+|x|^2)$ of Proposition 2.2.11, the same method gives the lower bound

$$p(x,x,t) \ge \frac{C}{t^{\frac{N}{2}}} \exp\{-t\log[1+(1+|x|)^2]\} = \frac{C}{t^{\frac{N}{2}}}(1+(1+|x|)^2)^{-Mt}.$$

Finally we consider the case of $V(x) = |x|^\alpha$ with $\alpha > 2$, see Proposition 2.2.8. As in [45] we have

$$p(x,x,t) = \sum_n e^{-\lambda_n t}\phi_n(x)^2 \geq e^{-\lambda_1 t}\phi_1(x)^2$$

where $(\phi_n), (\lambda_n)$ are the eigenfunctions and the eigenvalues of $-A$, respectively. Since

$$\phi_1(x) \geq C\exp\{-c|x|^{1+\alpha/2}\},$$

see [13, Corollary 4.5.7], we see that, for a fixed t, Proposition 2.2.8 gives the exact decay in the space variables. Also in this case we refer the reader to [45] and to the next chapter for more precise space-time estimates.

2.2.5 Large time estimates

As in [45], large time estimates are easily deduced from small time estimates.

Proposition 2.2.14. *Let λ_1 be the smallest eigenvalue of A. Then there exist positive constants C, c, δ such that for $t \geq 1, x \in \mathbb{R}^N$*

$$p(x,x,t) \leq Ce^{-\lambda_1 t}\exp\{-c|x|^{1+\frac{\alpha}{2}}\}$$

if $V(x) \geq M|x|^\alpha$ and $\alpha > 2$,

$$p(x,x,t) \leq Ce^{-\lambda_1 t}\exp\{-c(|x|^2+1)^{\frac{\alpha}{2}}\}$$

if $V(x) \geq M|x|^\alpha$ and $0 < \alpha \leq 2$,

$$p(x,x,t) \leq Ce^{-\lambda_1 t}\exp\{-c\exp\{c|x|^\alpha\}\}$$

if $V(x) \geq M\exp\{c_1|x|^\alpha\}$ and

$$p(x,x,t) \leq Ce^{-\lambda_1 t}(1+|x|^2)^{-\delta}$$

if $V(x) \geq M\log(1+|x|^2)$.

Proof. Let e^{-tA} be the semigroup generated by $-A$. We note that

$$\|e^{-tA}\|_{L^2\to L^2} = e^{-\lambda_1 t}; \tag{2.39}$$

$$e^{-tA}p(x,\cdot,s) = p(x,\cdot,s+t) \tag{2.40}$$

and

$$p(x,x,t) = \|p(x,\cdot,t/2)\|^2_{L^2} \tag{2.41}$$

for all $t, s > 0$ and $x \in \mathbb{R}^N$. Therefore, if $t > 1$, by (2.39), (2.40) and (2.41), we have

$$\begin{aligned} p(x,x,t) &= \left\|p(x,\cdot,\frac{t}{2})\right\|^2_{L^2} = \|e^{-(t/2-1/2)A}p(x,\cdot,1/2)\|^2_{L^2} \\ &\leq e^{-\lambda_1(t-1)}\|p(x,\cdot,1/2)\|^2_{L^2} = Ce^{-\lambda_1 t}p(x,x,1). \end{aligned}$$

Estimating $p(x,x,1)$ as in Propositions 2.2.8, 2.2.9, 2.2.10 and 2.2.11, the proof follows. ∎

Remark 2.2.15. Off-diagonal estimates for large times can be deduced from on-diagonal bounds by the following computation

$$\begin{aligned} |p(x,y,t)| &= \left| \int p(x,z,t/2)p(z,y,t/2) \right| dz \leq \|p(x,\cdot,t/2)\|_2 \|p(y,\cdot,t/2)\|_2 \\ &= p(x,x,t)^{\frac{1}{2}} p(y,y,t)^{\frac{1}{2}}. \end{aligned}$$

As in Remark 2.2.12, a Gaussian factor can be added to all the estimates of this section.