Chapter 1

Markov semigroups in $\mathbb{R}^N$

In this chapter we collect some preliminary results nedeed to develop the next theory. In particular we introduce elliptic operators with unbounded coefficients and we study the Markov semigroups associated with them.
We consider the operator

$$Au(x) = \sum_{i,j=1}^{N} a_{ij}(x)D_{ij}u(x) + \sum_{i=1}^{N} F_i(x)D_iu(x) - V(x)u(x)$$

under the hypotheses: (a_{ij}) symmetric matrix, $a_{ij},\ F_i,\ V$ real-valued functions, $V \geq 0$. Moreover we assume the ellipticity condition

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \lambda(x)|\xi|^2$$

for every $x,\ \xi \in \mathbb{R}^N$, with $\inf_K \lambda(x) > 0$ for every compact $K \subset \mathbb{R}^N$. The operator so defined is locally uniformly elliptic, that is uniformly elliptic on every compact subset of $\mathbb{R}^N$.
We introduce the realization of A in $C_b(\mathbb{R}^N)$ with $D_{max}(A)$ defined as follows

$$D_{max}(A) = \{u \in C_b(\mathbb{R}^N) \cap W^{2,p}_{loc}(\mathbb{R}^N) \quad \text{for all} \quad p < \infty : \ Au \in C_b(\mathbb{R}^N)\}.$$

In the first section, we prove existence results for bounded classical solutions of the Cauchy problem

$$\begin{cases} u_t(x,t) = Au(x,t) & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = f(x) & x \in \mathbb{R}^N \end{cases} \tag{1.1}$$

with initial datum $f \in C_b(\mathbb{R}^N)$ and under hölderianity assumptions on the coefficients. Since the coefficients of the operator are not bounded, the classical theory does not give a solution of the problem. The solution is constructed through an approximation procedure as limit of solutions of Cauchy Dirichlet

problems in suitable bounded domains and is given by a certain semigroup $T(t)$ applied to the initial datum f.
Moreover we prove that the solution can be represented by the formula

$$u(x,t) = \int_{\mathbb{R}^N} p(x,y,t) f(y)\, dy \qquad t > 0,\ x \in \mathbb{R}^N$$

where p is a positive function called the integral kernel. As above, p is obtained as limit of kernels of solutions in bounded domains.
A continuity property of the operators $T(t)$ is deduced.
In the second section we state and prove some results concerning the generator in a weak sense of the semigroup so constructed.
The last section is devoted to the study of a particular elliptic operator with unbounded coefficients, the so called Schrödinger operator. It is obtained in correspondence of vanishing drift term ($F = 0$) and constant diagonal matrix (a_{ij}). It's formal expression is given by $A = \Delta - V$ where V is an unbounded positive potential as before. The existence of the semigroup generated (in a weak sense) by such operator and of an integral kernel are obviously guaranteed by the theory developed in the first two sections under hölderianity hypothesis on the potential. Anyway we will see how a different approach, the quadratic form method, allows us to prove that, under the weaker assumption $V \in L^1_{loc}(\mathbb{R}^N)$, the Schrödinger operator generates a semigroup on $L^2(\mathbb{R}^N)$ that can be extrapolated to $L^p(\mathbb{R}^N)$ for $1 \le p \le \infty$ and admits an integral representation.

1.1 The Cauchy problem and the semigroup

Through this and the next section we assume the following hypothesis on the coefficients of the operator:

(i) $a_{ij} = a_{ji}$ for all $i,\, j = 1, ..., N$;

(ii) $\sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \ge \lambda(x)|\xi|^2$ for every $x,\ \xi \in \mathbb{R}^N$, with $\inf_K \lambda(x) > 0$ for every compact $K \subset \mathbb{R}^N$;

(iii) a_{ij}, F_i, V belong to $C^{\alpha}_{loc}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$;

(iv) $V(x) \ge 0$ for all $x \in \mathbb{R}^N$.

We will prove the following theorem.

Theorem 1.1.1. *There exists a positive semigroup $(T(t))_{t\ge 0}$ defined in $C_b(\mathbb{R}^N)$ such that, for any $f \in C_b(\mathbb{R}^N)$, $u(x,t) = T(t)f(x) \in C^{2+\alpha,1+\frac{\alpha}{2}}_{loc}(\mathbb{R}^N \times (0,+\infty))$ and satisfies the differential equation*

$$u_t(x,t) = \sum_{i,j=1}^N a_{ij}(x) D_{ij}u(x) + \sum_{i=1}^N F_i(x) D_i u(x) - V(x)u(x).$$

Let us fix a ball B_ρ in $\mathbb{R}^N$ and consider the problem

$$\begin{cases} u_t(x,t) = Au(x,t) & x \in B_\rho,\ \ t > 0, \\ u(x,t) = 0 & x \in \partial B_\rho,\ \ t > 0 \\ u(x,0) = f(x) & x \in \mathbb{R}^N. \end{cases} \tag{1.2}$$

Since the operator A is uniformly elliptic and the coefficients are bounded in B_ρ, there exists a unique solution u_ρ of the problem (1.2). In other words, the operator $A_\rho = (A, D_\rho(A))$ with

$$D_\rho(A) = \{u \in C_0(B_\rho) \cap W^{2,p}(B_\rho) \text{ for all } p < \infty : Au \in C(\overline{B}_\rho)\}$$

generates an analytic semigroups $(T_\rho(t))_{t\geq 0}$ in the space $C(\overline{B}_\rho)$ and the function $u_\rho(x,t) = T_\rho(t)f(x)$ solves (1.2).
Since the domain $D_\rho(A)$ is not dense in $C(\overline{B}_\rho)$, the semigroup is not strongly continuous at 0 indeed one can prove that $T_\rho(t)f$ converges uniformly to f in $\overline{B}_\rho$ as $t \to 0$ if and only if $f \in C_0(B_\rho)$. However the convergence is uniform in compact sets $\overline{B}_\sigma$ for every $\sigma < \rho$ and hence pointwise in B_ρ. The operators $T_\rho(t)$ are bounded in $L^p(B_\rho)$ for every $1 \leq p < \infty$ and are integral operators indeed, for every $\rho > 0$, there exists a kernel $p_\rho(x,y,t)$ such that

$$T_\rho(t)f(x) = \int_{B_\rho} p_\rho(x,y,t)f(y)\,dy \tag{1.3}$$

for every $f \in C(\overline{B}_\rho)$. The kernel p_ρ is positive and, for every fixed $y \in B_\rho$, $0 < \varepsilon < \tau$, it belongs to $C^{2+\alpha,1+\frac{\alpha}{2}}(B_\rho \times (\varepsilon,\tau))$ as a function of (x,t) and satisfies

$$\partial_t p_\rho = A p_\rho.$$

It follows that $T_\rho(t)$ are positive and satisfy the estimate $\|T_\rho(t)f\|_\infty \leq \|f\|_\infty$, moreover for every $f \in C(\overline{B}_\rho)$ the function $u_\rho(x,t)$ belongs to $C^{2+\alpha,1+\frac{\alpha}{2}}(B_\rho \times (\varepsilon,\tau))$. Finally, by the integral representation, we can immediately deduce a continuity property of the operator $T_\rho(t)$. If $(f_n) \subset C(\overline{B}_\rho)$, $f \in C(\overline{B}_\rho)$ satisfy $\|f_n\| \leq C$ for every $n \in \mathbb{N}$ and $f_n \to f$ pointwise, then $T_\rho(t)f_n \to T_\rho(t)f$ pointwise.
We refer to [25, Chapter 3] and [17, Chapter 3, Section 7] for a detailed description of the results mentioned above.
Now we would like to let ρ to infinity in order to define the semigroup associated with A in $\mathbb{R}^N$. To this aim we need an easy consequence of the parabolic maximum principle.

Lemma 1.1.2. *Let $0 \leq f \in C_b(\mathbb{R}^N)$ and let $\rho < \rho_1 < \rho_2$. Then for every $t \geq 0$ and $x \in B_\rho$ we have $0 \leq T_{\rho_1}(t)f(x) \leq T_{\rho_2}(t)f(x)$.*

Proof. First suppose that $f \equiv 0$ on the boundary ∂B_{ρ_1}. Then, since $T_\rho(t)f$ converges uniformly to f in $\overline{B}_{\rho_1}$ as $t \to 0$ if and only if $f \in C_0(B_{\rho_1})$, $w(x,t) = T_{\rho_2}(t)f(x) - T_{\rho_1}(t)f(x)$ is continuous on $\overline{B}_{\rho_1} \times [0,\infty)$, vanishes for $t = 0$, is nonnegative for $x \in \partial B_{\rho_1}$ and solves a parabolic equation. By the maximum principle $w(x,t) \geq 0$ in $\overline{B}_{\rho_1} \times [0,\infty)$. In general, if $f \in C_b(\mathbb{R}^N)$, we approximate it in the $L^2(B_{\rho_2})$ norm with continuous functions vanishing on ∂B_{ρ_1}. Using the first part of the proof and the boundedness of $T_{\rho_i}(t)$ in $L^2(B(\rho_i))$, $i = 1, 2$, the claim follows. □

PROOF (Theorem 1.1.1). If $f \in C_b(\mathbb{R}^N)$, $x \in \mathbb{R}^N$ we set

$$T(t)f(x) := \lim_{\rho\to\infty} T_\rho(t)f(x).$$

We know that this limit exists if $f \geq 0$ by monotonicity, otherwise we write a general f as $f^+ - f^-$. For the positive and the negative part of f the limit above exists and then, since $T_\rho(t)$ is linear, $T(t)f(x)$ is well defined. $T(t)$ are positive operators and $\|T(t)f\|_\infty \leq \|f\|_\infty$. Let us prove that the operators so defined satisfy the semigroup law. Consider $f \geq 0$. Let $t, s > 0$. Then

$$T(t+s)f(x) = \lim_{\rho\to\infty} T_\rho(t+s)f(x) = \lim_{\rho\to\infty} T_\rho(t)T_\rho(s)f(x) \leq T(t)T(s)f(x).$$

On the other hand, for every $\rho_1 > 0$ we have

$$T(t+s)f(x) = \lim_{\rho\to\infty} T_\rho(t)T_\rho(s)f(x) \geq \lim_{\rho\to\infty} T_{\rho_1(t)}T_\rho(s)f(x) = T_{\rho_1}(t)T(s)f(x)$$

and, letting $\rho_1 \to \infty$, it follows that $T(t+s)f(x) \geq T(t)T(s)f(x)$. Hence the semigroup law is true if the semigroup is applied to a positive function. The general case follows by linearity as above.
Set $u(x,t) = T(t)f(x)$, $u_\rho(x,t) = T(t)f(x)$ for $t \geq 0$ and $x \in \mathbb{R}^N$. Fix positive numbers ε, τ, σ with $0 < \varepsilon < \tau$. By the interior Schauder estimates ([17, Chapter 3, Section 2]) there exists a positive constant C such that for $\rho > \sigma$

$$\|u_\rho\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{B}_\sigma\times[\varepsilon,\tau])} \leq C\|u_\rho\|_\infty \leq C\|f\|_\infty.$$

So by Ascoli's Theorem it follows that u_ρ converges to u uniformly in $\overline{B}_\sigma \times [\varepsilon,\tau]$. Fix now $\sigma_1 < \sigma$, $\varepsilon < \varepsilon_1 < \tau_1 < \tau$ and apply again Schauder estimates. For $\rho_2 > \rho_1 > \sigma > \sigma_1$ we have

$$\|u_{\rho_2} - u_{\rho_1}\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{B}_{\sigma_1}\times[\varepsilon_1,\tau_1])} \leq C\|u_{\rho_2} - u_{\rho_1}\|_{L^\infty(\overline{B}_\sigma\times[\varepsilon,\tau])}.$$

Then $u \in C^{2+\alpha,1+\frac{\alpha}{2}}_{loc}(\mathbb{R}^N \times (0,\infty))$ and, letting $\rho \to \infty$ in the equation satisfied by u_ρ, it follows that $\partial_t u = Au$. □

We have observed that the semigroup $T(t)$ is not strongly continuous in $C_b(\mathbb{R}^N)$. We are interested now in the conditions under which the continuity at $t = 0$ holds.

Proposition 1.1.3. *For every $f \in C_0(\mathbb{R}^N)$*

$$\lim_{t\to 0} T(t)f = f$$

uniformly on $\mathbb{R}^N$.

Proof. Consider first $f \in C^2(\mathbb{R}^N)$ with support contained in B_σ and let $\rho > \sigma$. Then, for $x \in B_\rho$,

$$T_\rho(t)f(x) - f(x) = \int_0^t T_\rho(s)Af(x)\,ds$$

and, letting $\rho \to \infty$ by dominated convergence,

$$T(t)f(x) - f(x) = \int_0^t T(s)Af(x)\,ds.$$

By the arbitrarity of ρ, the equality above holds for every $x \in \mathbb{R}^N$ and, taking the supremum over $x \in \mathbb{R}^N$,

$$\|T(t)f - f\|_\infty \le t\|Af\|_\infty.$$

This implies that $T(t)f$ converges to f uniformly as $t \to 0$. By density the claim follows. □

Remark 1.1.4. By the previous proposition we cannot deduce that $(T(t))_{t\ge0}$ restricted to $C_0(\mathbb{R}^N)$ is strongly continuous since no invariance property of $C_0(\mathbb{R}^N)$ under the semigroup is guaranteed.

As we have seen before, $T_\rho(t)$ are integral operators, therefore they can be represented in integral form through a kernel p_ρ. In the next theorem we prove that also $T(t)$ is an integral operator and its kernel enjoies some regularity properties.

Theorem 1.1.5. *The following representation formula for $T(t)$ holds*

$$T(t)f(x) = \int_{\mathbb{R}^N} p(x, y, t)\,dy$$

for $f \in C_b(\mathbb{R}^N)$ and with p positive function such that for almost every $y \in \mathbb{R}^N$ it belongs to $C^{2+\alpha,1+\frac{\alpha}{2}}_{loc}(\mathbb{R}^N \times (0,\infty))$ as a function of (x,t) and solves $\partial_t p = Ap$.

Proof. Suppose $0 \le f \in C_b(\mathbb{R}^N)$. By Lemma 1.1.2, $T_\rho(t)f$ converges monotonically pointwise to $T(t)f$. Therefore, recalling that

$$T_\rho(t)f(x) = \int_{B_\rho} p_\rho(x, y, t)f(y)\,dy,$$

the kernels p_ρ increase with ρ. Then there exists

$$p(x, y, t) := \lim_{\rho\to\infty} p_\rho(x, y, t)$$

and, by monotone convergence,

$$T(t)f(x) = \lim_{\rho\to\infty} T_\rho(t)f(x) = \lim_{\rho\to\infty} \int_{B_\rho} p_\rho(x, y, t)f(y)\,dy = \int_{\mathbb{R}^N} p(x, y, t)f(y)\,dy.$$

The positivity of p immediately follows by the one of p_ρ. We show now the regularity properties of p.
We have $\int_{B_\rho} p_\rho(x,y,t)\,dy \le 1$ and, letting $\rho \to \infty$, $\int_{\mathbb{R}^N} p(x,y,t)\,dy \le 1$ so that $p(x,y,t)$ is finite for every $t > 0$, every $x \in \mathbb{R}^N$ and almost every $y \in \mathbb{R}^N$. Fix $t_1 > 0$, $\sigma > 0$, $x_0 \in B_\sigma$ and let $y_0 \in \mathbb{R}^N$ such that $p(x_0,y_0,t_1) < \infty$. If $\rho_2 > \rho_1 > \sigma+1$, the functions $p_{\rho_1}(\cdot,y_0,\cdot)$, $p_{\rho_2}(\cdot,y_0,\cdot)$ are solutions of the equation $\partial_t u = Au$ in $B_{\sigma+1}\times(0,\infty)$ and the difference $p_{\rho_2}-p_{\rho_1}$ is as well. By the parabolic Harnack inequality (see [24, Chapter VII]), for every fixed $0<\varepsilon<\tau<t_1$

$$\begin{aligned}\sup_{\varepsilon\le t\le\tau,\, x\in\overline{B}_\sigma}[p_{\rho_2}(x,y_0,t)-p_{\rho_1}(x,y_0,t)] &\le C\inf_{\overline{B}_\sigma}[p_{\rho_2}(x,y_0,t_1)-p_{\rho_1}(x,y_0,t_1)]\\ &\le C[p_{\rho_2}(x_0,y_0,t_1)-p_{\rho_1}(x_0,y_0,t_1)].\end{aligned}$$

Since $p(x_0,y_0,t_1)<\infty$, $p_\rho(\cdot,y_0,\cdot)$ is a Cauchy sequence in $C(\overline{B}_\sigma\times[\varepsilon,\tau])$. Then $p_\rho(\cdot,y_0,\cdot)$ converges uniformly to $p(\cdot,y_0,\cdot)$ in $\overline{B}_\sigma\times[\varepsilon,\tau]$. Fix now $\sigma_1<\sigma$, $\varepsilon<\varepsilon_1<\tau_1<\tau$ and apply the Schauder estimates. We have

$$\|p_{\rho_2}-p_{\rho_1}\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{B}_{\sigma_1}\times[\varepsilon_1,\tau_1])} \le C\|p_{\rho_2}-p_{\rho_1}\|_{L^\infty(\overline{B}_\sigma\times[\varepsilon,\tau])}.$$

Then $p\in C^{2+\alpha,1+\frac{\alpha}{2}}_{loc}(\mathbb{R}^N\times(0,\infty))$ and, letting $\rho\to\infty$ in the equation satisfied by p_ρ, it follows that and $\partial_t p = Ap$. □

Remark 1.1.6. By using the integral representation formula, we can extend the semigroup to the space of the bounded measurable functions. If $f\in B_b(\mathbb{R}^N)$, with $T(t)f$ we mean the $\int_{\mathbb{R}^N} p(x,y,t)f(y)\,dy$.

We now show the continuity up to $t=0$ of $u(x,t)$ and so we prove that we have built not only a solution of the parabolic equation but a solution of the Cauchy problem (1.1). Let us fix a notation. For any measurable set $E\subset\mathbb{R}^N$, with $p(x,E,t)$ we denote the $\int_E p(x,y,t)\,dy$.

Theorem 1.1.7. *Let $f\in C_b(\mathbb{R}^N)$. Then $T(t)f$ converges to f as $t\to 0$ uniformly on compact subsets of $\mathbb{R}^N$.*

Proof. Let $\rho>0$ and f_1, $f_2\in C_0(\mathbb{R}^N)$ such that $0\le\chi_{B_\rho}\le f_1\le\chi_{B_{2\rho}}\le f_2\le 1$. By the positivity of $T(t)$,

$$T(t)f_1(x)\le p(x,B_{2\rho},t)\le T(t)f_2(x)$$

for all $x\in\mathbb{R}^N$. By Proposition 1.1.3, $T(t)f_1\to f_1$, $T(t)f_2\to f_2$ uniformly on $\overline{B}_\rho$ as $t\to 0$. We observe that $f_1=f_2\equiv 1$ on $\overline{B}_\rho$. It follows that $p(x,B_{2\rho},t)\to 1$ on $\overline{B}_\rho$ as $t\to 0$. Then

$$0\le p(x,\mathbb{R}^N\setminus B_{2\rho},t)=p(x,\mathbb{R}^N,t)-p(x,B_{2\rho},t)\le 1-p(x,B_{2\rho},t)\to 0 \quad (1.4)$$

as $t\to 0$ uniformly on $\overline{B}_\rho$.
Let now $f\in C_b(\mathbb{R}^N)$ and $\eta\in C_0(\mathbb{R}^N)$ such that $0\le\eta\le 1$, $\eta=1$ on $B_{2\rho}$, $supp\,(\eta)\in B_{3\rho}$. Then

$$T(t)f-f=T(t)f-T(t)(\eta f)+T(t)(\eta f)-\eta f$$

on B_ρ. By Proposition 1.1.3, $\|T(t)(\eta f)-\eta f\|_\infty \to 0$ as $t\to 0$. Concerning the remaining terms, by (1.4) we have

$$\begin{aligned} |T(t)f(x)-T(t)(\eta f)(x)| &= T(t)((1-\eta)f)(x)\\ &= \int_{\mathbb{R}^N} p(x,y,t)((1-\eta(y))f(y))\,dy\\ &\le p(x,\mathbb{R}^N\setminus B_{2\rho},t)\|f\|_\infty \to 0\end{aligned}$$

uniformly on $\overline{B}_\rho$. We conclude therefore that $T(t)f\to f$ uniformly on $\overline{B}_\rho$ and by the arbitrarity of ρ the claim follows. □

Remark 1.1.8. We observe that, in general, the problem (1.1) is not uniquely solvable in $C_b(\mathbb{R}^N\times[0,+\infty))\cap C^{2+\alpha,1+\frac{\alpha}{2}}((0,+\infty)\times\mathbb{R}^N)$. Anyway we can say that the solution found above is the minimal among all the positive solutions of the given problem with positive initial datum. Infact, if $f\ge 0$ and v is another positive solution, then the maximum principle yields $v(x,t)\ge u_\rho(x,t)$ for all $t>0$, $x\in B_\rho$, u_ρ defined as before and, letting $\rho\to\infty$, $v\ge u$.

Now we prove some interesting continuity properties of the operators $T(t)$.

Proposition 1.1.9. *Let (g_n) be a bounded sequence in $C_b(\mathbb{R}^N)$, $g\in C_b(\mathbb{R}^N)$ and suppose that $g_n(x)\to g(x)$ for every $x\in\mathbb{R}^N$. Then, for every $0<\varepsilon<\tau$ and $\sigma>0$, $T(t)g_n(x)\to T(t)g(x)$ uniformly for $(x,t)\in\overline{B}_\sigma\times[\varepsilon,\tau]$. If $g_n\to g$ uniformly on compact sets, then $T(t)g_n(x)\to T(t)g(x)$ uniformly for $(x,t)\in\overline{B}_\sigma\times[0,\tau]$.*

Proof. Using the integral representation and the Lebesgue dominated convergence Theorem, we immediately deduce that $T(t)g_n(x)\to T(t)g(x)$ pointwise in $\mathbb{R}^N$. Let $K>0$ such that $\|g_n\|_\infty\le K$ for every $n\in\mathbb{N}$. Then $\|T(t)g_n\|_\infty\le K$ for every $n\in\mathbb{N}$ and, by the Schauder estimates, for every $0<\varepsilon<\tau$ and $\sigma>0$ there exists $C>0$ such that

$$\sup_n \|T(\cdot)g_n(\cdot)\|_{C^1(\overline{B}_\sigma\times[\varepsilon,\tau])}\le C.$$

By Ascoli's Theorem we deduce that the convergence is uniform in $\overline{B}_\sigma\times[\varepsilon,\tau]$. Let us prove the second statement. Without loss of generality we can suppose $g=0$ (otherwise we consider g_n-g) and $\|g_n\|_\infty\le 1$. Let σ, $\varepsilon>0$ and, for every $\rho>1$, consider $0\le f_\rho\in C_0(\mathbb{R}^N)$ such that $\chi_{B_{\rho-1}}\le f_\rho\le\chi_{B_\rho}$. Set

$$E=\{s\ge 0:\ \exists\,\rho>0\,\text{such that}\inf_{|x|\le\sigma,\ 0\le t\le s}T(t)(f_\rho(x)-\mathbf{1})\ge-\varepsilon\}.$$

Obviously $0\in E$. Now we prove that E is open and closed together and so we conclude that it coincides with the positive real axis. Let $s\in\overline{E}$, then there exists $(s_n)\subset E$, $s_n\to s$ for $n\to\infty$. Suppose that there exists $r\in\mathbb{N}$ such that $s_r\ge s$ and let ρ_r be such that

$$\inf_{|x|\le\sigma,\ 0\le t\le s_r}T(t)(f_{\rho_r}-\mathbf{1})(x)\ge-\varepsilon.$$

Then

$$\inf_{|x|\leq\sigma,\ 0\leq t\leq s} T(t)(f_{\rho_r}-\mathbf{1})(x) \geq \inf_{|x|\leq\sigma,\ 0\leq t\leq s_r} T(t)(f_{\rho_r}-\mathbf{1})(x) \geq -\varepsilon$$

and $s\in E$. Otherwise $s_n<s$ for every $n\in\mathbb{N}$. Since $s_1\in E$, there exists $\rho_1>0$ such that

$$\inf_{|x|\leq\sigma,\ 0\leq t\leq s_1} T(t)(f_{\rho_1}-\mathbf{1})(x) \geq -\varepsilon.$$

Recalling that $\{f_\rho\}$ is increasing, it turns out that the previous inequality is satisfied for every $\rho\geq\rho_1$. By the first part of the proof, we know that $T(\cdot)f_\rho \to T(\cdot)\mathbf{1}$ as $\rho\to\infty$ uniformly in $\overline{B}_\sigma\times[s_1,s]$. Therefore there exists $\rho_0>0$ such that

$$T(t)f_\rho(x)\geq T(t)\mathbf{1}-\varepsilon, \quad t\in[s_1,s],\ x\in\overline{B}_\sigma,\ \rho\geq\rho_0.$$

If we choose $\overline{\rho}=\max\{\rho_0,\ \rho_1\}$, then

$$T(t)f_{\overline{\rho}}(x)\geq T(t)\mathbf{1}-\varepsilon, \quad t\in[0,s],\ x\in\overline{B}_\sigma.$$

It follows that $s\in E$.
Now we prove that E is open. Let $s\in E$ and ρ as in the definition of E. Since $T(s)f_\rho\to T(s)\mathbf{1}$ as $\rho\to\infty$ uniformly in compact sets, there exists $\rho_0>0$ such that $T(s)f_\rho(x)\geq T(s)\mathbf{1}-\dfrac{\varepsilon}{2}$ for every $x\in\overline{B}_\sigma,\ \rho>\rho_0$. By Theorem 1.1.7, $T(s+\delta)f_\rho(x)\geq T(s)\mathbf{1}-\varepsilon$ for every $x\in\overline{B}_\sigma$ and δ sufficiently small. This shows that E is open. We conclude that $E=[0,\infty)$. In particular, if $\tau>0$ is fixed, we can find $\rho>0$ such that $p(x,B_\rho,t)\geq T(t)f_\rho(x)\geq T(t)\mathbf{1}-\varepsilon$ for every $x\in\overline{B}_\sigma$ and $t\in[0,\tau]$. Then we have

$$|T(t)g_n(x)|\leq\int_{B_\rho}p(x,y,t)|g_n(y)|\,dy+\int_{\mathbb{R}^N\setminus B_\rho}p(x,y,t)\,dy\leq\sup_{y\in B_\rho}|g_n(y)|+\varepsilon$$

for every $x\in\overline{B}_\sigma$ and $t\in[0,\tau]$. □

As consequence of the continuity result just proved, we deduce that $(T(t))_{t\geq0}$ is irreducible and satisfies the strong Feller property. We preliminary define these two properties.

Definition 1.1.10. *A semigroup* $((T(t))_{t\geq0}$ *in* $B_b(\mathbb{R}^N)$ *is irreducible if for any nonempty open set* $U\subset\mathbb{R}^N$, $T(t)\chi_U(x)>0$ *for every* $t>0$ *and* $x\in\mathbb{R}^N$.

Definition 1.1.11. *We say that* $(T(t))_{t\geq0}$ *satisfies the strong Feller property if* $T(t)f\in C_b(\mathbb{R}^N)$ *for any bounded Borel function* f.

Proposition 1.1.12. *The semigroup* $(T(t))_{t\geq0}$ *is irreducible and has the strong Feller property.*

PROOF. The irreducibility immediately follows since the integral kernel p is positive. Let f be a bounded Borel Function and let $(f_n)\in C_b(\mathbb{R}^N)$a bounded sequence such that $f_n(x)\to f(x)$ for almost every $x\in\mathbb{R}^N$. By dominated convergence, $T(t)f_n\to T(t)f$ pointwise in $\mathbb{R}^N$. Using the interior Schauder estimates, as in Proposition 1.1.9, we deduce that $T(t)f_n\to T(t)f$ uniformly on compact sets and then the limit $T(t)f\in C_b(\mathbb{R}^N)$. □

1.2 The weak generator of $T(t)$

In the previous section we have built a semigroup associated to the given elliptic operator with unbounded coefficients and we have observed that in general it is not strongly continuous in $C_b(\mathbb{R}^N)$, hence we cannot define it's generator in the usual sense. However, as we will see later, it is possible to define a generator in a weak sense.
In this section we state only some results useful in the following chapters, in particular we are interested in the conditions under which the domain of the weak generator coincides with the maximal one. For example this equality will be guaranted under the existence of suitable Lyapunov functions for the operator A.
First we enunciate an existence result for the solution of the elliptic equation associated with A.

Theorem 1.2.1. *For any $\lambda > 0$, $f \in C_b(\mathbb{R}^N)$, there exists $u \in D_{max}(A)$ such that*

$$\lambda u(x) - Au(x) = f(x), \qquad x \in \mathbb{R}^N.$$

Moreover the following estimate holds

$$\|u\|_\infty \leq \frac{1}{\lambda}\|f\|_\infty.$$

Finally, if $f \geq 0$, then $u \geq 0$.

We only sketch the proof. As in the parabolic case, the solution is obtained as limit of solutions of the analogous of the equation above for A_ρ, realization of the operator A with homogeneous Dirichlet boundary conditions in balls of $\mathbb{R}^N$ of radius ρ.
Set $A_\rho = (A, D_\rho(A))$ where

$$D_\rho(A) = \{u \in C_0(B_\rho) \cap W^{2,p}(B_\rho) \text{ for all } p < \infty : Au \in C(\overline{B}_\rho)\}$$

and $u_\rho = R(\lambda, A_\rho)f$. For any $\lambda > 0$ there exists a linear operator $R(\lambda)$ in $C_b(\mathbb{R}^N)$ such that for any $f \in C_b(\mathbb{R}^N)$ the solution is given by

$$u(x) = (R(\lambda)f)(x) = \lim_{\rho\to\infty} R(\lambda, A_\rho)f(x), \qquad x \in \mathbb{R}^N.$$

The family of operators $\{R(\lambda) :\ \lambda > 0\}$ satisfies the estimate

$$\|R(\lambda)f\|_\infty \leq \frac{1}{\lambda}\|f\|_\infty, \quad f \in C_b(\mathbb{R}^N),$$

moreover it is possible to prove that the operators $R(\lambda)$ are injective and satify the resolvent identity

$$R(\lambda)f - R(\mu)f = (\mu - \lambda)R(\mu)R(\lambda)f, \quad 0 < \lambda < \mu.$$

We refer to [4, Theorem 2.1.1, Theorem 2.1.3] or [29, Theorem 3.4] for a detailed proof of the last results. Then we can define the weak generator as the unique closed operator $(\hat{A}, \hat{D})$ such that $(0, +\infty) \subset \rho(\hat{A})$, $ImR(\lambda) = \hat{D}$ and $R(\lambda) = R(\lambda, \hat{A})$ for all $\lambda > 0$ (see [16, Chapter III, Proposition 4.6]). In some cases the following equivalent direct description of the weak generator can be more useful.

$$\begin{aligned} D(A_1) &= \Big\{ f \in C_b(\mathbb{R}^N) : \sup_{t\in(0,1)} \frac{\|T(t)f - f\|_\infty}{t} < \infty \text{ and } \exists\, g \in C_b(\mathbb{R}^N) : \\ &\quad \lim_{t\to 0^+} \frac{(T(t)f)(x) - f(x)}{t} = g(x) \ \forall\, x \in \mathbb{R}^N \Big\} \end{aligned}$$

and, for all $f \in D(A_1)$,

$$(A_1 f)(x) = \lim_{t\to 0^+} \frac{(T(t)f)(x) - f(x)}{t}, \quad x \in \mathbb{R}^N, \quad f \in D(A_1).$$

One can prove that $(\hat{A}, \hat{D}) = (A_1, D(A_1))$ (see for example [4, Proposition 2.3.1]). The weak generator enjoies similar properties to those of the infinitesimal generator. For example the following result remains true.

Proposition 1.2.2. *For any $f \in \hat{D}$, $T(t)f \in \hat{D}$ and for any fixed $x \in \mathbb{R}^N$ the function $(T(\cdot)f)(x)$ is continuously differentiable in $[0, +\infty)$ with*

$$\frac{d}{dt}(T(t)f)(x) = (\hat{A}T(t)f)(x) = (T(t)\hat{A}f)(x), \quad t \geq 0. \tag{1.5}$$

(See [4, Proposition 2.3.5]) for the proof.)
Next propositions show the connections between $D_{max}(A)$ and $\hat{D}$. We recall that our goal is to find some conditions under which the maximal domain and the domain of the weak generator coincide.

Proposition 1.2.3. *The following statements hold.*

(i) $\hat{D} \subset D_{max}(A)$ and $\hat{A}u = Au$ for $u \in \hat{D}$. The equality $\hat{D} = D_{max}(A)$ holds if and only if $\lambda - A$ is injective on $D_{max}(A)$ for some positive λ.

(ii) Set $D(A) = D_{max}(A) \cap C_0(\mathbb{R}^N)$, we have the inclusion $D(A) \subset \hat{D}$.

Proof. (i) The inclusion $\hat{D} \subset D_{max}(A)$ and the equality $\hat{A}u = Au$ for $u \in \hat{D}$ follow from the definition of $\hat{D}$ and Theorem 1.2.1. Concerning the second statement, obviously $\lambda - A$ is bijective from $\hat{D}$ onto $C_b(\mathbb{R}^N)$. If it is also injective on $D_{max}(A)$, then $\hat{D} = D_{max}(A)$.
(ii) Let $v \in D(A)$, $f = v - Av$ and $u = R(1, A)f$. If $u_\rho = R(1, A_\rho)f$, then $(u_\rho - v) - A(u_\rho - v) = 0$ in B_ρ and hence, by the maximum principle, $|u_\rho(x) - v(x)| \leq \sup_{|x|=\rho} |v(x)|$ for $|x| \leq \rho$. Letting $\rho \to \infty$ we obtain $u = v$ and hence $v \in \hat{D}$. □

Definition 1.2.4. *We say that W is a Lyapunov function for A if $W \in C^2(\mathbb{R}^N)$, $W \geq 0$, W goes to infinity as $|x| \to \infty$ and $\lambda W - AW \geq 0$ for some positive λ.*

Theorem 1.2.5. *Suppose that there exists a Lyapunov function W for A. Let $\lambda > 0$. If $u \in D_{max}(A)$ satisfies $\lambda u - Au \leq 0$ (≥ 0), then $u \leq 0$ ($u \geq 0$). In particular the operator $\lambda - A$ is injective and then $\hat{D} = D_{max}(A)$.*

We need the following maximum principle for solutions of elliptic equations. For the proof we refer to [25, Theorem 3.1.10].

Lemma 1.2.6. *Let $u \in W^{2,p}_{loc}(\mathbb{R}^N)$ for any $p < \infty$ and suppose that $Au \in C(\mathbb{R}^N)$. If u has a relative maximum (minimum) at the point x_0 then $Au(x_0) + V(x_0)u(x_0) \leq 0$ ($Au(x_0) + V(x_0)u(x_0) \geq 0$).*

Proof (Theorem 1.2.5). For every $\varepsilon > 0$ set $u_\varepsilon = u - \varepsilon W$. Obviously $\lambda u_\varepsilon - Au_\varepsilon \leq 0$ in $\mathbb{R}^N$ and $\lim_{|x|\to\infty} u_\varepsilon(x) = -\infty$. Let $(x_n) \subset \mathbb{R}^N$ be such that $\sup_{x\in\mathbb{R}^N} u_\varepsilon(x) = \lim_{n\to\infty} u_\varepsilon(x_n)$. Then (x_n) is bounded and, without restriction, we may assume that $\lim_{n\to\infty} x_n = x_0$. By Lemma 1.2.6, $Au_\varepsilon(x_0) \leq -V(x_0)u_\varepsilon(x_0)$, then

$$\lambda u_\varepsilon(x_0) \leq Au_\varepsilon(x_0) \leq -V(x_0)u_\varepsilon(x_0)$$

and hence

$$(\lambda + V(x_0))u_\varepsilon(x_0) \leq 0.$$

Since V is a positive potential, it follows $u_\varepsilon(x_0) \leq 0$ and then

$$u_\varepsilon \leq \max_{x\in\mathbb{R}^N} u_\varepsilon(x) = u_\varepsilon(x_0) \leq 0.$$

Letting $\varepsilon \to 0$, we obtain $u \leq 0$. □

1.3 Schrödinger operators via form method

In this section we sketch the construction of the semigroup associated with the Schrödinger operator $A = \Delta - V$ by means of the method of the quadratic forms. Moreover we will see how it is possible to represent this semigroup in integral form through a kernel. All over the section we only require V positive potential in $L^1_{loc}(\mathbb{R}^N)$.

1.3.1 From forms to semigroups

Let W a Hilbert space over the field $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$. A sesquilinear from $a : W \times W \to \mathbb{K}$ is a mapping satisfying

$$\begin{aligned} &a(u+v, w) = a(u,w) + a(v,w)\\ &a(\lambda u, w) = \lambda a(u,w)\\ &a(u, v+w) = a(u,v) + a(u,w)\\ &a(u, \lambda v) = \overline{\lambda} a(u,v) \end{aligned}$$

for $u,\ v,\ w \in W,\ \lambda \in \mathbb{K}$. In other words, a is linear in the first and antilinear in the second variable. If $\mathbb{K} = \mathbb{R}$, then we say that a is bilinear.

Definition 1.3.1. *The form a is called continuous if there exists $M \geq 0$ such that*

$$|a(u,v)| \leq M\|u\|_W\|v\|_W \qquad u,\, v \in W.$$

The form is called coercive if there exists $\alpha > 0$ such that

$$Re\, a(u,u) \geq \alpha\|u\|_W^2, \qquad u \in W.$$

The form a is called symmetric if

$$a(u,v) = \overline{a(v,u)} \qquad \forall\, u, v \in W.$$

Assume from now on that the Hilbert space W is continuously and densely embedded into another Hilbert space H and consider the *operator A associated with the form* on H so defined

$$D(A) = \{u \in W : \exists\, f \in H \quad \text{such that} \quad a(u,v) = (f|v)_H \quad \text{for all} \quad v \in W\}$$
$$Au = f.$$

Observe that f is uniquely determined by u since W is dense in H. The following theorem allows us to construct a semigroup associated with the form. For its proof we refer to [49].

Theorem 1.3.2. *Assume that $a : W \times W \to \mathbb{K}$ is a continuous, coercive form where $W \hookrightarrow H$ densely. Then the operator $-A$ above defined generates a strongly continuous holomorphic semigroup on H.*

Unless we make a rescaling, we can prove that an assumption weaker than the coercivity is sufficient to get a generation result.

Definition 1.3.3. *Let W, H be Hilbert spaces over $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ such that $W \hookrightarrow H$. Let $a : W \times W \to \mathbb{K}$ a sesquilinear form. We call a elliptic (or more precisely H-elliptic) if*

$$Re\, a(u,u) + \omega\|u\|_H^2 \geq \alpha\|u\|_W^2$$

for some $\omega \in \mathbb{R}$, $\alpha > 0$ and for all $u \in W$.

The last definition is equivalent to saying that the form $a_\omega : W \times W \to K$ defined by

$$a_\omega(u,v) := a(u,v) + \omega(u|v)_H \qquad u, v \in W$$

is coercive.

Remark 1.3.4. If A is the operator associated with the form a, then $A + \omega$ is the operator associated with the form a_ω. It follows that if $W \hookrightarrow H$ densely and $a : W \times W \to \mathbb{K}$ is a continuous, elliptic form with ellipticity constant ω, then the operator $-(A + \omega)$ generates a holomorphic strongly continuous semigroup T_ω. Consequently $-A$ generates the semigroup T given by $T(t) = e^{\omega t}T_\omega(t)$. So the assumption of coercivity on a in Theorem 1.3.2 can be replaced by the ellipticity.

It is possible to prove the following density result on the domain.

Proposition 1.3.5. *The domain $D(A)$ of A is dense in W.*

We are ready to prove a generation result for Schrödinger operators.

Example 1.3.6. Let $\mathbb{K} = \mathbb{R}$, $H = L^2(\mathbb{R}^N)$, $0 \le V \in L^1_{loc}(\mathbb{R}^N)$,

$$\begin{aligned} a_1(u,v) &= \int_{\mathbb{R}^N} \nabla u \nabla v \, dx, \qquad u, v \in W_1 := W^{1,2}(\mathbb{R}^N), \\ a_2(u,v) &= \int_{\mathbb{R}^N} V uv \, dx, \qquad u, v \in W_2 := L^2(\mathbb{R}^N, (1+V(x))dx) \end{aligned}$$

and consider the form sum

$$a(u,v) = \int_{\mathbb{R}^N} \nabla u \nabla v \, dx + \int_{\mathbb{R}^N} V uv \, dx$$

defined on $W = W_1 \cap W_2$ with the scalar product

$$(u|v)_W := (u|v)_{W_1} + (u|v)_{W_2}.$$

First, let us observe that W is complete indeed $\|u\|^2_W = \|u\|^2_{W_1} + \|u\|^2_{W_2}$ and it is dense in $L^2(\mathbb{R}^N)$. Moreover a is a symmetric, continuous, elliptic form on $L^2(\mathbb{R}^N)$ infact

$$\begin{aligned} a(u,v) &= \int_{\mathbb{R}^N} \nabla u \nabla v + \int_{\mathbb{R}^N} V uv = \int_{\mathbb{R}^N} \nabla v \nabla u + \int_{\mathbb{R}^N} V vu = a(v,u); \\ |a(u,v)| &\le M(\|\nabla u\|_{L^2(\mathbb{R}^N)} \|\nabla v\|_{L^2(\mathbb{R}^N)} + \|V^{\frac{1}{2}} u\|_{L^2(\mathbb{R}^N)} \|V^{\frac{1}{2}} v\|_{L^2(\mathbb{R}^N)}) \\ &\le M(\|u\|_{W_1}\|v\|_{W_1} + \|u\|_{W_2}\|v\|_{W_2}) \le M\|u\|_W\|v\|_W; \\ a(u,u) + 2\|u\|^2_{L^2(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} |u|^2 + \int_{\mathbb{R}^N} (V+1)u^2 \\ &= \|u\|^2_{W_1} + \|u\|^2_{W_2} \end{aligned}$$

By Remark 1.3.4, we deduce that the operator $-A$ associated with a given by

$$\begin{aligned} &D(A) = \{u \in W^{1,2}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, (1+V(x))dx) : \ -\Delta\, u + Vu \in L^2(\mathbb{R}^N)\}, \\ &Au = -\Delta u + Vu \end{aligned}$$

(where, for $u \in L^2(\mathbb{R}^N)$, $-\Delta u + Vu \in L^2(\mathbb{R}^N)$ is considered in the distributional sense) generates a strongly continuous holomorphic semigroup.

We can immediately prove the positivity of the semigroup generated by the Schrödinger operator.

Proposition 1.3.7. *Let $V \ge 0, \in L^1_{loc}(\mathbb{R}^N)$ a positive potential, then the semigroup $(T(t))_{t\ge 0}$ generated by $-A = \Delta - V$ is positive.*

Proof. Let $f \in L^2(\mathbb{R}^N),\ f \le 0,\ \lambda > 0$, set $u = (\lambda + A)^{-1} f \in W^{1,2}(\mathbb{R}^N)$ (The invertibility of $\lambda + A$ is guaranteed by the Lax- Milgram Theorem). Then

$$\lambda u - \Delta u + V u = f.$$

If we multiply both sides of the previous equality by u^+ and integrate by parts over $\mathbb{R}^N$, we obtain

$$\lambda \int_{\mathbb{R}^N} (u^+)^2 + \int_{\mathbb{R}^N} (\nabla u^+)^2 + \int_{\mathbb{R}^N} V (u^+)^2 = \int_{\mathbb{R}^N} f u^+ \le 0.$$

This implies $u^+ \equiv 0$ and so $u \le 0$. Recalling now that

$$T(t) f = \lim_{n\to\infty} \left(I + \frac{t}{n} A \right)^{-n} f$$

(see [16, Corollary 5.5]), we have the claim. □

From the proposition above it immediately follows that a comparison principle holds for semigroups generated by Schrödinger operators.

Corollary 1.3.8. *Let $(T_1(t))_{t\ge 0},\ (T_2(t))_{t\ge 0}$ be respectively the semigroups generated by the operators $-A_1 = \Delta - V_1$ and $-A_2 = \Delta - V_2$. If $V_1 \le V_2$, then for every $0 \le f \in L^2(\mathbb{R}^N)$ and for all $t \ge 0$, $T_1(t) f \ge T_2(t) f$.*

Proof. Let $\lambda > 0$, $0 \le f \in L^2(\mathbb{R}^N)$ and set $u_1 = (\lambda + A_1)^{-1} f$, $u_2 = (\lambda + A_2)^{-1} f$. As in the proof of the Proposition 1.3.7, in virtue of the approximation formula of the semigroup via the resolvent, it is sufficient to prove that $u_1 \ge u_2$. The functions u_1, u_2 satisfy

$$\lambda u_1 - \Delta u_1 + V_1 u_1 = f$$

and

$$\lambda u_2 - \Delta u_2 + V_2 u_2 = f.$$

Therefore the difference satisfies

$$\lambda(u_1 - u_2) - \Delta(u_1 - u_2) + V_1(u_1 - u_2) = (V_2 - V_1) u_2.$$

Since $f \ge 0$, by Proposition 1.3.7, $u_2 \ge 0$ and then, by the assumption, $(V_2 - V_1) u_2 \ge 0$. By Proposition 1.3.7 again it follows $u_1 \ge u_2$. □

1.3.2 Contractivity properties

In light of the construction of the semigroup via forms method, some nice properties for $(T(t))_{t\ge 0}$ can be deduced by keeping suitable assumptions on a. We establish a contractivity result.
We need the following preliminary proposition.

Proposition 1.3.9. *Let B be the generator of a strongly continuous semigroups $(T(t))_{t\ge 0}$ on H. Then $\|T(t)\| \le 1$ for all $t \ge 0$ if and only if B is dissipative.*

Proof. Assume that B is dissipative, i.e.

$$Re\,(Bu,u) \le 0 \qquad u \in D(B).$$

Let $u \in D(B)$. Then

$$\begin{aligned}\frac{d}{dt}\|T(t)u\|_H^2 &= \frac{d}{dt}(T(t)u|T(t)u)_H = (BT(t)u|T(t)u)_H + (T(t)u|BT(t)u)_H \\ &= 2Re(BT(t)u|T(t)u)_H \le 0.\end{aligned}$$

It follows that $\|T(\cdot)u\|_H^2$ is decreasing. In particular $\|T(t)u\|_H \le \|u\|_H$ for all $t \ge 0$, $u \in D(B)$. Since $D(B)$ is dense in H, the claim follows.
Conversely, assume that T is contractive. Let $u \in D(B)$. Then

$$\|T(t+s)u\|_H = \|T(t)T(s)u\|_H \le \|T(s)u\|_H \quad t,\, s \ge 0.$$

We deduce that $\|T(\cdot)u\|_H^2$ is decreasing and then

$$Re(Bu|u)_H = \frac{1}{2}\frac{d}{dt}_{|t=0}\|T(t)u\|_H^2 \le 0.$$

□

Definition 1.3.10. *We say that the sesquilinear form a is accretive if*

$$Re\,a(u,u) \ge 0 \quad u \in W.$$

Proposition 1.3.11. *Let $(T(t))_{t\ge0}$ the semigroup on H associated with the form a. Then $(T(t))_{t\ge0}$ is contractive if and only if a is accretive.*

Proof. Suppose a accretive. Then $Re(Au,u) = a(u,u) \ge 0$ for all $u \in D(A)$. Thus $-A$ is dissipative and the semigroup is contractive by Proposition 1.3.9. Viceversa, suppose that the semigroup is contractive, then, by Proposition 1.3.9 again, $-A$ is dissipative, hence

$$Re\,a(u,u) = Re(Au|u)_H \ge 0 \quad u \in D(A).$$

Since $D(A)$ is dense in W (see Proposition 1.3.5), $Re\,a(u,u) \ge 0$ for all $u \in W$. □

Example 1.3.12. The form associated with the Schrödinger operator defined in Example 1.3.6 is accretive infact for all $u \in W$

$$a(u,u) = \int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} Vu^2 \ge 0.$$

Therefore the semigroup generated by $\Delta - V$ is contractive on $L^2(\mathbb{R}^N)$.

1.3.3 Symmetric forms

Our next goal is to prove that symmetric forms are associated with symmetric operators and symmetric semigroups.
Let H be a Hilbert space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let A be a densely defined operator on H with domain $D(A)$. Then the adjoint A^* of A is defined by

$$\begin{aligned} &D(A^*) := \{u \in H : \exists\, f \in H \text{ s.t. } (Av|u)_H = (v|f)_H \ \forall\, u \in D(A)\}, \\ &A^* u := f. \end{aligned}$$

Since $D(A)$ is dense in H, the element f is uniquely determined by u. It is easy to prove the following preliminary proposition whose proof is omitted.

Proposition 1.3.13. *Assume that $\lambda \in \rho(A) \cap \mathbb{R}$.*
Then $\lambda \in \rho(A^*)$ and $R(\lambda, A)^* = R(\lambda, A^*)$. Moreover the following are equivalent

(a) $A = A^*$*;*

(b) *A is symmetric;*

(c) $R(\lambda, A)^* = R(\lambda, A)$.

If (a) holds, then we say that A is selfadjoint.

Let now a be a continuous, elliptic, sesquilinear form defined as before on a dense Hilbert space W continuously embedded in H and let A, $(T(t))_{t \geq 0}$ be the associated operator and semigroup respectively. Since $-A$ is the generator of a holomorphic semigroup, $\rho(A) \cap \mathbb{R}$ is nonempty and we can apply Proposition 1.3.13. Denote by $a^* : W \times W \to \mathbb{K}$ the adjoint form of a given by

$$a^*(u, v) := \overline{a(v, u)} \quad u, v \in W.$$

It is natural to investigate about the relations between a^* and the adjoint operator A^*. The following result can be found in [49, Lemma 2.2.3].

Proposition 1.3.14. *The adjoint A^* of A coincides with the operator on H associated with a^*.*

By Proposition 1.3.13 and the Post Widder inversion formula the following proposition immediately follows.

Proposition 1.3.15. *The adjoint operator $-A^*$ generates the adjoint semigroup $(T(t)^*)_{t \geq 0}$ of $(T(t))_{t \geq 0}$.*

Proof. It is sufficient to recall that for every strongly continuous semigroup $(T(t))_{t \geq 0}$ on H with generator $(A, D(A))$ one has

$$T(t)u = \lim_{n \to \infty} \left(I - \frac{t}{n} A \right)^{-n} u \qquad \forall\, u \in H.$$

See [16, Corollary 5.5] for the last formula. □

Remark 1.3.16. In particular we obtained that if $a = a^*$, then $A = A^*$ and $T(t) = T(t)^*$ for every $t \geq 0$. In the case of the Schrödinger operator, we have therefore that it generates a symmetric semigroup.

1.3.4 Ultracontractivity

We finally prove, by using the Berling-Deny conditions and some extrapolation theorems, that the semigroup generated by $\Delta - V$ is ultracontractive and so, by the Dunford-Pettis Theorem, it admits an integral kernel. We state the key ultracontractivity result keeping in mind the application to Schrödinger operators, however it remain true in a slightly more general setting.
Let $H = L^2(\mathbb{R}^N)$, W be a Hilbert space such that $W \hookrightarrow L^2(\mathbb{R}^N)$ is dense. We assume that $u \in W$ implies $u \wedge 1 \in W$. Furthermore we assume that $N \geq 2$ and $W \hookrightarrow L^q(\mathbb{R}^N)$ where $\frac{1}{q} = \frac{1}{2} - \frac{1}{N}$.

Theorem 1.3.17. *Let $a : W \times W \to \mathbb{R}$ be a bilinear, continuous, symmetric form such that for some $\mu > 0$*

$$a(u,u) \geq \mu \|u\|_W^2$$

and $a(u \wedge 1, (u-1)^+) \geq 0$ for all $u \in W$. Denote by T the semigroup associated with a on $L^2(\mathbb{R}^N)$. Then there exists a constant $c > 0$ which depends on W such that

$$\|T(t)\|_{\mathcal{L}(L^1, L^\infty)} \leq c\mu^{-\frac{N}{2}} t^{-\frac{N}{2}} \qquad t > 0.$$

Proof. Since W is continuously embedded in L^q, there exists a positive constant c such that

$$\|u\|_{L^q} \leq c\|u\|_W \qquad \forall\, u \in W.$$

Observe that, by the Berling Deny conditions and since a is symmetric and so A selfadjoint, $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is invariant under the semigroup and $(T(t))_{t \geq 0} = (T(t)^*)_{t \geq 0}$ defined on $L^2(\mathbb{R}^N)$ extends to a positive contraction semigroup $T_p(t)$ on $L^p(\mathbb{R}^N)$ for all $1 \leq p \leq \infty$ (see [13, Theorem 1.4.1]). In particular we have $\|T(t)\|_{\mathcal{L}(L^q)} \leq 1$, hence $\|T(\cdot)f\|_{L^q}$ is decreasing for all $f \in L^q(\mathbb{R}^N)$. Consequently, for $f \in W$, we have

$$\begin{aligned} t\|T(t)f\|_{L^q}^2 &= \int_0^t \|T(t)f\|_{L^q}^2 ds \leq \int_0^t \|T(s)f\|_{L^q}^2 ds \leq c^2 \int_0^t \|T(s)f\|_W^2 ds \\ &\leq \frac{c^2}{\mu} \int_0^t a(T(s)f, T(s)f) ds = \frac{c^2}{\mu} \int_0^t (AT(s)f | T(s)f)_{L^2} ds \\ &= -\frac{c^2}{2\mu} \int_0^t \frac{d}{ds} \|T(s)f\|_{L^2}^2 = \frac{c^2}{2\mu} (\|f\|_{L^2}^2 - \|T(t)f\|_{L^2}^2) \\ &\leq \frac{c^2}{2\mu} \|f\|_{L^2}^2. \end{aligned}$$

So we obtained that

$$\|T(t)f\|_{L^q} \leq \frac{c}{\sqrt{2\mu}} t^{-\frac{1}{2}} \|f\|_{L^2}.$$

By [12, Lemma II.1] it follows that

$$\|T(t)\|_{\mathcal{L}(L^1, L^\infty)} \leq C\mu^{-\frac{N}{2}} t^{-\frac{N}{2}} \qquad \forall\; t > 0.$$

□

Remark 1.3.18. If a is a bilinear, continuous, symmetric and elliptic form with positive ellipticity constant ω, such that $a(u \wedge 1, (u-1)^+) \geq 0$ for all $u \in W$, after a rescaling we obtain that there exists a positive constant c such that

$$\|T(t)\|_{\mathcal{L}(L^1,L^\infty)} \leq ce^{\omega t}t^{-\frac{N}{2}} \qquad t > 0.$$

Example 1.3.19. The form associated with the Schrödinger operator is continuous, symmetric and elliptic with positive ellipticity constant. Moreover if $u \in W^{1,2}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, (1+V(x))dx)$ then $(u \wedge 1)$ belongs to the same space indeed we have

$$\begin{aligned}
&\nabla(u \wedge 1) = \nabla u \chi_{\{u \leq 1\}};\\
&\int_{\mathbb{R}^N} (u \wedge 1)^2 = \int_{\{u \leq 1\}} u^2 + \int_{\{u>1\}} 1 \leq 2 \int_{\mathbb{R}^N} u^2 < \infty;\\
&\int_{\mathbb{R}^N} (1+V)(u \wedge 1)^2 = \int_{\{u \leq 1\}} (1+V)u^2 + \int_{\{u>1\}} (1+V)\\
&\leq 2 \int_{\mathbb{R}^N} (1+V)u^2 < \infty.
\end{aligned}$$

By Stampacchia's Lemma and some straightforward computations,

$$\begin{aligned}
&\nabla(u-1)^+ = \nabla u \chi_{\{u \geq 1\}};\\
&\nabla u(x) = 0 \quad \text{a.e. on} \quad \{u = 1\};\\
&a(u \wedge 1, (u-1)^+) = \int_{\mathbb{R}^N} \nabla(u \wedge 1)\nabla(u-1)^+ + \int_{\mathbb{R}^N} V(u \wedge 1)(u-1)^+\\
&= \int_{\{u \geq 1\}} V(u-1)^+ \geq 0.
\end{aligned}$$

It follows that there exist C, ω positive constants such that the semigroup generated by $\Delta - V$ satisfies

$$\|T(t)\|_{\mathcal{L}(L^1,L^\infty)} \leq ce^{\omega t}t^{-\frac{N}{2}} \qquad \forall\, t > 0.$$

Thanks to the Dunford-Pettis criterion we are finally able to deduce the existence of an integral kernel.
Given $p \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$,

$$(B_p f)(x) = \int_{\mathbb{R}^N} p(x,y)f(y)\,dy$$

defines a bounded operator $B_p \in \mathcal{L}(L^1(\mathbb{R}^N), L^\infty(\mathbb{R}^N))$ and

$$\|B_p\|_{\mathcal{L}(L^1,L^\infty)} \leq \|p\|_{L^\infty(\mathbb{R}^N \times \mathbb{R}^N)}.$$

A kind of converse is true. The proof of the following result can be found in [1, Theorem 1.3].

Theorem 1.3.20. (Dunford- Pettis) Let $1 \le r < \infty$, $B \in \mathcal{L}(L^r(\mathbb{R}^N))$ such that $\|B\|_{\mathcal{L}(L^1(\mathbb{R}^N),L^\infty(\mathbb{R}^N))} < \infty$. Then there exists $p \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ such that

$$(Bf)(x) = \int_{\mathbb{R}^N} p(x,y)f(y)\,dy$$

almost everywhere for all $f \in L^1(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$. In that case $B \ge 0$ if and only if $p \ge 0$.

Summarizing, through this section, we proved that, without assuming hölderianity assumptions, but only requiring local integrability on the positive potential, the semigroup generated by the Schrödinger operator is an integral operator. There exists therefore a positive kernel $p(x,y,t)$ such that

$$(T(t)f)(x) = \int_{\mathbb{R}^N} p(x,y,t)f(y)dy \qquad \forall\ x \in \mathbb{R}^N,\ t > 0,\ f \in L^1(\mathbb{R}^N).$$

Moreover there exists $C,\ \omega > 0$ such that

$$\|p(\cdot,\cdot,t)\|_{L^\infty(\mathbb{R}^N \times \mathbb{R}^N)} \le Ce^{\omega t}t^{-\frac{N}{2}}$$

for all $t > 0$.

Remark 1.3.21. By Corollary 1.3.8, it follows that, if p_1 and p_2 are the kernels corresponding respectively to the Schrödinger operators $\Delta - V_1$ and $\Delta - V_2$ with $V_1 \le V_2$, then $p_1 \ge p_2$. In particular, choosing $V_1 \equiv 0$, it follows that the kernel of the semigroup generated by the Schrödinger operator is pointwise dominated by the heat kernel of the Laplacian.

Remark 1.3.22. By the representation formula and the symmetry of the semigroup generated by a Schrödinger operator, it follows that the kernel is symmetric with respect to the variables x and y, moreover the contractivity of $(T(t))_{t\ge 0}$ in $L^\infty(\mathbb{R}^N)$ yields $\int_{\mathbb{R}^N} p(x,y,t)\,dy \le 1$ for all $t > 0$ and $x \in \mathbb{R}^N$.