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Kernel Estimates for Markov Semigroups and Parabolic Schrödinger Operators

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Introduction

In the last years, owing to their connections with probability and stochastic analysis, there has been an increasing interest towards linear elliptic and parabolic operators with unbounded coefficients. In literature, one can find a careful theory concerning solutions of Cauchy problems associated with the above mentioned operators in several function spaces. Many aspects such as existence, uniqueness, regularity, integral representation are object of study for numerous authors.
We will deal with elliptic operators of form

$$Au(x) = \sum_{i,j=1}^{N} a_{ij}(x) D_{ij} u(x) + \sum_{i=1}^{N} F_i(x) D_i u(x) - V(x) u(x)$$

with (a_{ij}) symmetric matrix satisfying the ellipticity condition, $a_{ij},\ F_i,\ V$ real-valued functions, V positive potential. Under hölderianity assumptions on the coefficients, an existence result for bounded classical solutions of the Cauchy problem

$$\begin{cases} u_t(x,t) = Au(x,t) & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = f(x) & x \in \mathbb{R}^N \end{cases}$$

with initial datum $f \in C_b(\mathbb{R}^N)$ holds (see [29], [4]). The solution is constructed through an approximation procedure as the limit of solutions of Cauchy Dirichlet problems in suitable bounded domains and is given by a certain semigroup $T(t)$ applied to the initial datum f.
Moreover it can be represented by the formula

$$u(x,t) = \int_{\mathbb{R}^N} p(x,y,t) f(y)\, dy \qquad t > 0,\ x \in \mathbb{R}^N$$

where p is a positive function called integral kernel. In the first four chapters of this work, our attention is mainly devoted to the study of the integral kernel p just introduced. In particular we prove upper bounds on these kernels. We examined separately operators containing only the second and the first order parts and Schrödinger operators characterized by a vanishing drift term ($F = 0$) and second order part given by the Laplacian. The case of the whole operator is not contemplated. The semigroup associated with the Schrödinger operator can be built under weaker assumptions on the potential by means of the quadratic form method. It is sufficient the requirement $V \in L^1_{loc}(\mathbb{R}^N)$ to obtain a strongly continuous analytic semigroup on $L^2(\mathbb{R}^N)$ that can be extrapolated to $L^p(\mathbb{R}^N)$ for $1 \leq p \leq \infty$ and that admits an integral representation.

If A is given by $\Delta - V$, the kernel p is pointwise dominated by the heat kernel of the Laplacian in $\mathbb{R}^N$, that is

$$p(x, y, t) \leq \frac{1}{(4\pi t)^{\frac{N}{2}}} \exp\left\{-\frac{|x-y|^2}{4t}\right\}, \qquad \forall\, x,\ y \in \mathbb{R}^N.$$

For the presence of the positive potential, one expects more decay in the space variables.

Deeper upper bounds for $V(x) = |x|^\alpha$ with $\alpha > 2$ can be found for example in [13, Section 4.5]. Davies and Simon prove that $p(x, y, t) \leq c(t)\psi(x)\psi(y)$, where ψ is the ground state of $-A$, that is the eigenfunction corresponding to the smallest eigenvalue, and c has an explicit behaviour near 0. Similar estimates can be found in [28] where upper bounds like $p(x, y, t) \leq c(t)\phi(x)\phi(y)$ are obtained for a large class of potential tending to infinity as $|x| \to \infty$ under the main assumption that $\omega = 1/\phi$ satisfies $\omega(x) \to \infty$ as $|x| \to \infty$ and $-A\omega \geq g \circ \omega$ where g is a convex function growing faster then linearly. The behaviour of $c(t)$ near 0 is also shown to be precise. The authors are able to deduce estimates for $V(x) = |x|^\alpha$ for every $\alpha > 0$ but the Davies and Simon bounds cannot be achieved since the ground state does not satisfy their assumptions.

Sikora proves an other kind of estimates for $V(x) = |x|^\alpha$, $\alpha > 0$, see[45] where also lower bounds are proved. He obtains precise on-diagonal bounds of the form $p(x, x, t) \leq h(x, t)$ and then he deduces off-diagonal bounds from the semigroups law.

Potentials unbounded only in certain directions (like $x_1^2 x_2^2 x_3^2$ in $\mathbb{R}^3$) are considered by Kurata in [22] where upper bounds are proved. Such estimates are not sharp but their main concern is the applicability to degenerate non homogeneous potentials.

In the case of $V(x) = |x|^\alpha$ we obtain estimates similar to those of Sikora ([45]). However our method is not confined to special polynomial potentials but applies also to logarithmic, exponential growths or more generally to radial increasing potentials and potentials consisting of a radial part and lower order terms. Moreover our approach allows us to obtain more precise bounds.

On the other hand we consider also bounds similar to the Davies and Simon ones and, using the similarity between Schr̆odinger and Kolmogorov operators, we improve the estimates obtained by Davies and Simon for $V(x) = |x|^\alpha$ with $\alpha > 2$ and we show that the same techinque works for other potentials too. As nice application, we see how the Sikora type estimates combined with a Tauberian theorem due to Karamata allow us to deduce some interesting information about the asymptotic distribution of the eigenvalues of $-A$. When V has a polynomial behaviour these results have been proved by Titchmarsh (see [51]) using cube-decomposition methods. Our approach allows us to treat also potentials with different growth.

Kolmogorov operators, that is elliptic operators with unbounded drift term and vanishing potential, have also been studied. Some results concerning pointwise upper bounds for their kernels can be found for example in [27] where the authors use Lyapunov functions techniques to prove estimates of the form $p(x, y, t) \leq c(t)\omega(y)$. We get inspiration from this paper to prove upper bounds like $p(x, y, t) \leq c(t)\omega(y, t)$.
In recent papers (see [6], [7] and [8]), Bogachev, Krylov, Röckner and Shaposhnikov prove existence and regularity properties for parabolic problems having measures as initial data, they also deduce uniform boundedness of solutions but we cannot compare their estimates with our results since the fundamental solution p is singular for $t = 0$.

Besides the kernel estimates, other aspects of Schr̆odinger operators were widely investigated. For example, an interesting problem is the characterization of the domain in which the operator generates a strongly continuous or an analytic semigroup. A natural question is under which conditions on the potential V the domain of $\Delta - V$ in $L^p(\mathbb{R}^N)$ coincides with the intersection of the domain of the Laplacian and the domain of the potential that is $W^{2,p}(\mathbb{R}^N) \cap D(V)$ where $D(V) = \{u \in L^p(\mathbb{R}^N) : \; Vu \in L^p(\mathbb{R}^N)\}$. By the classical theory for elliptic operators with bounded coefficients, the last description of the domain is true for bounded potentials but in general a greater effort is needed to get information on the domain in the unbounded case and additional assumptions have to be required.
Cannarsa and Vespri (see [10]) prove that, assuming an oscillation condition on the potential, namely $|\nabla V| = o(V^{\frac{3}{2}})$, the operator generates an analytic semigroup in $L^p(\mathbb{R}^N)$ for $1 \leq p \leq \infty$. Moreover with their approach they characterize for $1 < p < \infty$. We remark that they consider a more general operator containing also a drift term.
Metafune, Pruss, Rhandi and Schnaubelt (see [31]) improve the previous generation result. In particular they establish that, under suitable assumptions on the drift term and the oscillation assumption above on the potential, the whole elliptic operator A endowed with the natural domain $D(\Delta) \cap D(V)$ generates an analytic and contractive strongly continuous semigroup on $L^p(\mathbb{R}^N)$, $1 \leq p < \infty$, and on $C_0(\mathbb{R}^N)$. The precise description of the domain corresponds to good apriori estimates for the elliptic problem $\lambda u - Au = f$. Moreover the maximal regularity of type L^q for the inhomogeneus parabolic problem associated with the given operator is deduced.
On the other hand the equality $D(\Delta - V) = D(\Delta) \cap D(V)$ holds even if V belongs to suitable Reverse Hölder classes (see for example [41] and [3]). The oscillation condition and the reverse Hölder one are incomparable, it is easy to find examples of polynomials which satisfy a reverse Hölder inequality for which the oscillation condition fails and viceversa. The potential $V(x, y) = x^2y^2$ does not satisfy $|DV| \leq \gamma V^{\frac{3}{2}}$ for any γ but it belongs to the reverse Hölder class B_p for every $1 < p \leq \infty$. The potential $V(x) = e^x$ in $\mathbb{R}$ does not satisfy the

doubling property and then it does not belong to any reverse Hölder class but the oscillation condition obviously holds.
In [41] Shen proves the L^p boundedness of $D^2(-\Delta+V)^{-1}$ on $\mathbb{R}^N$ for $1 < p < \infty$, assuming $V \in B_p$ and under the restrictions $N \geq 3$, $p \geq \frac{N}{2}$, he introduces an auxiliary function $m(x, V)$, which is well defined for $p \geq \frac{N}{2}$ and allows him to estimate the fundamental solution.
In a recent work, P. Auscher and B. Ben Ali , see [3], extend Shen's result removing the original restrictions on the space dimension and on p. In their proof they use a criterion to prove the L^p boundedness of certain operators in absence of kernels, see [42, Theorem 3.1], [2, Theorem 3.14], and some weighted mean value inequalities for nonnegative subharmonic functions with respect to Muckenhoupt weights.
Following Shen's approach, W. Gao and Y. Jiang extend the previous results to the parabolic case. In [18], they consider the parabolic operator $\partial_t - \Delta + V$ where $V \in B_p$ is a nonnegative potential depending only on the space variables and, under the assumptions $N \geq 3$ and $p > (N + 2)/2$, they prove the boundedness of $V(\partial_t - \Delta + V)^{-1}$ in L^p.
We consider the parabolic Schrödinger operator, in particular we focus our attention on the validity of apriori estimates for solutions of $\lambda u - \partial_t u + \Delta u - V u = f$ in $L^p(\mathbb{R}^{N+1})$ and consequently on the characterization of the domain. We improve the results of Gao and Jiang indeed a larger class of potentials is allowed. We obtain the L^p boundedness of $V(\partial_t - \Delta + V)^{-1}$ (and consequently of $\partial_t(\partial_t - \Delta + V)^{-1}$ and $D^2(\partial_t - \Delta + V)^{-1}$) if the potential V belongs to some parabolic Reverse Hölder class B_p for $1 < p < \infty$, without any restriction on the space dimension and on p; moreover we remark that our potentials may also depend on the time variable. Our approach is similar to that of [3]. We use a more general version of the boundedness criterion in absence of kernels in homogeneous spaces (see Theorem D.1.1) and the Harnack inequality for subsolutions of the heat equation. A crucial role is played by some properties of the B_p weights originally proved in the classical case that is when $\mathbb{R}^N$ is equipped with the Lebesgue measure and the Euclidean distance. Since we need parabolic cylinders instead of balls of $\mathbb{R}^N$, we use the more general theory of B_p weights in homogeneous spaces, as treated in [48, Chapter I].

The first chapter contains some introductory and known results. Specifically, following [29, Section 4], we assume local uniform ellipticity and local hölderianity on the coefficients to prove that there exists a positive semigroup $(T(t))_{t\geq 0}$ such that, for any $f \in C_b(\mathbb{R}^N)$, $u(x, t) = T(t)f(x)$ is a classical solution of the Cauchy problem associated with $A = \sum_{i,j=1}^N a_{ij} D_{ij} + \sum_{i=1}^N F_i D_i - V$. $T(t)$ is the semigroup generated by A in a weak sense. The semigroup $(T(t))_{t\geq 0}$ has a smooth integral kernel whose behaviour will be examined later.
After that, in a special case we show how a different approach is possible. We sketch the construction of the semigroup generated by Schrödinger operators with locally integrable potentials by means of the quadratic form theory (see [13]). The semigroup generated by $\Delta - V$ is ultracontractive and, by the Dun-

ford Pettis Theorem, it admits an integral kernel.

In Chapter 2 we prove upper and lower bounds for heat kernels of Schrödinger semigroups and upper bounds for Kolmogorov semigroups. In both cases we consider the semigroup built under hölderianity assumptions on the coefficients. First we analyse Kolmogorov operators. We assume the existence of a Lyapunov function for the operator A, i.e. a positive and smooth function V going to infinity for $|x| \to \infty$ such that $AV \leq \lambda V$ for some positive λ. This requirement is not restrictive since for the operators we are interested in through this chapter a function satisfying this property exists (see [27, Section 2]). This assumption insures that the domain of the weak generator coincides with the maximal domain.
We introduce Lyapunov functions for the parabolic operator $L = \partial_t + A$. The definition is a little bit different from the one given in the elliptic case. We say that a continuous function $W : [0,T] \times \mathbb{R}^N \to [0,+\infty)$ is a Lyapunov function for the operator L if it belongs to $C^{2,1}(Q_T)$, $\lim_{|x|\to\infty} W(x,t) = +\infty$ uniformly with respect to t in compact sets of $(0,T]$ and there exists $h : [0,T] \to [0,\infty)$ integrable in a neighborhood of 0 such that $LW(x,t) \leq h(t)W(x,t)$ for all $(x,t) \in Q_T$. Note that we do not require that $W(x,0)$ tends to ∞ as $|x| \to \infty$. We prove that a similar functions is integrable with respect to the kernel p, more precisely $\int_{\mathbb{R}^N} p(x,y,t)W(y,t)\,dy \leq e^{\int_0^t h(s)ds} W(x,0)$. Assuming growth assumptions on the radial component of the drift, we provide a class of Lyapunov functions for L. To achieve the main result, we preliminary establish some integrability and regularity results for the kernel. Then, by using the estimate of the L^1-norm of Lyapunov functions stated before, we prove pointwise estimates of kernels of the form $p(x,y,t) \leq c(t)\omega(y,t)$. The main ingredient is an estimate of the L^∞-norm of solutions of certain parabolic problems. We explicitly write the bounds so obtained in correspondence of some particular choices of the drift.
A similar method based upon the Lyapunov functions technique works also for Schrödinger operators. In the second part of the chapter we deduce upper bounds for Schrödinger semigroups even if a different approach gives sometimes more refined estimates as it will be shown in Chapter 3. Here we assume that the potential satisfies the oscillation hypothesis $|DV| \leq \gamma V^{\frac{3}{2}} + C_\gamma$ for small values of γ.
The integrability of Lyapunov functions, a parabolic regularity result and an interpolative estimate of the sup norm of functions in parabolic Sobolev spaces play a crucial role in the proof of the wished estimates which are of the Sikora form $p(x,y,t) \leq c(t)\omega(x,t)\omega(y,t)$ (see [45]). As application we see that this method enables us to deduce small times upper bounds for potentials growing in a polynomial, exponential or logarithmic way. The sharpness is discussed. For $V(x) = |x|^\alpha$, $0 < \alpha < 2$, $V(x) = \exp\{c|x|^\alpha\}$ and $V(x) = M\log(1+|x|^2)$ our estimates are sharp, the method does not give a precise estimate of certain constants in ω which however will be obtained in the next chapter. The estimate for $V(x) = |x|^\alpha$, $\alpha > 2$, is exact concerning the decay in the space variable for a

fixed time, sharp estimates for such potential are proved in Chapter 3 by considering suitable space-time regions. Finally large time estimates are deduced by the previous ones by means of the simmetry of the kernel and by the semigroup law.

The third chapter is devoted to the study of upper and lower bounds of Schrödinger kernels. In some cases, the results here obtained cover the ones in the previous chapter.
Given a positive potential V, for each positive s we consider the new potential V_s equal to s in the level set corresponding to s and V otherwise. To obtain the bound on p, as in [45], we estimate the difference between the kernels p and p_s and then we use the triangle inequality. In [45], Sikora uses the functional calculus to estimate such a difference for the potential $V(x) = |x|^\alpha$. Our approach, though more elementary, yields more precise bounds and a wider class of potentials can be studied. Once the difference is estimated, we observe that, for radial potentials and in correspondence of a particular choice of s depending on the potential, the measure of the level set is known and the bound can be explicitely written as follows

$$p(x,x,t) \le \frac{1}{(4\pi t)^{\frac{N}{2}}} \exp\{-tV(cx)\} + \frac{C(N)}{t^{\frac{N}{2}}} \frac{c^N \omega_N}{(1-c)^N} \exp\left\{-\frac{(1-c)^2|x|^2}{4t}\right\}$$

for all $0 < c < 1$.
Low-order perturbations of the potentials above can be estimated in similar way. We remark that we first obtain on diagonal estimates and then by the semigroup law we deduce off diagonal estimates.
The natural question is whether such estimates are sharp. Considering suitable space-time regions, one can control the gaussian term with the first addendum, moreover in these regions similar lower estimates are true and the sharpness follows.
As consequence we deduce a result concerning the asymptotic distribution of the eigenvalues of $-\Delta + V$. Denoted by $N(\lambda)$ the number of eigenvalues less then λ and λ_n the eigenvalues of $-\Delta+V$, the Karamata Theorem relates the asymptotic behaviour of $N(\lambda)$ for $\lambda \to \infty$ with the behaviour of $\sum_n e^{-\lambda_n t}$ for small values of t, by Mercer's Theorem we know that $\int_{\mathbb{R}^N} p(x,x,t) = \sum_{n=1}^\infty e^{-\lambda_n t}$, therefore we can use the upper and lower estimates for p to achieve information on $N(\lambda)$.

In Chapter 4, we prove once again upper bounds for Schrödinger semigroups. But this time we obtain Davies-type estimates. We recall that by a result due to Davies, if $V(x) = |x|^\alpha$, $\alpha > 2$, then for all $\frac{\alpha+2}{\alpha-2} < b < \infty$, $p(x,y,t) \le c_1 \exp\{c_2 t^{-b}\}\psi(x)\psi(y)$ for all $x,\ y \in \mathbb{R}^N,\ 0 < t \le 1$, where ψ is the ground state of $-\Delta + |x|^\alpha$. Moreover the lower bound on b is sharp in the sense that if $p(x,y,t) \le c(t)\psi(x)\psi(y)$ then $c(t) \ge c_1 \exp\{c_2 t^{-\frac{\alpha+2}{\alpha-2}}\}$. We improve this estimate indeed we show that $p(x,y,t) \le c_1 \exp\{c_2 t^{-\frac{\alpha+2}{\alpha-2}}\}\psi(x)\psi(y)$ by using the similarity between Schrödinger and Kolmogorov operators. If the function

$|\nabla\phi|^2 - 2\Delta\phi$ is bounded from below in $\mathbb{R}^N$, then the operator $\Delta - \nabla\phi \cdot \nabla$ in $L^2(\mathbb{R}^N)$ is unitarily equivalent to the Schrödinger operator $\Delta - V$ with potential $V = \frac{1}{4}|\nabla\phi|^2 - \frac{1}{2}\Delta\phi$ in $L^2(\mathbb{R}^N)$ (with respect to the Lebesgue measure), see [26, Proposition 2.2]. In particular $\Delta - \nabla\phi \cdot \nabla = -T(\Delta - V)T^{-1}$ where T is the multiplication operator $Tu = e^{\frac{\phi}{2}}u$. Consequently the problems of finding estimates for the kernels of the two operators are equivalent. We prove estimates for the Kolmogorov kernel as in [27] and then we deduce estimates for the Schrödinger kernel.

The last chapter is aimed at the description of the domain of parabolic Schrödinger operators. As main result, we prove that, if the potential V is in a parabolic Reverse Hölder class B_p, then $\|Vu\|_{L^p(\mathbb{R}^{N+1})} \leq C\|\partial_t u - \Delta u + Vu\|_{L^p(\mathbb{R}^{N+1})}$ for all u in the maximal domain of the operator. By difference and by parabolic regularity, the estimates for the L^p norm of D^2u and $\partial_t u$ follow. Consequently we deduce that the domain of $\partial_t - \Delta + V$ is $W_p^{2,1}(\mathbb{R}^{N+1}) \cap D(V)$ where $D(V) = \{u \in L^p(\mathbb{R}^{N+1}) : \ Vu \in L^p(\mathbb{R}^{N+1})\}$.
Through this chapter, we define the parabolic reverse Hölder classes by replacing cubes or balls of $\mathbb{R}^N$ in the classical definition with parabolic cylinders and we state some useful properties enjoyed by them. For istance B_p weights are in some Muckenhoupt classes A_p and satisfy a self improvement property due to Gehring. Some examples of B_p weights are provided.
We take care of giving a meaning to the operator. We get inspiration by an elliptic Kato's result (see [19]) to endow $\partial_t - \Delta + V$ in L^p with the maximal domain $\{u \in L^p(\mathbb{R}^{N+1}) : \ Vu \in L^1_{loc}(\mathbb{R}^{N+1}), \ (\partial_t - \Delta + V)u \in L^p(\mathbb{R}^{N+1})\}$. We prove that for every $\lambda > 0$, $\lambda + \partial_t - \Delta + V$ is invertible and, for every $1 \leq p < \infty$, C_c^∞ is a core for the operator. The main tool is a parabolic version of Kato's inequality originally proved in the elliptic case and which we generalized to the parabolic one.
Then we consider the operator on L^1 and we prove the apriori estimates. This is an easy task, indeed the claimed estimates for $p = 1$ immediately follow by approximation and integration by parts. These estimates will play a key role in the proof of the apriori estimates in the general case which is more involved and requires a greater effort. We use a powerful criterion to prove the boundedness of certain operators in absence of kernels. We turn our attention toward the operator $T = V(\partial_t + \Delta - V)^{-1}|\cdot|$. Its boundedness in L^1, which follows by the previous apriori estimates, and a sort of reverse Hölder inequality which follows by the properties of the B_p weights and by the Harnack inequality for subsolution of the heat equation, thanks to the criterion mentioned above, give the boundedness in L^p. The main result immediately follows.

Appendix A, B and C contain respectively the Karamata Theorem and a weaker version of it used in Chapter 3 to study the asymptotic distribution of the eigenvalues of the Schrödinger operators, a preliminary inequality needed to prove an integration by parts formula (see [32]) and used in Chapter 5 to study the parabolic Schrödinger operator in an infinite cylinder $Q(S,T)$ and some Embedding Theorems for parabolic Sobolev spaces useful in the second chapter.

The whole Appendix D is devoted to the boundedness criterion used in Chapter 3. It's worth it aiming the attention to such result which is extremely helpful and of own interest. A weaker version of such theorem appears in [42, Theorem 3.1], it is confined to the elliptic case and it is more restrictive concerning the exponents involved. Namely, Shen, inspired by a paper of Caffarelli and Peral (see [9]), proved that if T is a sublinear bounded operator on $L^2(\mathbb{R}^N)$ such that, given $p > 2$, there exist some positive constants $\alpha_2 > \alpha_1 > 1,\ N > 0$ for which

$$\left\{ \frac{1}{|B|} \int_B |Tf|^p \, dx \right\}^{\frac{1}{p}} \le N \left\{ \left(\frac{1}{|\alpha_1 B|} \int_{\alpha_1 B} |Tf|^2 \, dx \right)^{\frac{1}{2}} + \sup_{B' \supset B} \left(\frac{1}{|B'|} \int_{B'} |f|^2 \, dx \right)^{\frac{1}{2}} \right\}$$

for any ball $B \subset \mathbb{R}^N$ and any bounded measurable function f with compact support contained in $\mathbb{R}^N \setminus \alpha_2 B$ then T is bounded in $L^q(\mathbb{R}^N)$ for any $2 < q < p$. Following [42, Theorem 3.1], we prove the result stated above in a more general setting, i.e. we replace balls of $\mathbb{R}^N$ with parabolic cylinders and a whatever L^{p_0} space plays the role of the L^2 space in the assumptions. For the proof we need a revisited theory in the parabolic case concerning the Maximal Hardy-Littlewood functions, the Lebesgue points and a Calderón-Zygmund decomposition.
We remark that, since $\mathbb{R}^{N+1}$ endowed with the parabolic distance is a homogeneous space, the result can be deduced by a more general version of this theorem formulated by Auscher and Martell (see [2, Section5]).
As application we provide an alternative proof of the classical apriori estimates for the operator $\partial_t - \Delta$ and of the classical Calderón-Zygmund Theorem. These operators are both bounded in L^2 and satisfy the assumption of Shen's Theorem, this can be proved by means of Cacioppoli-type estimates and by Sobolev Embedding Theorems in the parabolic case and by the mean value Theorem for harmonic functions in the elliptic one.

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Contents

Chapter 1

Markov semigroups in $\mathbb{R}^N$

In this chapter we collect some preliminary results nedeed to develop the next theory. In particular we introduce elliptic operators with unbounded coefficients and we study the Markov semigroups associated with them.
We consider the operator

$$Au(x) = \sum_{i,j=1}^{N} a_{ij}(x)D_{ij}u(x) + \sum_{i=1}^{N} F_i(x)D_iu(x) - V(x)u(x)$$

under the hypotheses: (a_{ij}) symmetric matrix, $a_{ij},\ F_i,\ V$ real-valued functions, $V \geq 0$. Moreover we assume the ellipticity condition

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \lambda(x)|\xi|^2$$

for every $x,\ \xi \in \mathbb{R}^N$, with $\inf_K \lambda(x) > 0$ for every compact $K \subset \mathbb{R}^N$. The operator so defined is locally uniformly elliptic, that is uniformly elliptic on every compact subset of $\mathbb{R}^N$.
We introduce the realization of A in $C_b(\mathbb{R}^N)$ with $D_{max}(A)$ defined as follows

$$D_{max}(A) = \{u \in C_b(\mathbb{R}^N) \cap W^{2,p}_{loc}(\mathbb{R}^N) \quad \text{for all} \quad p < \infty :\ Au \in C_b(\mathbb{R}^N)\}.$$

In the first section, we prove existence results for bounded classical solutions of the Cauchy problem

$$\begin{cases} u_t(x,t) = Au(x,t) & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = f(x) & x \in \mathbb{R}^N \end{cases} \tag{1.1}$$

with initial datum $f \in C_b(\mathbb{R}^N)$ and under hölderianity assumptions on the coefficients. Since the coefficients of the operator are not bounded, the classical theory does not give a solution of the problem. The solution is constructed through an approximation procedure as limit of solutions of Cauchy Dirichlet

problems in suitable bounded domains and is given by a certain semigroup $T(t)$ applied to the initial datum f.
Moreover we prove that the solution can be represented by the formula

$$u(x,t) = \int_{\mathbb{R}^N} p(x,y,t)f(y)\,dy \qquad t > 0,\ x \in \mathbb{R}^N$$

where p is a positive function called the integral kernel. As above, p is obtained as limit of kernels of solutions in bounded domains.
A continuity property of the operators $T(t)$ is deduced.
In the second section we state and prove some results concerning the generator in a weak sense of the semigroup so constructed.
The last section is devoted to the study of a particular elliptic operator with unbounded coefficients, the so called Schrödinger operator. It is obtained in correspondence of vanishing drift term ($F = 0$) and constant diagonal matrix (a_{ij}). It's formal expression is given by $A = \Delta - V$ where V is an unbounded positive potential as before. The existence of the semigroup generated (in a weak sense) by such operator and of an integral kernel are obviously guaranted by the theory developed in the first two sections under hölderianity hypothesis on the potential. Anyway we will see how a different approach, the quadratic form method, allows us to prove that, under the weaker assumption $V \in L^1_{loc}(\mathbb{R}^N)$, the Schrödinger operator generates a semigroup on $L^2(\mathbb{R}^N)$ that can be extrapolated to $L^p(\mathbb{R}^N)$ for $1 \leq p \leq \infty$ and admits an integral representation.

1.1 The Cauchy problem and the semigroup

Through this and the next section we assume the following hypothesis on the coefficients of the operator:

(i) $a_{ij} = a_{ji}$ for all $i,\ j = 1, ..., N$;

(ii) $\sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \geq \lambda(x)|\xi|^2$ for every $x,\ \xi \in \mathbb{R}^N$, with $\inf_K \lambda(x) > 0$ for every compact $K \subset \mathbb{R}^N$;

(iii) a_{ij}, F_i, V belong to $C^\alpha_{loc}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$;

(iv) $V(x) \geq 0$ for all $x \in \mathbb{R}^N$.

We will prove the following theorem.

Theorem 1.1.1. *There exists a positive semigroup $(T(t))_{t\geq 0}$ defined in $C_b(\mathbb{R}^N)$ such that, for any $f \in C_b(\mathbb{R}^N)$, $u(x,t) = T(t)f(x) \in C^{2+\alpha,1+\frac{\alpha}{2}}_{loc}(\mathbb{R}^N \times (0,+\infty))$ and satisfies the differential equation*

$$u_t(x,t) = \sum_{i,j=1}^N a_{ij}(x)D_{ij}u(x) + \sum_{i=1}^N F_i(x)D_iu(x) - V(x)u(x).$$

Let us fix a ball B_ρ in $\mathbb{R}^N$ and consider the problem

$$\begin{cases} u_t(x,t) = Au(x,t) & x \in B_\rho,\ \ t>0, \\ u(x,t) = 0 & x \in \partial B_\rho,\ \ t>0 \\ u(x,0) = f(x) & x \in \mathbb{R}^N. \end{cases} \tag{1.2}$$

Since the operator A is uniformly elliptic and the coefficients are bounded in B_ρ, there exists a unique solution u_ρ of the problem (1.2). In other words, the operator $A_\rho = (A, D_\rho(A))$ with

$$D_\rho(A) = \{u \in C_0(B_\rho) \cap W^{2,p}(B_\rho) \text{ for all } p < \infty : Au \in C(\overline{B}_\rho)\}$$

generates an analytic semigroups $(T_\rho(t))_{t\geq 0}$ in the space $C(\overline{B}_\rho)$ and the function $u_\rho(x,t) = T_\rho(t)f(x)$ solves (1.2).
Since the domain $D_\rho(A)$ is not dense in $C(\overline{B}_\rho)$, the semigroup is not strongly continuous at 0 indeed one can prove that $T_\rho(t)f$ converges uniformly to f in $\overline{B}_\rho$ as $t \to 0$ if and only if $f \in C_0(B_\rho)$. However the convergence is uniform in compact sets $\overline{B}_\sigma$ for every $\sigma < \rho$ and hence pointwise in B_ρ. The operators $T_\rho(t)$ are bounded in $L^p(B_\rho)$ for every $1 \leq p < \infty$ and are integral operators indeed, for every $\rho > 0$, there exists a kernel $p_\rho(x,y,t)$ such that

$$T_\rho(t)f(x) = \int_{B_\rho} p_\rho(x,y,t)f(y)\,dy \tag{1.3}$$

for every $f \in C(\overline{B}_\rho)$. The kernel p_ρ is positive and, for every fixed $y \in B_\rho$, $0 < \varepsilon < \tau$, it belongs to $C^{2+\alpha,1+\frac{\alpha}{2}}(B_\rho \times (\varepsilon,\tau))$ as a function of (x,t) and satisfies

$$\partial_t p_\rho = Ap_\rho.$$

It follows that $T_\rho(t)$ are positive and satisfy the estimate $\|T_\rho(t)f\|_\infty \leq \|f\|_\infty$, moreover for every $f \in C(\overline{B}_\rho)$ the function $u_\rho(x,t)$ belongs to $C^{2+\alpha,1+\frac{\alpha}{2}}(B_\rho \times (\varepsilon,\tau))$. Finally, by the integral representation, we can immediately deduce a continuity property of the operator $T_\rho(t)$. If $(f_n) \subset C(\overline{B}_\rho)$, $f \in C(\overline{B}_\rho)$ satisfy $\|f_n\| \leq C$ for every $n \in \mathbb{N}$ and $f_n \to f$ pointwise, then $T_\rho(t)f_n \to T_\rho(t)f$ pointwise.
We refer to [25, Chapter 3] and [17, Chapter 3, Section 7] for a detailed description of the results mentioned above.
Now we would like to let ρ to infinity in order to define the semigroup associated with A in $\mathbb{R}^N$. To this aim we need an easy consequence of the parabolic maximum principle.

Lemma 1.1.2. *Let $0 \leq f \in C_b(\mathbb{R}^N)$ and let $\rho < \rho_1 < \rho_2$. Then for every $t \geq 0$ and $x \in B_\rho$ we have $0 \leq T_{\rho_1}(t)f(x) \leq T_{\rho_2}(t)f(x)$.*

PROOF. First suppose that $f \equiv 0$ on the boundary ∂B_{ρ_1}. Then, since $T_\rho(t)f$ converges uniformly to f in $\overline{B}_{\rho_1}$ as $t \to 0$ if and only if $f \in C_0(B_{\rho_1})$, $w(x,t) = T_{\rho_2}(t)f(x) - T_{\rho_1}(t)f(x)$ is continuous on $\overline{B}_{\rho_1} \times [0,\infty)$, vanishes for $t=0$, is nonnegative for $x \in \partial B_{\rho_1}$ and solves a parabolic equation. By the maximum principle $w(x,t) \geq 0$ in $\overline{B}_{\rho_1} \times [0,\infty)$. In general, if $f \in C_b(\mathbb{R}^N)$, we approximate it in the $L^2(B_{\rho_2})$ norm with continuous functions vanishing on ∂B_{ρ_1}. Using the first part of the proof and the boundedness of $T_{\rho_i}(t)$ in $L^2(B(\rho_i))$, $i = 1$, 2, the claim follows. □

Proof (Theorem 1.1.1). If $f \in C_b(\mathbb{R}^N)$, $x \in \mathbb{R}^N$ we set

$$T(t)f(x) := \lim_{\rho\to\infty} T_\rho(t)f(x).$$

We know that this limit exists if $f \geq 0$ by monotonicity, otherwise we write a general f as $f^+ - f^-$. For the positive and the negative part of f the limit above exists and then, since $T_\rho(t)$ is linear, $T(t)f(x)$ is well defined. $T(t)$ are positive operators and $\|T(t)f\|_\infty \leq \|f\|_\infty$. Let us prove that the operators so defined satisfy the semigroup law. Consider $f \geq 0$. Let $t, s > 0$. Then

$$T(t+s)f(x) = \lim_{\rho\to\infty} T_\rho(t+s)f(x) = \lim_{\rho\to\infty} T_\rho(t)T_\rho(s)f(x) \leq T(t)T(s)f(x).$$

On the other hand, for every $\rho_1 > 0$ we have

$$T(t+s)f(x) = \lim_{\rho\to\infty} T_\rho(t)T_\rho(s)f(x) \geq \lim_{\rho\to\infty} T_{\rho_1(t)}T_\rho(s)f(x) = T_{\rho_1}(t)T(s)f(x)$$

and, letting $\rho_1 \to \infty$, it follows that $T(t+s)f(x) \geq T(t)T(s)f(x)$. Hence the semigroup law is true if the semigroup is applied to a positive function. The general case follows by linearity as above.
Set $u(x,t) = T(t)f(x)$, $u_\rho(x,t) = T(t)f(x)$ for $t \geq 0$ and $x \in \mathbb{R}^N$. Fix positive numbers ε, τ, σ with $0 < \varepsilon < \tau$. By the interior Schauder estimates ([17, Chapter 3, Section 2]) there exists a positive constant C such that for $\rho > \sigma$

$$\|u_\rho\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{B}_\sigma\times[\varepsilon,\tau])} \leq C\|u_\rho\|_\infty \leq C\|f\|_\infty.$$

So by Ascoli's Theorem it follows that u_ρ converges to u uniformly in $\overline{B}_\sigma \times [\varepsilon, \tau]$. Fix now $\sigma_1 < \sigma$, $\varepsilon < \varepsilon_1 < \tau_1 < \tau$ and apply again Schauder estimates. For $\rho_2 > \rho_1 > \sigma > \sigma_1$ we have

$$\|u_{\rho_2} - u_{\rho_1}\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{B}_{\sigma_1}\times[\varepsilon_1,\tau_1])} \leq C\|u_{\rho_2} - u_{\rho_1}\|_{L^\infty(\overline{B}_\sigma\times[\varepsilon,\tau])}.$$

Then $u \in C^{2+\alpha,1+\frac{\alpha}{2}}_{loc}(\mathbb{R}^N \times (0,\infty))$ and, letting $\rho \to \infty$ in the equation satisfied by u_ρ, it follows that $\partial_t u = Au$. □

We have observed that the semigroup $T(t)$ is not strongly continuous in $C_b(\mathbb{R}^N)$. We are interested now in the conditions under which the continuity at $t = 0$ holds.

Proposition 1.1.3. *For every* $f \in C_0(\mathbb{R}^N)$

$$\lim_{t\to 0} T(t)f = f$$

uniformly on $\mathbb{R}^N$.

PROOF. Consider first $f \in C^2(\mathbb{R}^N)$ with support contained in B_σ and let $\rho > \sigma$. Then, for $x \in B_\rho$,

$$T_\rho(t)f(x) - f(x) = \int_0^t T_\rho(s)Af(x)\,ds$$

and, letting $\rho \to \infty$ by dominated convergence,

$$T(t)f(x) - f(x) = \int_0^t T(s)Af(x)\,ds.$$

By the arbitrarity of ρ, the equality above holds for every $x \in \mathbb{R}^N$ and, taking the supremum over $x \in \mathbb{R}^N$,

$$\|T(t)f - f\|_\infty \le t\|Af\|_\infty.$$

This implies that $T(t)f$ converges to f uniformly as $t \to 0$. By density the claim follows. □

Remark 1.1.4. By the previous proposition we cannot deduce that $(T(t))_{t\ge 0}$ restricted to $C_0(\mathbb{R}^N)$ is strongly continuous since no invariance property of $C_0(\mathbb{R}^N)$ under the semigroup is guaranteed.

As we have seen before, $T_\rho(t)$ are integral operators, therefore they can be represented in integral form through a kernel p_ρ. In the next theorem we prove that also $T(t)$ is an integral operator and its kernel enjoies some regularity properties.

Theorem 1.1.5. *The following representation formula for $T(t)$ holds*

$$T(t)f(x) = \int_{\mathbb{R}^N} p(x, y, t)\,dy$$

for $f \in C_b(\mathbb{R}^N)$ and with p positive function such that for almost every $y \in \mathbb{R}^N$ it belongs to $C^{2+\alpha,1+\frac{\alpha}{2}}_{loc}(\mathbb{R}^N \times (0,\infty))$ as a function of (x,t) and solves $\partial_t p = Ap$.

PROOF. Suppose $0 \le f \in C_b(\mathbb{R}^N)$. By Lemma 1.1.2, $T_\rho(t)f$ converges monotonically pointwise to $T(t)f$. Therefore, recalling that

$$T_\rho(t)f(x) = \int_{B_\rho} p_\rho(x, y, t)f(y)\,dy,$$

the kernels p_ρ increase with ρ. Then there exists

$$p(x, y, t) := \lim_{\rho\to\infty} p_\rho(x, y, t)$$

and, by monotone convergence,

$$T(t)f(x) = \lim_{\rho\to\infty} T_\rho(t)f(x) = \lim_{\rho\to\infty} \int_{B_\rho} p_\rho(x, y, t)f(y)\,dy = \int_{\mathbb{R}^N} p(x, y, t)f(y)\,dy.$$

The positivity of p immediately follows by the one of p_ρ. We show now the regularity properties of p.
We have $\int_{B_\rho} p_\rho(x,y,t)\,dy \le 1$ and, letting $\rho \to \infty$, $\int_{\mathbb{R}^N} p(x,y,t)\,dy \le 1$ so that $p(x,y,t)$ is finite for every $t > 0$, every $x \in \mathbb{R}^N$ and almost every $y \in \mathbb{R}^N$. Fix $t_1 > 0$, $\sigma > 0$, $x_0 \in B_\sigma$ and let $y_0 \in \mathbb{R}^N$ such that $p(x_0,y_0,t_1) < \infty$. If $\rho_2 > \rho_1 > \sigma + 1$, the functions $p_{\rho_1}(\cdot,y_0,\cdot)$, $p_{\rho_2}(\cdot,y_0,\cdot)$ are solutions of the equation $\partial_t u = Au$ in $B_{\sigma+1} \times (0,\infty)$ and the difference $p_{\rho_2} - p_{\rho_1}$ is as well. By the parabolic Harnack inequality (see [24, Chapter VII]), for every fixed $0 < \varepsilon < \tau < t_1$

$$\begin{aligned}\sup_{\varepsilon \le t \le \tau,\, x \in \overline{B}_\sigma} [p_{\rho_2}(x,y_0,t) - p_{\rho_1}(x,y_0,t)] &\le C \inf_{\overline{B}_\sigma} [p_{\rho_2}(x,y_0,t_1) - p_{\rho_1}(x,y_0,t_1)] \\ &\le C[p_{\rho_2}(x_0,y_0,t_1) - p_{\rho_1}(x_0,y_0,t_1)].\end{aligned}$$

Since $p(x_0,y_0,t_1) < \infty$, $p_\rho(\cdot,y_0,\cdot)$ is a Cauchy sequence in $C(\overline{B}_\sigma \times [\varepsilon,\tau])$. Then $p_\rho(\cdot,y_0,\cdot)$ converges uniformly to $p(\cdot,y_0,\cdot)$ in $\overline{B}_\sigma \times [\varepsilon,\tau]$. Fix now $\sigma_1 < \sigma$, $\varepsilon < \varepsilon_1 < \tau_1 < \tau$ and apply the Schauder estimates. We have

$$\|p_{\rho_2} - p_{\rho_1}\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{B}_{\sigma_1} \times [\varepsilon_1,\tau_1])} \le C \|p_{\rho_2} - p_{\rho_1}\|_{L^\infty(\overline{B}_\sigma \times [\varepsilon,\tau])}.$$

Then $p \in C^{2+\alpha,1+\frac{\alpha}{2}}_{loc}(\mathbb{R}^N \times (0,\infty))$ and, letting $\rho \to \infty$ in the equation satisfied by p_ρ, it follows that and $\partial_t p = Ap$. ☐

Remark 1.1.6. By using the integral representation formula, we can extend the semigroup to the space of the bounded measurable functions. If $f \in B_b(\mathbb{R}^N)$, with $T(t)f$ we mean the $\int_{\mathbb{R}^N} p(x,y,t) f(y)\,dy$.

We now show the continuity up to $t = 0$ of $u(x,t)$ and so we prove that we have built not only a solution of the parabolic equation but a solution of the Cauchy problem (1.1). Let us fix a notation. For any measurable set $E \subset \mathbb{R}^N$, with $p(x,E,t)$ we denote the $\int_E p(x,y,t)\,dy$.

Theorem 1.1.7. *Let $f \in C_b(\mathbb{R}^N)$. Then $T(t)f$ converges to f as $t \to 0$ uniformly on compact subsets of $\mathbb{R}^N$.*

Proof. Let $\rho > 0$ and f_1, $f_2 \in C_0(\mathbb{R}^N)$ such that $0 \le \chi_{B_\rho} \le f_1 \le \chi_{B_{2\rho}} \le f_2 \le 1$. By the positivity of $T(t)$,

$$T(t)f_1(x) \le p(x,B_{2\rho},t) \le T(t)f_2(x)$$

for all $x \in \mathbb{R}^N$. By Proposition 1.1.3, $T(t)f_1 \to f_1$, $T(t)f_2 \to f_2$ uniformly on $\overline{B}_\rho$ as $t \to 0$. We observe that $f_1 = f_2 \equiv 1$ on $\overline{B}_\rho$. It follows that $p(x,B_{2\rho},t) \to 1$ on $\overline{B}_\rho$ as $t \to 0$. Then

$$0 \le p(x,\mathbb{R}^N \setminus B_{2\rho},t) = p(x,\mathbb{R}^N,t) - p(x,B_{2\rho},t) \le 1 - p(x,B_{2\rho},t) \to 0 \quad (1.4)$$

as $t \to 0$ uniformly on $\overline{B}_\rho$.
Let now $f \in C_b(\mathbb{R}^N)$ and $\eta \in C_0(\mathbb{R}^N)$ such that $0 \le \eta \le 1$, $\eta = 1$ on $B_{2\rho}$, *supp* $(\eta) \in B_{3\rho}$. Then

$$T(t)f - f = T(t)f - T(t)(\eta f) + T(t)(\eta f) - \eta f$$

on B_ρ. By Proposition 1.1.3, $\|T(t)(\eta f) - \eta f\|_\infty \to 0$ as $t \to 0$. Concerning the remaining terms, by (1.4) we have

$$\begin{aligned} |T(t)f(x) - T(t)(\eta f)(x)| &= T(t)((1-\eta)f)(x) \\ &= \int_{\mathbb{R}^N} p(x,y,t)((1-\eta(y))f(y))\,dy \\ &\leq p(x, \mathbb{R}^N \setminus B_{2\rho}, t)\|f\|_\infty \to 0 \end{aligned}$$

uniformly on $\overline{B}_\rho$. We conclude therefore that $T(t)f \to f$ uniformly on $\overline{B}_\rho$ and by the arbitrarity of ρ the claim follows. ☐

Remark 1.1.8. We observe that, in general, the problem (1.1) is not uniquely solvable in $C_b(\mathbb{R}^N \times [0,+\infty)) \cap C^{2+\alpha, 1+\frac{\alpha}{2}}((0,+\infty) \times \mathbb{R}^N)$. Anyway we can say that the solution found above is the minimal among all the positive solutions of the given problem with positive initial datum. Infact, if $f \geq 0$ and v is another positive solution, then the maximum principle yields $v(x,t) \geq u_\rho(x,t)$ for all $t > 0$, $x \in B_\rho$, u_ρ defined as before and, letting $\rho \to \infty$, $v \geq u$.

Now we prove some interesting continuity properties of the operators $T(t)$.

Proposition 1.1.9. *Let (g_n) be a bounded sequence in $C_b(\mathbb{R}^N)$, $g \in C_b(\mathbb{R}^N)$ and suppose that $g_n(x) \to g(x)$ for every $x \in \mathbb{R}^N$. Then, for every $0 < \varepsilon < \tau$ and $\sigma > 0$, $T(t)g_n(x) \to T(t)g(x)$ uniformly for $(x,t) \in \overline{B}_\sigma \times [\varepsilon, \tau]$. If $g_n \to g$ uniformly on compact sets, then $T(t)g_n(x) \to T(t)g(x)$ uniformly for $(x,t) \in \overline{B}_\sigma \times [0, \tau]$.*

Proof. Using the integral representation and the Lebesgue dominated convergence Theorem, we immediately deduce that $T(t)g_n(x) \to T(t)g(x)$ pointwise in $\mathbb{R}^N$. Let $K > 0$ such that $\|g_n\|_\infty \leq K$ for every $n \in \mathbb{N}$. Then $\|T(t)g_n\|_\infty \leq K$ for every $n \in \mathbb{N}$ and, by the Schauder estimates, for every $0 < \varepsilon < \tau$ and $\sigma > 0$ there exists $C > 0$ such that

$$\sup_n \|T(\cdot)g_n(\cdot)\|_{C^1(\overline{B}_\sigma \times [\varepsilon,\tau])} \leq C.$$

By Ascoli's Theorem we deduce that the convergence is uniform in $\overline{B}_\sigma \times [\varepsilon, \tau]$. Let us prove the second statement. Without loss of generality we can suppose $g = 0$ (otherwise we consider $g_n - g$) and $\|g_n\|_\infty \leq 1$. Let σ, $\varepsilon > 0$ and, for every $\rho > 1$, consider $0 \leq f_\rho \in C_0(\mathbb{R}^N)$ such that $\chi_{B_{\rho-1}} \leq f_\rho \leq \chi_{B_\rho}$. Set

$$E = \{s \geq 0 : \exists\, \rho > 0 \,\text{such that} \inf_{|x|\leq\sigma,\ 0\leq t\leq s} T(t)(f_\rho(x) - \mathbf{1}) \geq -\varepsilon\}.$$

Obviously $0 \in E$. Now we prove that E is open and closed together and so we conclude that it coincides with the positive real axis. Let $s \in \overline{E}$, then there exists $(s_n) \subset E$, $s_n \to s$ for $n \to \infty$. Suppose that there exists $r \in \mathbb{N}$ such that $s_r \geq s$ and let ρ_r be such that

$$\inf_{|x|\leq\sigma,\ 0\leq t\leq s_r} T(t)(f_{\rho_r} - \mathbf{1})(x) \geq -\varepsilon.$$

Then

$$\inf_{|x|\le\sigma,\ 0\le t\le s} T(t)(f_{\rho_r}-\mathbf{1})(x) \ge \inf_{|x|\le\sigma,\ 0\le t\le s_r} T(t)(f_{\rho_r}-\mathbf{1})(x) \ge -\varepsilon$$

and $s \in E$. Otherwise $s_n < s$ for every $n \in \mathbb{N}$. Since $s_1 \in E$, there exists $\rho_1 > 0$ such that

$$\inf_{|x|\le\sigma,\ 0\le t\le s_1} T(t)(f_{\rho_1}-\mathbf{1})(x) \ge -\varepsilon.$$

Recalling that $\{f_\rho\}$ is increasing, it turns out that the previous inequality is satisfied for every $\rho \ge \rho_1$. By the first part of the proof, we know that $T(\cdot)f_\rho \to T(\cdot)\mathbf{1}$ as $\rho \to \infty$ uniformly in $\overline{B}_\sigma \times [s_1, s]$. Therefore there exists $\rho_0 > 0$ such that

$$T(t)f_\rho(x) \ge T(t)\mathbf{1} - \varepsilon, \quad t \in [s_1, s],\ x \in \overline{B}_\sigma,\ \rho \ge \rho_0.$$

If we choose $\overline{\rho} = \max\{\rho_0,\ \rho_1\}$, then

$$T(t)f_{\overline{\rho}}(x) \ge T(t)\mathbf{1} - \varepsilon, \quad t \in [0, s],\ x \in \overline{B}_\sigma.$$

It follows that $s \in E$.

Now we prove that E is open. Let $s \in E$ and ρ as in the definition of E. Since $T(s)f_\rho \to T(s)\mathbf{1}$ as $\rho \to \infty$ uniformly in compact sets, there exists $\rho_0 > 0$ such that $T(s)f_\rho(x) \ge T(s)\mathbf{1} - \dfrac{\varepsilon}{2}$ for every $x \in \overline{B}_\sigma$, $\rho > \rho_0$. By Theorem 1.1.7, $T(s+\delta)f_\rho(x) \ge T(s)\mathbf{1} - \varepsilon$ for every $x \in \overline{B}_\sigma$ and δ sufficiently small. This shows that E is open. We conclude that $E = [0, \infty)$. In particular, if $\tau > 0$ is fixed, we can find $\rho > 0$ such that $p(x, B_\rho, t) \ge T(t)f_\rho(x) \ge T(t)\mathbf{1} - \varepsilon$ for every $x \in \overline{B}_\sigma$ and $t \in [0, \tau]$. Then we have

$$|T(t)g_n(x)| \le \int_{B_\rho} p(x,y,t)|g_n(y)|\,dy + \int_{\mathbb{R}^N\setminus B_\rho} p(x,y,t)\,dy \le \sup_{y\in B_\rho} |g_n(y)| + \varepsilon$$

for every $x \in \overline{B}_\sigma$ and $t \in [0, \tau]$. ☐

As consequence of the continuity result just proved, we deduce that $(T(t))_{t\ge0}$ is irreducible and satisfies the strong Feller property. We preliminary define these two properties.

Definition 1.1.10. *A semigroup* $((T(t))_{t\ge0}$ *in* $B_b(\mathbb{R}^N)$ *is irreducible if for any nonempty open set* $U \subset \mathbb{R}^N$, $T(t)\chi_U(x) > 0$ *for every* $t > 0$ *and* $x \in \mathbb{R}^N$.

Definition 1.1.11. *We say that* $(T(t))_{t\ge0}$ *satisfies the strong Feller property if* $T(t)f \in C_b(\mathbb{R}^N)$ *for any bounded Borel function* f.

Proposition 1.1.12. *The semigroup* $(T(t))_{t\ge0}$ *is irreducible and has the strong Feller property.*

PROOF. The irreducibility immediately follows since the integral kernel p is positive. Let f be a bounded Borel Function and let $(f_n) \in C_b(\mathbb{R}^N)$a bounded sequence such that $f_n(x) \to f(x)$ for almost every $x \in \mathbb{R}^N$. By dominated convergence, $T(t)f_n \to T(t)f$ pointwise in $\mathbb{R}^N$. Using the interior Schauder estimates, as in Proposition 1.1.9, we deduce that $T(t)f_n \to T(t)f$ uniformly on compact sets and then the limit $T(t)f \in C_b(\mathbb{R}^N)$. ☐

1.2 The weak generator of $T(t)$

In the previous section we have built a semigroup associated to the given elliptic operator with unbounded coefficients and we have observed that in general it is not strongly continuous in $C_b(\mathbb{R}^N)$, hence we cannot define it's generator in the usual sense. However, as we will see later, it is possible to define a generator in a weak sense.

In this section we state only some results useful in the following chapters, in particular we are interested in the conditions under which the domain of the weak generator coincides with the maximal one. For example this equality will be guaranted under the existence of suitable Lyapunov functions for the operator A.

First we enunciate an existence result for the solution of the elliptic equation associated with A.

Theorem 1.2.1. *For any $\lambda > 0$, $f \in C_b(\mathbb{R}^N)$, there exists $u \in D_{max}(A)$ such that*

$$\lambda u(x) - Au(x) = f(x), \qquad x \in \mathbb{R}^N.$$

Moreover the following estimate holds

$$\|u\|_\infty \leq \frac{1}{\lambda}\|f\|_\infty.$$

Finally, if $f \geq 0$, then $u \geq 0$.

We only sketch the proof. As in the parabolic case, the solution is obtained as limit of solutions of the analogous of the equation above for A_ρ, realization of the operator A with homogeneous Dirichlet boundary conditions in balls of $\mathbb{R}^N$ of radius ρ.

Set $A_\rho = (A, D_\rho(A))$ where

$$D_\rho(A) = \{u \in C_0(B_\rho) \cap W^{2,p}(B_\rho) \text{ for all } p < \infty : Au \in C(\overline{B}_\rho)\}$$

and $u_\rho = R(\lambda, A_\rho)f$. For any $\lambda > 0$ there exists a linear operator $R(\lambda)$ in $C_b(\mathbb{R}^N)$ such that for any $f \in C_b(\mathbb{R}^N)$ the solution is given by

$$u(x) = (R(\lambda)f)(x) = \lim_{\rho \to \infty} R(\lambda, A_\rho)f(x), \qquad x \in \mathbb{R}^N.$$

The family of operators $\{R(\lambda) : \lambda > 0\}$ satisfies the estimate

$$\|R(\lambda)f\|_\infty \leq \frac{1}{\lambda}\|f\|_\infty, \quad f \in C_b(\mathbb{R}^N),$$

moreover it is possible to prove that the operators $R(\lambda)$ are injective and satisfy the resolvent identity

$$R(\lambda)f - R(\mu)f = (\mu - \lambda)R(\mu)R(\lambda)f, \quad 0 < \lambda < \mu.$$

We refer to [4, Theorem 2.1.1, Theorem 2.1.3] or [29, Theorem 3.4] for a detailed proof of the last results. Then we can define the weak generator as the unique closed operator $(\hat{A}, \hat{D})$ such that $(0, +\infty) \subset \rho(\hat{A})$, $ImR(\lambda) = \hat{D}$ and $R(\lambda) = R(\lambda, \hat{A})$ for all $\lambda > 0$ (see [16, Chapter III, Proposition 4.6]). In some cases the following equivalent direct description of the weak generator can be more useful.

$$\begin{aligned} D(A_1) \quad = \quad & \Big\{ f \in C_b(\mathbb{R}^N) : \sup_{t \in (0,1)} \frac{\|T(t)f - f\|_\infty}{t} < \infty \text{ and } \exists\ g \in C_b(\mathbb{R}^N) : \\ & \lim_{t \to 0^+} \frac{(T(t)f)(x) - f(x)}{t} = g(x)\ \forall\ x \in \mathbb{R}^N \Big\} \end{aligned}$$

and, for all $f \in D(A_1)$,

$$(A_1 f)(x) = \lim_{t \to 0^+} \frac{(T(t)f)(x) - f(x)}{t}, \quad x \in \mathbb{R}^N, \quad f \in D(A_1).$$

One can prove that $(\hat{A}, \hat{D}) = (A_1, D(A_1))$ (see for example [4, Proposition 2.3.1]). The weak generator enjoies similar properties to those of the infinitesimal generator. For example the following result remains true.

Proposition 1.2.2. *For any* $f \in \hat{D}$, $T(t)f \in \hat{D}$ *and for any fixed* $x \in \mathbb{R}^N$ *the function* $(T(\cdot)f)(x)$ *is continuously differentiable in* $[0, +\infty)$ *with*

$$\frac{d}{dt}(T(t)f)(x) = (\hat{A}T(t)f)(x) = (T(t)\hat{A}f)(x), \quad t \geq 0. \tag{1.5}$$

(See [4, Proposition 2.3.5]) for the proof.)

Next propositions show the connections between $D_{max}(A)$ and $\hat{D}$. We recall that our goal is to find some conditions under which the maximal domain and the domain of the weak generator coincide.

Proposition 1.2.3. *The following statements hold.*

(i) $\hat{D} \subset D_{max}(A)$ *and* $\hat{A}u = Au$ *for* $u \in \hat{D}$. *The equality* $\hat{D} = D_{max}(A)$ *holds if and only if* $\lambda - A$ *is injective on* $D_{max}(A)$ *for some positive* λ.

(ii) Set $D(A) = D_{max}(A) \cap C_0(\mathbb{R}^N)$, *we have the inclusion* $D(A) \subset \hat{D}$.

Proof. (i) The inclusion $\hat{D} \subset D_{max}(A)$ and the equality $\hat{A}u = Au$ for $u \in \hat{D}$ follow from the definition of $\hat{D}$ and Theorem 1.2.1. Concerning the second statement, obviously $\lambda - A$ is bijective from $\hat{D}$ onto $C_b(\mathbb{R}^N)$. If it is also injective on $D_{max}(A)$, then $\hat{D} = D_{max}(A)$.
(ii) Let $v \in D(A)$, $f = v - Av$ and $u = R(1, A)f$. If $u_\rho = R(1, A_\rho)f$, then $(u_\rho - v) - A(u_\rho - v) = 0$ in B_ρ and hence, by the maximum principle, $|u_\rho(x) - v(x)| \leq \sup_{|x|=\rho} |v(x)|$ for $|x| \leq \rho$. Letting $\rho \to \infty$ we obtain $u = v$ and hence $v \in \hat{D}$. □

Definition 1.2.4. *We say that* W *is a Lyapunov function for* A *if* $W \in C^2(\mathbb{R}^N)$, $W \geq 0$, W *goes to infinity as* $|x| \to \infty$ *and* $\lambda W - AW \geq 0$ *for some positive* λ.

Theorem 1.2.5. *Suppose that there exists a Lyapunov function* W *for* A*. Let* $\lambda > 0$*. If* $u \in D_{max}(A)$ *satisfies* $\lambda u - Au \leq 0$ (≥ 0)*, then* $u \leq 0$ $(u \geq 0)$*. In particular the operator* $\lambda - A$ *is injective and then* $\hat{D} = D_{max}(A)$*.*

We need the following maximum principle for solutions of elliptic equations. For the proof we refer to [25, Theorem 3.1.10].

Lemma 1.2.6. *Let* $u \in W^{2,p}_{loc}(\mathbb{R}^N)$ *for any* $p < \infty$ *and suppose that* $Au \in C(\mathbb{R}^N)$*. If* u *has a relative maximum (minimum) at the point* x_0 *then* $Au(x_0) + V(x_0)u(x_0) \leq 0$ *(*$Au(x_0) + V(x_0)u(x_0) \geq 0$*).*

PROOF (Theorem 1.2.5). For every $\varepsilon > 0$ set $u_\varepsilon = u - \varepsilon W$. Obviously $\lambda u_\varepsilon - Au_\varepsilon \leq 0$ in $\mathbb{R}^N$ and $\lim_{|x|\to\infty} u_\varepsilon(x) = -\infty$. Let $(x_n) \subset \mathbb{R}^N$ be such that $\sup_{x\in\mathbb{R}^N} u_\varepsilon(x) = \lim_{n\to\infty} u_\varepsilon(x_n)$. Then (x_n) is bounded and, without restriction, we may assume that $\lim_{n\to\infty} x_n = x_0$. By Lemma 1.2.6, $Au_\varepsilon(x_0) \leq -V(x_0)u_\varepsilon(x_0)$, then

$$\lambda u_\varepsilon(x_0) \leq Au_\varepsilon(x_0) \leq -V(x_0)u_\varepsilon(x_0)$$

and hence

$$(\lambda + V(x_0))u_\varepsilon(x_0) \leq 0.$$

Since V is a positive potential, it follows $u_\varepsilon(x_0) \leq 0$ and then

$$u_\varepsilon \leq \max_{x\in\mathbb{R}^N} u_\varepsilon(x) = u_\varepsilon(x_0) \leq 0.$$

Letting $\varepsilon \to 0$, we obtain $u \leq 0$. ☐

1.3 Schrödinger operators via form method

In this section we sketch the construction of the semigroup associated with the Schrödinger operator $A = \Delta - V$ by means of the method of the quadratic forms. Moreover we will see how it is possible to represent this semigroup in integral form through a kernel. All over the section we only require V positive potential in $L^1_{loc}(\mathbb{R}^N)$.

1.3.1 From forms to semigroups

Let W a Hilbert space over the field $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$. A sesquilinear from $a : W \times W \to \mathbb{K}$ is a mapping satisfying

$$\begin{aligned} &a(u + v, w) = a(u, w) + a(v, w)\\ &a(\lambda u, w) = \lambda a(u, w)\\ &a(u, v + w) = a(u, v) + a(u, w)\\ &a(u, \lambda v) = \overline{\lambda} a(u, v) \end{aligned}$$

for $u,\ v,\ w \in W,\ \lambda \in \mathbb{K}$. In other words, a is linear in the first and antilinear in the second variable. If $\mathbb{K} = \mathbb{R}$, then we say that a is bilinear.

Definition 1.3.1. *The form a is called continuous if there exists $M \geq 0$ such that*

$$|a(u,v)| \leq M\|u\|_W\|v\|_W \qquad u,\, v \in W.$$

The form is called coercive if there exists $\alpha > 0$ such that

$$Re\, a(u,u) \geq \alpha\|u\|_W^2, \qquad u \in W.$$

The form a is called symmetric if

$$a(u,v) = \overline{a(v,u)} \qquad \forall\, u, v \in W.$$

Assume from now on that the Hilbert space W is continuously and densely embedded into another Hilbert space H and consider the *operator A associated with the form* on H so defined

$$\begin{aligned} &D(A) = \{u \in W : \exists\, f \in H \quad \text{such that} \quad a(u,v) = (f|v)_H \quad \text{for all} \quad v \in W\} \\ &Au = f. \end{aligned}$$

Observe that f is uniquely determined by u since W is dense in H. The following theorem allows us to construct a semigroup associated with the form. For its proof we refer to [49].

Theorem 1.3.2. *Assume that $a : W \times W \to \mathbb{K}$ is a continuous, coercive form where $W \hookrightarrow H$ densely. Then the operator $-A$ above defined generates a strongly continuous holomorphic semigroup on H.*

Unless we make a rescaling, we can prove that an assumption weaker than the coercivity is sufficient to get a generation result.

Definition 1.3.3. *Let $W,\ H$ be Hilbert spaces over $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ such that $W \hookrightarrow H$. Let $a : W \times W \to \mathbb{K}$ a sesquilinear form. We call a elliptic (or more precisely H-elliptic) if*

$$Re\, a(u,u) + \omega\|u\|_H^2 \geq \alpha\|u\|_W^2$$

for some $\omega \in \mathbb{R}$, $\alpha > 0$ and for all $u \in W$.

The last definition is equivalent to saying that the form $a_\omega : W \times W \to K$ defined by

$$a_\omega(u,v) := a(u,v) + \omega(u|v)_H \qquad u, v \in W$$

is coercive.

Remark 1.3.4. If A is the operator associated with the form a, then $A + \omega$ is the operator associated with the form a_ω. It follows that if $W \hookrightarrow H$ densely and $a : W \times W \to \mathbb{K}$ is a continuous, elliptic form with ellipticity constant ω, then the operator $-(A+\omega)$ generates a holomorphic strongly continuous semigroup T_ω. Consequently $-A$ generates the semigroup T given by $T(t) = e^{\omega t}T_\omega(t)$. So the assumption of coercivity on a in Theorem 1.3.2 can be replaced by the ellipticity.

It is possible to prove the following density result on the domain.

Proposition 1.3.5. *The domain $D(A)$ of A is dense in W.*

We are ready to prove a generation result for Schrödinger operators.

Example 1.3.6. Let $\mathbb{K}=\mathbb{R}$, $H=L^2(\mathbb{R}^N)$, $0\le V\in L^1_{loc}(\mathbb{R}^N)$,

$$\begin{aligned}
a_1(u,v) &= \int_{\mathbb{R}^N}\nabla u\nabla v\,dx, \qquad u,v\in W_1:=W^{1,2}(\mathbb{R}^N),\\
a_2(u,v) &= \int_{\mathbb{R}^N} Vuv\,dx, \qquad u,v\in W_2:=L^2(\mathbb{R}^N,(1+V(x))dx)
\end{aligned}$$

and consider the form sum

$$a(u,v)=\int_{\mathbb{R}^N}\nabla u\nabla v\,dx+\int_{\mathbb{R}^N}Vuv\,dx$$

defined on $W=W_1\cap W_2$ with the scalar product

$$(u|v)_W:=(u|v)_{W_1}+(u|v)_{W_2}.$$

First, let us observe that W is complete indeed $\|u\|_W^2=\|u\|_{W_1}^2+\|u\|_{W_2}^2$ and it is dense in $L^2(\mathbb{R}^N)$. Moreover a is a symmetric, continuous, elliptic form on $L^2(\mathbb{R}^N)$ infact

$$\begin{aligned}
a(u,v) &= \int_{\mathbb{R}^N}\nabla u\nabla v+\int_{\mathbb{R}^N}Vuv=\int_{\mathbb{R}^N}\nabla v\nabla u+\int_{\mathbb{R}^N}Vvu=a(v,u);\\
|a(u,v)| &\le M(\|\nabla u\|_{L^2(\mathbb{R}^N)}\|\nabla v\|_{L^2(\mathbb{R}^N)}+\|V^{\frac12}u\|_{L^2(\mathbb{R}^N)}\|V^{\frac12}v\|_{L^2(\mathbb{R}^N)})\\
&\le M(\|u\|_{W_1}\|v\|_{W_1}+\|u\|_{W_2}\|v\|_{W_2})\le M\|u\|_W\|v\|_W;\\
a(u,u)+2\|u\|^2_{L^2(\mathbb{R}^N)} &= \int_{\mathbb{R}^N}|\nabla u|^2+\int_{\mathbb{R}^N}|u|^2+\int_{\mathbb{R}^N}(V+1)u^2\\
&= \|u\|_{W_1}^2+\|u\|_{W_2}^2
\end{aligned}$$

By Remark 1.3.4, we deduce that the operator $-A$ associated with a given by

$$\begin{aligned}
&D(A)=\{u\in W^{1,2}(\mathbb{R}^N)\cap L^2(\mathbb{R}^N,(1+V(x))dx):\ -\Delta\,u+Vu\in L^2(\mathbb{R}^N)\},\\
&Au=-\Delta u+Vu
\end{aligned}$$

(where, for $u\in L^2(\mathbb{R}^N)$, $-\Delta u+Vu\in L^2(\mathbb{R}^N)$ is considered in the distributional sense) generates a strongly continuous holomorphic semigroup.

We can immediately prove the positivity of the semigroup generated by the Schrödinger operator.

Proposition 1.3.7. *Let $V\ge 0,\in L^1_{loc}(\mathbb{R}^N)$ a positive potential, then the semigroup $(T(t))_{t\ge0}$ generated by $-A=\Delta-V$ is positive.*

Proof. Let $f \in L^2(\mathbb{R}^N),\ f \leq 0,\ \lambda > 0$, set $u = (\lambda + A)^{-1} f \in W^{1,2}(\mathbb{R}^N)$ (The invertibility of $\lambda + A$ is guaranteed by the Lax- Milgram Theorem). Then

$$\lambda u - \Delta u + Vu = f.$$

If we multiply both sides of the previous equality by u^+ and integrate by parts over $\mathbb{R}^N$, we obtain

$$\lambda \int_{\mathbb{R}^N} (u^+)^2 + \int_{\mathbb{R}^N} (\nabla u^+)^2 + \int_{\mathbb{R}^N} V(u^+)^2 = \int_{\mathbb{R}^N} f u^+ \leq 0.$$

This implies $u^+ \equiv 0$ and so $u \leq 0$. Recalling now that

$$T(t)f = \lim_{n \to \infty} \left(I + \frac{t}{n} A \right)^{-n} f$$

(see [16, Corollary 5.5]), we have the claim. □

From the proposition above it immediately follows that a comparison principle holds for semigroups generated by Schrödinger operators.

Corollary 1.3.8. *Let $(T_1(t))_{t \geq 0}$, $(T_2(t))_{t \geq 0}$ be respectively the semigroups generated by the operators $-A_1 = \Delta - V_1$ and $-A_2 = \Delta - V_2$. If $V_1 \leq V_2$, then for every $0 \leq f \in L^2(\mathbb{R}^N)$ and for all $t \geq 0$, $T_1(t)f \geq T_2(t)f$.*

Proof. Let $\lambda > 0$, $0 \leq f \in L^2(\mathbb{R}^N)$ and set $u_1 = (\lambda + A_1)^{-1} f$, $u_2 = (\lambda + A_2)^{-1} f$. As in the proof of the Proposition 1.3.7, in virtue of the approximation formula of the semigroup via the resolvent, it is sufficient to prove that $u_1 \geq u_2$. The functions u_1, u_2 satisfy

$$\lambda u_1 - \Delta u_1 + V_1 u_1 = f$$

and

$$\lambda u_2 - \Delta u_2 + V_2 u_2 = f.$$

Therefore the difference satisfies

$$\lambda(u_1 - u_2) - \Delta(u_1 - u_2) + V_1(u_1 - u_2) = (V_2 - V_1)u_2.$$

Since $f \geq 0$, by Proposition 1.3.7, $u_2 \geq 0$ and then, by the assumption, $(V_2 - V_1)u_2 \geq 0$. By Proposition 1.3.7 again it follows $u_1 \geq u_2$. □

1.3.2 Contractivity properties

In light of the construction of the semigroup via forms method, some nice properties for $(T(t))_{t \geq 0}$ can be deduced by keeping suitable assumptions on a. We establish a contractivity result.
We need the following preliminary proposition.

Proposition 1.3.9. *Let B be the generator of a strongly continuous semigroups $(T(t))_{t \geq 0}$ on H. Then $\|T(t)\| \leq 1$ for all $t \geq 0$ if and only if B is dissipative.*

Proof. Assume that B is dissipative, i.e.

$$Re\,(Bu,u) \le 0 \qquad u \in D(B).$$

Let $u \in D(B)$. Then

$$\begin{aligned}\frac{d}{dt}\|T(t)u\|_H^2 &= \frac{d}{dt}(T(t)u|T(t)u)_H = (BT(t)u|T(t)u)_H + (T(t)u|BT(t)u)_H \\ &= 2Re(BT(t)u|T(t)u)_H \le 0.\end{aligned}$$

It follows that $\|T(\cdot)u\|_H^2$ is decreasing. In particular $\|T(t)u\|_H \le \|u\|_H$ for all $t \ge 0$, $u \in D(B)$. Since $D(B)$ is dense in H, the claim follows.
Conversely, assume that T is contractive. Let $u \in D(B)$. Then

$$\|T(t+s)u\|_H = \|T(t)T(s)u\|_H \le \|T(s)u\|_H \quad t,\, s \ge 0.$$

We deduce that $\|T(\cdot)u\|_H^2$ is decreasing and then

$$Re(Bu|u)_H = \frac{1}{2}\frac{d}{dt}_{|t=0}\|T(t)u\|_H^2 \le 0.$$

□

Definition 1.3.10. *We say that the sesquilinear form a is accretive if*

$$Re\,a(u,u) \ge 0 \quad u \in W.$$

Proposition 1.3.11. *Let $(T(t))_{t\ge0}$ the semigroup on H associated with the form a. Then $(T(t))_{t\ge0}$ is contractive if and only if a is accretive.*

Proof. Suppose a accretive. Then $Re(Au,u) = a(u,u) \ge 0$ for all $u \in D(A)$. Thus $-A$ is dissipative and the semigroup is contractive by Proposition 1.3.9. Viceversa, suppose that the semigroup is contractive, then, by Proposition 1.3.9 again, $-A$ is dissipative, hence

$$Re\,a(u,u) = Re(Au|u)_H \ge 0 \quad u \in D(A).$$

Since $D(A)$ is dense in W (see Proposition 1.3.5), $Re\,a(u,u) \ge 0$ for all $u \in W$. □

Example 1.3.12. The form associated with the Schrödinger operator defined in Example 1.3.6 is accretive infact for all $u \in W$

$$a(u,u) = \int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} Vu^2 \ge 0.$$

Therefore the semigroup generated by $\Delta - V$ is contractive on $L^2(\mathbb{R}^N)$.

1.3.3 Symmetric forms

Our next goal is to prove that symmetric forms are associated with symmetric operators and symmetric semigroups.
Let H be a Hilbert space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let A be a densely defined operator on H with domain $D(A)$. Then the adjoint A^* of A is defined by

$$\begin{aligned} &D(A^*) := \{u \in H : \exists\, f \in H \text{ s.t. } (Av|u)_H = (v|f)_H \ \forall\, u \in D(A)\},\\ &A^*u := f. \end{aligned}$$

Since $D(A)$ is dense in H, the element f is uniquely determined by u. It is easy to prove the following preliminary proposition whose proof is omitted.

Proposition 1.3.13. *Assume that* $\lambda \in \rho(A) \cap \mathbb{R}$.
Then $\lambda \in \rho(A^*)$ *and* $R(\lambda, A)^* = R(\lambda, A^*)$. *Moreover the following are equivalent*

- *(a)* $A = A^*$*;*
- *(b)* A *is symmetric;*
- *(c)* $R(\lambda, A)^* = R(\lambda, A)$.

If (a) holds, then we say that A is selfadjoint.

Let now a be a continuous, elliptic, sesquilinear form defined as before on a dense Hilbert space W continuously embedded in H and let A, $(T(t))_{t\geq 0}$ be the associated operator and semigroup respectively. Since $-A$ is the generator of a holomorphic semigroup, $\rho(A) \cap \mathbb{R}$ is nonempty and we can apply Proposition 1.3.13. Denote by $a^* : W \times W \to \mathbb{K}$ the adjoint form of a given by

$$a^*(u, v) := \overline{a(v, u)} \quad u, v \in W.$$

It is natural to investigate about the relations between a^* and the adjoint operator A^*. The following result can be found in [49, Lemma 2.2.3].

Proposition 1.3.14. *The adjoint* A^* *of* A *coincides with the operator on* H *associated with* a^*.

By Proposition 1.3.13 and the Post Widder inversion formula the following proposition immediately follows.

Proposition 1.3.15. *The adjoint operator* $-A^*$ *generates the adjoint semigroup* $(T(t)^*)_{t\geq 0}$ *of* $(T(t))_{t\geq 0}$.

Proof. It is sufficient to recall that for every strongly continuous semigroup $(T(t))_{t\geq 0}$ on H with generator $(A, D(A))$ one has

$$T(t)u = \lim_{n\to\infty} \left(I - \frac{t}{n}A\right)^{-n} u \qquad \forall\, u \in H.$$

See [16, Corollary 5.5] for the last formula. □

Remark 1.3.16. In particular we obtained that if $a = a^*$, then $A = A^*$ and $T(t) = T(t)^*$ for every $t \geq 0$. In the case of the Schrödinger operator, we have therefore that it generates a symmetric semigroup.

1.3.4 Ultracontractivity

We finally prove, by using the Berling-Deny conditions and some extrapolation theorems, that the semigroup generated by $\Delta - V$ is ultracontractive and so, by the Dunford-Pettis Theorem, it admits an integral kernel. We state the key ultracontractivity result keeping in mind the application to Schrödinger operators, however it remain true in a slightly more general setting.
Let $H = L^2(\mathbb{R}^N)$, W be a Hilbert space such that $W \hookrightarrow L^2(\mathbb{R}^N)$ is dense. We assume that $u \in W$ implies $u \wedge 1 \in W$. Furthermore we assume that $N \geq 2$ and $W \hookrightarrow L^q(\mathbb{R}^N)$ where $\frac{1}{q} = \frac{1}{2} - \frac{1}{N}$.

Theorem 1.3.17. *Let $a : W \times W \to \mathbb{R}$ be a bilinear, continuous, symmetric form such that for some $\mu > 0$*

$$a(u,u) \geq \mu \|u\|_W^2$$

and $a(u \wedge 1, (u-1)^+) \geq 0$ for all $u \in W$. Denote by T the semigroup associated with a on $L^2(\mathbb{R}^N)$. Then there exists a constant $c > 0$ which depends on W such that

$$\|T(t)\|_{\mathcal{L}(L^1, L^\infty)} \leq c\mu^{-\frac{N}{2}} t^{-\frac{N}{2}} \qquad t > 0.$$

Proof. Since W is continuously embedded in L^q, there exists a positive constant c such that

$$\|u\|_{L^q} \leq c\|u\|_W \qquad \forall\, u \in W.$$

Observe that, by the Berling Deny conditions and since a is symmetric and so A selfadjoint, $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is invariant under the semigroup and $(T(t))_{t\geq 0} = (T(t)^*)_{t \geq 0}$ defined on $L^2(\mathbb{R}^N)$ extends to a positive contraction semigroup $T_p(t)$ on $L^p(\mathbb{R}^N)$ for all $1 \leq p \leq \infty$ (see [13, Theorem 1.4.1]). In particular we have $\|T(t)\|_{\mathcal{L}(L^q)} \leq 1$, hence $\|T(\cdot)f\|_{L^q}$ is decreasing for all $f \in L^q(\mathbb{R}^N)$. Consequently, for $f \in W$, we have

$$\begin{aligned}
t\|T(t)f\|_{L^q}^2 &= \int_0^t \|T(t)f\|_{L^q}^2 ds \leq \int_0^t \|T(s)f\|_{L^q}^2 ds \leq c^2 \int_0^t \|T(s)f\|_W^2 ds \\
&\leq \frac{c^2}{\mu} \int_0^t a(T(s)f, T(s)f) ds = \frac{c^2}{\mu} \int_0^t (AT(s)f | T(s)f)_{L^2} ds \\
&= -\frac{c^2}{2\mu} \int_0^t \frac{d}{ds} \|T(s)f\|_{L^2}^2 = \frac{c^2}{2\mu} (\|f\|_{L^2}^2 - \|T(t)f\|_{L^2}^2) \\
&\leq \frac{c^2}{2\mu} \|f\|_{L^2}^2.
\end{aligned}$$

So we obtained that

$$\|T(t)f\|_{L^q} \leq \frac{c}{\sqrt{2\mu}} t^{-\frac{1}{2}} \|f\|_{L^2}.$$

By [12, Lemma II.1] it follows that

$$\|T(t)\|_{\mathcal{L}(L^1, L^\infty)} \leq C\mu^{-\frac{N}{2}} t^{-\frac{N}{2}} \qquad \forall\, t > 0.$$

□

Remark 1.3.18. If a is a bilinear, continuous, symmetric and elliptic form with positive ellipticity constant ω, such that $a(u \wedge 1, (u-1)^+) \geq 0$ for all $u \in W$, after a rescaling we obtain that there exists a positive constant c such that

$$\|T(t)\|_{\mathcal{L}(L^1,L^\infty)} \leq ce^{\omega t}t^{-\frac{N}{2}} \qquad t > 0.$$

Example 1.3.19. The form associated with the Schrödinger operator is continuous, symmetric and elliptic with positive ellipticity constant. Moreover if $u \in W^{1,2}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, (1+V(x))dx)$ then $(u \wedge 1)$ belongs to the same space indeed we have

$$\begin{aligned}
&\nabla(u \wedge 1) = \nabla u \chi_{\{u \leq 1\}};\\
&\int_{\mathbb{R}^N} (u \wedge 1)^2 = \int_{\{u \leq 1\}} u^2 + \int_{\{u>1\}} 1 \leq 2\int_{\mathbb{R}^N} u^2 < \infty;\\
&\int_{\mathbb{R}^N} (1+V)(u \wedge 1)^2 = \int_{\{u \leq 1\}} (1+V)u^2 + \int_{\{u>1\}} (1+V)\\
&\leq 2\int_{\mathbb{R}^N} (1+V)u^2 < \infty.
\end{aligned}$$

By Stampacchia's Lemma and some straightforward computations,

$$\begin{aligned}
&\nabla(u-1)^+ = \nabla u \chi_{\{u \geq 1\}};\\
&\nabla u(x) = 0 \quad \text{a.e. on} \quad \{u = 1\};\\
&a(u \wedge 1, (u-1)^+) = \int_{\mathbb{R}^N} \nabla(u \wedge 1)\nabla(u-1)^+ + \int_{\mathbb{R}^N} V(u \wedge 1)(u-1)^+\\
&= \int_{\{u \geq 1\}} V(u-1)^+ \geq 0.
\end{aligned}$$

It follows that there exist C, ω positive constants such that the semigroup generated by $\Delta - V$ satisfies

$$\|T(t)\|_{\mathcal{L}(L^1,L^\infty)} \leq ce^{\omega t}t^{-\frac{N}{2}} \qquad \forall\ t > 0.$$

Thanks to the Dunford-Pettis criterion we are finally able to deduce the existence of an integral kernel.
Given $p \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$,

$$(B_p f)(x) = \int_{\mathbb{R}^N} p(x,y)f(y)\,dy$$

defines a bounded operator $B_p \in \mathcal{L}(L^1(\mathbb{R}^N), L^\infty(\mathbb{R}^N))$ and

$$\|B_p\|_{\mathcal{L}(L^1,L^\infty)} \leq \|p\|_{L^\infty(\mathbb{R}^N \times \mathbb{R}^N)}.$$

A kind of converse is true. The proof of the following result can be found in [1, Theorem 1.3].

Theorem 1.3.20. (Dunford- Pettis) Let $1 \leq r < \infty$, $B \in \mathcal{L}(L^r(\mathbb{R}^N))$ such that $\|B\|_{\mathcal{L}(L^1(\mathbb{R}^N), L^\infty(\mathbb{R}^N))} < \infty$. Then there exists $p \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ such that

$$(Bf)(x) = \int_{\mathbb{R}^N} p(x,y) f(y)\, dy$$

almost everywhere for all $f \in L^1(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$. In that case $B \geq 0$ if and only if $p \geq 0$.

Summarizing, through this section, we proved that, without assuming hölderianity assumptions, but only requiring local integrability on the positive potential, the semigroup generated by the Schrödinger operator is an integral operator. There exists therefore a positive kernel $p(x,y,t)$ such that

$$(T(t)f)(x) = \int_{\mathbb{R}^N} p(x,y,t) f(y) dy \qquad \forall\, x \in \mathbb{R}^N,\ t > 0,\ f \in L^1(\mathbb{R}^N).$$

Moreover there exists $C,\ \omega > 0$ such that

$$\|p(\cdot,\cdot,t)\|_{L^\infty(\mathbb{R}^N \times \mathbb{R}^N)} \leq C e^{\omega t} t^{-\frac{N}{2}}$$

for all $t > 0$.

Remark 1.3.21. By Corollary 1.3.8, it follows that, if p_1 and p_2 are the kernels corresponding respectively to the Schrödinger operators $\Delta - V_1$ and $\Delta - V_2$ with $V_1 \leq V_2$, then $p_1 \geq p_2$. In particular, choosing $V_1 \equiv 0$, it follows that the kernel of the semigroup generated by the Schrödinger operator is pointwise dominated by the heat kernel of the Laplacian.

Remark 1.3.22. By the representation formula and the symmetry of the semigroup generated by a Schrödinger operator, it follows that the kernel is symmetric with respect to the variables x and y, moreover the contractivity of $(T(t))_{t \geq 0}$ in $L^\infty(\mathbb{R}^N)$ yields $\int_{\mathbb{R}^N} p(x,y,t)\, dy \leq 1$ for all $t > 0$ and $x \in \mathbb{R}^N$.

Chapter 2

Kernel estimates for Markov semigroups

This chapter is devoted to the study of kernels of elliptic operators. As we have seen in Chapter 1, even if the coefficients of the operators are unbounded, the semigroup generated in the space of continuous and bounded functions admits an integral representation through a kernel p. We are interested in finding pointwise upper bounds for such kernels. However we will not consider the whole operator, our attention will be first turned toward Kolmogorov operators not containing a zero order derivative term. In a second moment we will analyse also Schrödinger operators not containing a drift term.
In both cases we use Lyapunov function techniques.

2.1 Kernel estimates for a class of Kolmogorov semigroups

We consider the second order elliptic operator

$$A=\sum_{i,j=1}^{N} a_{ij}D_{ij}+\sum_{i=1}^{N}F_iD_i=A_0+F\cdot D$$

where $A_0=\sum_{i,j=1}^{N} a_{ij}D_{ij}$ and the associated parabolic problem

$$\begin{cases} u_t(x,t)=Au(x,t), & x\in\mathbb{R}^N,\ t>0,\\ u(x,0)=f(x) & x\in\mathbb{R}^N \end{cases} \tag{2.1}$$

with initial datum $f\in C_b(\mathbb{R}^N)$.
The operator A is endowed with the maximal domain in $C_b(\mathbb{R}^N)$ given by

$$D_{max}(A)=\{u\in C_b(\mathbb{R}^N)\cap W^{2,p}_{loc}(\mathbb{R}^N)\quad\text{for all}\quad p<\infty:\ Au\in C_b(\mathbb{R}^N)\}.$$

As proved in Chapter 1, assuming that (a_{ij}) is a symmetric matrix, $a_{ij} \in C^{\alpha}_{loc}(\mathbb{R}^N)$, $F_i \in C^{\alpha}_{loc}(\mathbb{R}^N)$ for some $0 < \alpha < 1$ and the ellipticity condition

$$\lambda|\xi|^2 \le \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2$$

for every $x,\ \xi \in \mathbb{R}^N$ and suitable $0 < \lambda \le \Lambda$, it is possible to prove the existence of a bounded classical solution of such problem, i.e. a function $u \in C(\mathbb{R}^N \times [0,+\infty)) \cap C^{1,2}(\mathbb{R}^N \times (0,+\infty))$ which is bounded in $\mathbb{R}^N \times [0,T]$ for any $T > 0$ and satisfies $\partial_t u,\ D^2 u \in C^{\alpha}(\mathbb{R}^N \times (0,+\infty))$ and (2.1). In their work, Metafune, Pallara and Rhandi (see [27]), using Lyapunov functions independent of t, prove estimates of the form

$$p(x,y,t) \le c(t)\omega(y).$$

For instance, if the drift term is given by $F(x) = -|x|^r \dfrac{x}{|x|}$ and the second order part is the Laplacian, they prove that, for any $\gamma < 1/(r+1)$ and for some positive constants c_1 and c_2, $p(x,y,t) \le c_1 \exp\left(c_2 t^{-\frac{r+1}{r-1}}\right)\exp(-\gamma|y|^{r+1})$ for small times t and for all $x,\ y \in \mathbb{R}^N$.
Following their idea, but considering Lyapunov functions depending also on the time variable for the operator $\partial_t + A$, we deduce estimates of the form

$$p(x,y,t) \le c(t)\omega(y,t).$$

In particular, in the special case mentioned above, for small times, we obtain

$$p(x,y,t) \le c_1 t^{-\delta}\exp(-t^{\alpha}\gamma|y|^{r+1}).$$

We remark that, although for $0 < t \le 1$ $\exp\{-c|y|^{r+1}\} \le \exp\{-ct^{\alpha}|y|^{r+1}\}$, the function $c(t)$ blows up polynomially in our estimates and exponentially in [27]. Therefore, using Lyapunov functions for the parabolic operator depending also on the time variable t, we gain a better behaviour for the function $c(t)$.
We start by proving the integrability of certain Lyapunov functions with respect to the measure $p(x,\cdot,t)dy$. Moreover an estimate of the L^1-norm of the Lyapunov functions with respect to the measure above is obtained. Assuming suitable assumptions on the radial component of the drift F, examples of Lyapunov functions for the parabolic operator are given.
Following [27, Section 3], it is proved how, underthe hypothesis of integrability of some power k of the drift with respect to the measure p, the kernel is in some Lebesgue spaces L^r or in some other spaces embedded in L^{∞} for k large enough. Then the main result is proved, we apply an estimate for the L^{∞}-norm of solutions of certain parabolic problems to deduce the claimed result. An useful tool employed here is a result of Sobolev regularity for transition probabilities.
In some recent papers, Bogachev, Krylov, Röckner and Shaposhnikov (see [6], [7] and [8]) have proved existence and regularity properties for parabolic problems

having measures as initial data. The authors assume also integrability properties of the drift term, comparables to ours, and deduce the uniform boundedness of the solutions in $\mathbb{R}^N \times [0,T]$ whenever $T < 1$. Their results do not apply to our situation since the fundamental solution p is singular for $t = 0$.

All over the section we will assume the existence of a Lyapunov function for the operator A, that is a function $0 \leq V \in C^2(\mathbb{R}^N)$ such that $\lim_{|x|\to\infty} V(x) = +\infty$ and $AV(x) \leq \lambda V(x)$ for some positive λ. We recall that this assumption insures that the domain of the weak generator $\hat{D}$ coincides with the maximal domain $D_{max}(A)$ (see Theorem 1.2.5). We will see later that Lyapunov functions exist for the operators we are interested in.
Moreover, since we will deal with differential quotients and we have to apply the integration by parts formula, we suppose that the coefficients a_{ij} of the operator are of class $C_b^1(\mathbb{R}^N)$.

2.1.1 L^1- estimates of some Lyapunov functions

In this section we show how to obtain the integrability of certain unbounded functions with respect to the kernel p. Later pointwise estimates will be deduced from L^1-bounds.
Our technique rests on the following definition, where $L = \partial_t + A$.

We say that a continuous function $W : [0,T] \times \mathbb{R}^N \to [0,+\infty)$ is a Lyapunov function for the operator L if it belongs to $C^{2,1}(Q_T)$, $\lim_{|x|\to\infty} W(x,t) = +\infty$ uniformly with respect to t in compact sets of $(0,T]$ and there exists $h : (0,T] \to [0,\infty)$ integrable in a neighborhood of 0 such that $LW(x,t) \leq h(t)W(x,t)$ for all $(x,t) \in Q_T$. Note that we do not require that $W(x,0)$ tends to ∞ as $|x| \to \infty$.

We refer the reader to [30] for results similar to the next proposition, when the Lyapunov function is independent of t.

Proposition 2.1.1. *For each $t \in [0,T]$, a Lyapunov function $W(\cdot,t)$ is integrable with respect to the measure $p(x,\cdot,t)$. Moreover, setting*

$$\xi_W(x,t) = \int_{\mathbb{R}^N} p(x,y,t)W(y,t)dy, \tag{2.2}$$

the inequality

$$\xi_W(x,t) \leq e^{\int_0^t h(s)ds} W(x,0) \tag{2.3}$$

holds.

Proof. Let us consider, for every $\alpha \geq 0$, $\psi_\alpha \in C_b^\infty(\mathbb{R})$ such that $\psi_\alpha(s) = s$ for $s \leq \alpha$, ψ_α is constant in $[\alpha+1,\infty)$, $\psi'_\alpha \geq 0$ and $\psi''_\alpha \leq 0$. From the concavity of ψ_α it follows that

$$s\psi'_\alpha(s) \leq \psi_\alpha(s) \qquad \forall\ s \geq 0. \tag{2.4}$$

Obviously $\psi_\alpha \circ W \in BUC(Q_T)$ and, moreover, it belongs to $BUC^{2,1}(Q(\varepsilon,T))$ for every $\varepsilon > 0$, since is constant for $t \geq \varepsilon > 0$ and large $|x|$. We set $\xi_\alpha(x,t) = \int_{\mathbb{R}^N} p(x,y,t)\psi_\alpha(W(y,t))dy$. For every fixed $t \geq \varepsilon$, the function $(\psi_\alpha \circ W)(\cdot,t)$ belongs to $D_{max}(A)$, which coincides with the domain of the generator by the assumption of the existence of Lyapunov functions for A. It follows that

$$\partial_t \xi_\alpha(\cdot,t) = e^{tA}A(\psi_\alpha \circ W)(\cdot,t) + e^{tA}\partial_t(\psi_\alpha \circ W)(\cdot,t)$$

and then

$$\partial_t \xi_\alpha(x,t) = \int_{\mathbb{R}^N} p(x,y,t)L(\psi_\alpha \circ W)(y,t)dy.$$

By a straightforward computation we obtain

$$\begin{aligned} L(\psi_\alpha \circ W)(x,t) =& \psi'_\alpha(W(x,t))LW(x,t) \\ &+ \psi''_\alpha(W(x,t)) \sum_{i,j=1}^N a_{ij} D_j W(x,t) D_i W(x,t) \\ \leq& \psi'_\alpha(W(x,t))LW(x,t). \end{aligned}$$

Thus, for $t \geq \varepsilon$,

$$\partial_t \xi_\alpha(x,t) \quad \leq \quad \int_{\mathbb{R}^N} p(x,y,t)\psi'_\alpha(W(y,t))LW(y,t)dy.$$

Using the property of W, the positivity of ψ' and (2.4) we get

$$\partial_t \xi_\alpha(x,t) \leq h(t) \int_{\mathbb{R}^N} p(x,y,t)\psi_\alpha(W(y,t))dy = h(t)\xi_\alpha(x,t).$$

Therefore for $t \geq \varepsilon$

$$\xi_\alpha(x,t) \leq e^{\int_\varepsilon^t h(s)ds} \xi_\alpha(x,\varepsilon). \tag{2.5}$$

Now we prove that $\xi_\alpha(x,\varepsilon) \to \psi_\alpha(W(x,0))$ as $\varepsilon \to 0$. We have

$$\begin{aligned} |\xi_\alpha(x,\varepsilon) - \psi_\alpha(W(x,0))| \quad &= \quad \left| \int_{\mathbb{R}^N} p(x,y,\varepsilon)\psi_\alpha(W(y,\varepsilon))dy - \psi_\alpha(W(x,0))\, dy \right| \\ &\leq \quad \int_{\mathbb{R}^N} p(x,y,\varepsilon)|\psi_\alpha(W(y,\varepsilon)) - \psi_\alpha(W(y,0))|\, dy \\ &+ \quad |T(\varepsilon)\psi_\alpha(W(x,0)) - \psi_\alpha(W(x,0))|. \end{aligned}$$

The second term in the right member obviously goes to 0 as $\varepsilon \to 0$ since $\psi_\alpha \circ W \in C_b(\mathbb{R}^N)$ and $T(t)f \to f$ as $t \to 0$ uniformly on compact sets of $\mathbb{R}^N$ for $f \in C_b(\mathbb{R}^N)$ (see Theorem 1.1.7). Concerning the first addend, we fixe $R > |x| + 1$ and we split it in the integral over B_R and the integral over the complementary of B_R. We have

$$\begin{aligned}\int_{\mathbb{R}^N} p(x,y,\varepsilon)|\psi_\alpha(W(y,\varepsilon))-\psi_\alpha(W(y,0))|\,dy \\ = \int_{B_R} p(x,y,\varepsilon)|\psi_\alpha(W(y,\varepsilon))-\psi_\alpha(W(y,0))|\,dy \\ + \int_{\mathbb{R}^N\setminus B_R} p(x,y,\varepsilon)|\psi_\alpha(W(y,\varepsilon))-\psi_\alpha(W(y,0))|\,dy.\end{aligned}$$

The integral on B_R tends to 0 as $\varepsilon \to 0$ since $\psi_\alpha(W(y,\varepsilon)) \to \psi_\alpha(W(y,0))$ uniformly on B_R. Consider the integral on the complementary of B_R. Let h_R be a smooth function on $\mathbb{R}^N$ such that $\chi_{\mathbb{R}^N\setminus B_R} \le h_R \le \chi_{\mathbb{R}^N\setminus B_{R-1}}$. Observe that $h_R \in D_{max}(A)$ and hence $T(\varepsilon)h_R \to h_R$ uniformly in $\mathbb{R}^N$ since

$$T(\varepsilon)h_R(x) - h_R(x) = \int_0^\varepsilon T(s)Ah_R(x)\,ds$$

for all $x \in \mathbb{R}^N$ (see [38, Proposition 3.2]). Therefore, given $\delta > 0$, there exists $\varepsilon_0 > 0$ such that, for $\varepsilon \le \varepsilon_0$, $T(\varepsilon)h_R \le \delta + h_R$. By means of the previous remarks, since $|x| < R-1$, we deduce

$$\begin{aligned}\int_{\mathbb{R}^N\setminus B_R} p(x,y,\varepsilon)|\psi_\alpha(W(y,\varepsilon))-\psi_\alpha(W(y,0))| &\le 2(\alpha+1)\int_{\mathbb{R}^N\setminus B_R} p(x,y,\varepsilon) \\ &= 2(\alpha+1)T(\varepsilon)\chi_{\mathbb{R}^N\setminus B_R}(x) \\ &\le 2(\alpha+1)T(\varepsilon)h_R(x) \\ &\le 2(\alpha+1)[\delta+h_R(x)] \\ &= 2(\alpha+1)\delta\end{aligned}$$

for $\varepsilon \le \varepsilon_0$. Letting $\varepsilon \to 0$ in (2.5) we obtain

$$\xi_\alpha(x,t) \le e^{\int_0^t h(s)ds}\psi_\alpha(W(x,0)).$$

Letting $\alpha \to \infty$ in the previous inequality and using Fatou's Lemma we get

$$\int_{\mathbb{R}^N} p(x,y,t)W(y,t)dy \le \liminf_{\alpha\to\infty}\xi_\alpha(x,t) \le e^{\int_0^t h(s)ds}W(x,0).$$

□

In the next proposition we prove that suitable exponential functions in x and t are of Lyapunov for a class of Kolmogorov operators.

Proposition 2.1.2. *Let $L = \partial_t + A_0 + F\cdot D$ such that*

$$\limsup_{|x|\to\infty} |x|^{-r}F(x)\cdot\frac{x}{|x|} < -c \tag{2.6}$$

for some positive c and $r > 1$. Then, if $\alpha > \frac{r+1}{r-1}$, $\delta < \frac{c}{\Lambda(r+1)}$ and $0 < t \le 1$, $W(x,t) = \exp\{\delta t^\alpha |x|^{r+1}\}$ is a Lyapunov function for L. Moreover $\xi_W(x,t) \le CW(x,0) = C$ for some positive constant C and for all $x \in \mathbb{R}^N$ and $0 < t \le 1$.

Proof. An easy computation gives

$$\begin{aligned}
LW(x,t) &= \delta(r+1)t^{\alpha}W(x,t)\Big[\alpha\frac{|x|^{r+1}}{t(r+1)}+(r-1)|x|^{r-3}\sum_{i,j=1}^{N}a_{ij}x_ix_j \\
&+ |x|^{r-1}\sum_{i=1}^{N}a_{ii}+\delta(r+1)t^{\alpha}|x|^{2r-2}\sum_{i,j=1}^{N}a_{ij}x_ix_j+|x|^{r}F\cdot\frac{x}{|x|}\Big] \\
&\le \delta(r+1)t^{\alpha}W(x,t)\Big[\alpha\frac{|x|^{r+1}}{t(r+1)}+\big[\Lambda(r-1)+\sum_{i=1}^{N}a_{ii}\big]|x|^{r-1} \\
&+ \Lambda\delta(r+1)t^{\alpha}|x|^{2r}+|x|^{r}F\cdot\frac{x}{|x|}\Big].
\end{aligned}$$

Considering suitable space-time regions it is possible to estimate the right hand side in the previous inequality.
Let $\gamma > \dfrac{1}{r-1}$. If $|x| > \dfrac{1}{t^{\gamma}}$, $0<t<1$,

$$\begin{aligned}
LW(x,t) &\le \delta(r+1)t^{\alpha}W(x,t)\Big[\alpha\frac{|x|^{r+1}}{t(r+1)}+\big[\Lambda(r-1)+\sum_{i=1}^{N}a_{ii}\big]|x|^{r-1} \\
&+ \Lambda\delta(r+1)t^{\alpha}|x|^{2r}+|x|^{r}F\cdot\frac{x}{|x|}\Big] \\
&\le \delta(r+1)t^{\alpha}W(x,t)\Big[\frac{\alpha}{r+1}|x|^{r+1+\frac{1}{\gamma}}+\big[\Lambda(r-1)+\sum_{i=1}^{N}a_{ii}\big]|x|^{r-1} \\
&+ \Lambda\delta(r+1)|x|^{2r}+|x|^{r}F\cdot\frac{x}{|x|}\Big] \\
&\le \delta(r+1)t^{\alpha}|x|^{2r}W(x,t)\Big[\frac{\alpha}{r+1}|x|^{r+1+\frac{1}{\gamma}-2r} \\
&+ \big[\Lambda(r-1)+\sum_{i=1}^{N}a_{ii}\big]|x|^{-r-1}+\Lambda\delta(r+1)+|x|^{-r}F\cdot\frac{x}{|x|}\Big].
\end{aligned}$$

By assumption (2.6), if $|x|$ is large enough,

$$\begin{aligned}
LW(x,t) \le \delta(r+1)t^{\alpha}|x|^{2r}W(x,t)\Big[&\frac{\alpha}{r+1}|x|^{-r+1+\frac{1}{\gamma}} \\
&+\big[\Lambda(r-1)+\sum_{i=1}^{N}a_{ii}\big]|x|^{-r-1}+\Lambda\delta(r+1)-c\Big].
\end{aligned}$$

Since $\delta < \dfrac{c}{\Lambda(r+1)}$ and $\gamma > \dfrac{1}{r-1}$, for $|x|$ large enough and belonging to the considered region $LW \le 0$. For the remaining small values of x in this region $LW(x,t) \le C \le CW(x,t)$.

If $|x| \leq \dfrac{1}{t^\gamma}$ and is large enough in order that the term containing the drift is negative,

$$\begin{aligned} &LW(x,t) \leq \\ &\qquad W(x,t)\Bigg[\frac{\delta\alpha}{t^{\gamma(r+1)+1-\alpha}} + \delta(r+1)\left(\Lambda(r-1)+\sum_{i=1}^N a_{ii}\right)\frac{1}{t^{\gamma(r-1)-\alpha}} \\ &\qquad + \Lambda\delta^2(r+1)^2\frac{1}{t^{2\gamma r-2\alpha}}\Bigg]. \end{aligned}$$

If we choose $\gamma < \frac{\alpha}{r+1}$, we have $\gamma(r+1)-\alpha+1<1$ and $2r\gamma-2\alpha<0$. If $|x|$ is small we obtain the estimate as in the other region. In any case

$$LW(x,t) \leq h(t)W(x,t)$$

with h integrable near 0. Observe moreover that the conditions on γ are compatible since $\alpha > \frac{r+1}{r-1}$. The existence of Lyapunov functions for the elliptic operator is guaranteed under the assumption (2.6) (see [27, Prop. 2.6]). Then by Proposition 2.1.1 the estimate of $\xi_W(x,t)$ follows. □

Example 2.1.3. In particular, Proposition 2.1.2 applies if

$$L = \partial_t + \Delta - |x|^r\frac{x}{|x|}\cdot D$$

with $r>1$. Then, for $\alpha > \dfrac{r+1}{r-1}$, $\delta < \frac{1}{r+1}$ and $0<t\leq 1$, $W(x,t) = \exp\{\delta t^\alpha |x|^{r+1}\}$ is a Lyapunov function for L and $\xi_W(x,t) \leq CW(x,0) = C$ for some positive constant C, for all $x\in\mathbb{R}^N$ and $0<t\leq 1$.

2.1.2 Integrability and regularity results for the kernel

Following [27, Section 3 and Appendix A], in this subsection we collect some useful and of independent interest results. We prove embedding theorems for the spaces $\mathcal{H}^{k,1}$ due to Krylov (see [21]) and, using the same methods, we deduce also embedding theorems for the spaces Θ^k (see definitions below).

Then we fix $T>0$, $0<a_0<a<b<b_0\leq T$, assume $b_0-b\geq a-a_0$ and consider p as a function depending on $(y,t)\in\mathbb{R}^N\times(0,T)$ for arbitrary, but fixed, $x\in\mathbb{R}^N$.

Setting

$$\Gamma(k,x,a_0,b_0) = \left(\int_{Q(a_0,b_0)} |F(y)|^k p(x,y,t)dy\,dt\right)^{\frac{1}{k}}$$

and making use of the embeddings above, we show global regularity result for p with respect to the variables (y,t) assuming $\Gamma(k,x,a_0,b_0)<\infty$ for suitable $k\geq 1$.

Definition 2.1.4. *Given $k \geq 1$, $\mathcal{H}^{k,1}(Q_T)$ denotes the space of all functions $u \in W_k^{1,0}(Q_T)$ with $\partial_t u \in (W_{k'}^{1,0}(Q_T))'$, the dual space of $W_{k'}^{1,0}(Q_T)$, endowed with the norm*

$$\|u\|_{\mathcal{H}^{k,1}(Q_T)} := \|\partial_t u\|_{(W_{k'}^{1,0}(Q_T))'} + \|u\|_{W_k^{1,0}(Q_T)},$$

with $\frac{1}{k} + \frac{1}{k'} = 1$.

Definition 2.1.5. *For $k > 2$, $\Theta^k(Q_T)$ is the space of all functions u belonging to $W_k^{1,0}(Q_T)$ such that there exists $C > 0$ for which*

$$\left| \int_{Q_T} u \partial_t \phi \, dx \, dt \right| \leq C \left(\|\phi\|_{L^{\frac{k}{k-2}}(Q_T)} + \|D\phi\|_{L^{\frac{k}{k-1}}(Q_T)} \right)$$

for every ϕ such that the right hand side above is finite. Observe that $\frac{k}{k-1} = k'$ and $\frac{k}{k-2} = \left(\frac{k}{2}\right)'$. $\Theta^k(Q_T)$ is a Banach space endowed with the norm

$$\|u\|_{\Theta^k(Q_T)} = \|u\|_{W_k^{1,0}(Q_T)} + \|\partial_t u\|_{\frac{k}{2},k;Q_T},$$

where $\|\partial_t u\|_{\frac{k}{2},k;Q_T}$ is the best constant such that the above estimate holds.

By using a reflection argument and standard approximation by smooth functions methods one can prove the following extension and density results.

Lemma 2.1.6. *There exists linear, continuous extension operators*

$$E_1 : \mathcal{H}^{k,1}(Q_T) \to \mathcal{H}^{k,1}(\mathbb{R}^{N+1})$$

and

$$E_2 : \Theta^k(Q_T) \to \Theta^k(\mathbb{R}^{N+1}).$$

Lemma 2.1.7. *The restrictions of functions in $C_c^\infty(\mathbb{R}^{N+1})$ to Q_T are dense in $\mathcal{H}^{k,1}(Q_T)$ and in $\Theta^k(Q_T)$.*

Theorem 2.1.8. *The following embeddings of $\mathcal{H}^{k,1}$ in L^r spaces hold.*

- *(i) If $1 < k < N+2$, then $\mathcal{H}^{k,1}(Q_T)$ is continuously embedded in $L^r(Q_T)$ for $\frac{1}{r} = \frac{1}{k} - \frac{1}{N+2}$.*
- *(ii) If $k = N+2$, then $\mathcal{H}^{k,1}(Q_T)$ is continuously embedded in $L^r(Q_T)$ for $N+2 \leq r < \infty$.*
- *(iii) If $k > N+2$, then $\mathcal{H}^{k,1}(Q_T)$ is continuously embedded in $L^\infty(Q_T)$.*

PROOF. Since the restrictions of functions in the space $C_c^\infty(\mathbb{R}^{N+1})$ are dense in $\mathcal{H}^{k,1}(Q_T)$, in any case we will prove the estimate

$$\|u\|_{L^r(Q_T)} \leq \|u\|_{\mathcal{H}^{k,1}(Q_T)} \tag{2.7}$$

for every function $u \in C_c^\infty(\mathbb{R}^{N+1})$ and some positive constant C independent of u. Let G be the fundamental solution of the operator $\partial_t - \Delta$ in $\mathbb{R}^{N+1}$ given by

$$G(x,t) = \begin{cases} \frac{1}{(4\pi t)^{\frac{N}{2}}} \exp\left(-\frac{1}{4t}|x|^2\right) & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t \le 0. \end{cases} \tag{2.8}$$

Let $u \in C_c^\infty(\mathbb{R}^{N+1})$, $\psi \in C_c^\infty(Q_T)$ and set $\phi = G * \psi$. Then $\phi \in C^2(\mathbb{R}^{N+1})$ and, by [20, Theorem 8.4.2], it satisfies $\partial_t\phi - \Delta\phi = \psi$. Moreover, since ψ has support in $\mathbb{R}^N \times [0,T]$, then $G * \psi = G_T * \psi$ where $G_T = G\chi_{[0,T]}$. By simple computations it immediately follows that $G_T \in L^s(\mathbb{R}^{N+1})$ for $1 \le s < \frac{N+2}{N}$ and $DG_T \in L^s(\mathbb{R}^{N+1})$ for $1 \le s < \frac{N+2}{N+1}$ where the gradient is understood with respect to the space variable. Young's inequality yields $\|\phi\|_{W_s^{1,0}(Q_T)} \le C\|\psi\|_{L^1(Q_T)}$.
We have

$$\begin{aligned} \left|\int_{Q_T} u\psi \, dx\, dt\right| &= \left|\int_{Q_T} u(\partial_t\phi - \Delta\phi)\, dx\, dt\right| \\ &= \left|\int_{Q_T} u\partial_t\phi + Du \cdot D\phi)\, dx\, dt\right| \\ &\le C\|u\|_{\mathcal{H}^{k,1}(Q_T)}\|\phi\|_{W_{k'}^{1,0}(Q_T)}. \end{aligned} \tag{2.9}$$

Let us prove (i). Let $1 < k < N+2$, r such that $\frac{1}{r} = \frac{1}{k} - \frac{1}{N+2}$. By Theorem A.0.8, $\|\phi\|_{W_{r'}^{2,1}(Q_T)} \le c\|\psi\|_{L^{r'}(Q_T)}$, by the embedding $W_{r'}^{2,1}(Q_T) \subset W_{k'}^{1,0}(Q_T)$ (see Theorem A.0.9) and the previous inequality (2.9), we obtain

$$\begin{aligned} \left|\int_{Q_T} u\psi\, dx\, dt\right| &\le C\|u\|_{\mathcal{H}^{k,1}(Q_T)}\|\phi\|_{W_{k'}^{1,0}(Q_T)} \le C\|u\|_{\mathcal{H}^{k,1}(Q_T)}\|\phi\|_{W_{r'}^{2,1}(Q_T)} \\ &\le C\|u\|_{\mathcal{H}^{k,1}(Q_T)}\|\psi\|_{L^{r'}(Q_T)}. \end{aligned}$$

This implies (2.7).
Let now $k = N+2$, $N+2 \le r < \infty$ and choose $1 < s < \frac{N+2}{N+1}$ such that

$$\frac{1}{k'} = \frac{1}{s} + \frac{1}{r'} - 1.$$

Young's inequality yields $\|\phi\|_{W_{k'}^{1,0}(Q_T)} \le C\|\psi\|_{L^{r'}(Q_T)}$ and then by 2.9 we deduce (ii). Finally, if $k > N+2$, then $k' < \frac{N+2}{N+1}$ and by Young's inequality we get $\|\phi\|_{W_{k'}^{1,0}(Q_T)} \le C\|\psi\|_{L^1(Q_T)}$. By (2.9),

$$\left|\int_{Q_T} u\psi\, dx\, dt\right| \le C\|u\|_{\mathcal{H}^{k,1}(Q_T)}\|\phi\|_{W_{k'}^{1,0}(Q_T)} \le C\|u\|_{\mathcal{H}^{k,1}(Q_T)}\|\psi\|_{L^1(Q_T)}.$$

□

Theorem 2.1.9. *If $k > N+2$, then $\Theta^k(Q_T)$ is continuously embedded in $L^\infty(Q_T)$. Moreover the following estimate holds*

$$\|u\|_{L^\infty(Q_T)} \le C(\|Du\|_{L^k(Q_T)} + \|\partial_t u\|_{\frac{k}{2},k;Q_T}).$$

Proof. Let $u \in C_c^\infty(\mathbb{R}^{N+1})$ and ϕ, ψ as in the proof of the previous theorem. As before we have

$$\left|\int_{Q_T} u\psi\, dx\, dt\right| = \left|\int_{Q_T} u(\partial_t\phi - \Delta\phi)\, dx\, dt\right| = \left|\int_{Q_T} (u\partial_t\phi + Du\cdot D\phi)\, dx\, dt\right| \tag{2.10}$$

$$\leq (\|Du\|_{L^k(Q_T)} + \|\partial_t u\|_{\frac{k}{2},k;Q_T})\left(\|D\phi\|_{L^{\frac{k}{k-1}}(Q_T)} + \|\phi\|_{L^{\frac{k}{k-2}}(Q_T)}\right).$$

Now, since $k > N+2$, $k' < \frac{N+2}{N+1}$ and $\frac{k}{k-2} < \frac{N+2}{N}$. By Young's inequality we get $\|\phi\|_{W^{1,0}_{k'}(Q_T)} \leq C\|\psi\|_{L^1(Q_T)}$ and $\|\phi\|_{L^{\frac{k}{k-2}}(Q_T)} \leq C\|\psi\|_{L^1(Q_T)}$. Therefore

$$\left|\int_{Q_T} u\psi\, dx\, dt\right| \leq (\|Du\|_{L^k(Q_T)} + \|\partial_t u\|_{\frac{k}{2},k;Q_T})\|\psi\|_{L^1(Q_T)}.$$

and the claim follows. □

The embedding theorems above allow us to prove some integrability and regularity properties for the kernel p. A preliminary lemma is needed.

Lemma 2.1.10. *Let $0 \leq t_1 < t_2$ and $\phi \in C^{2,1}(Q(t_1,t_2))$ such that $\phi(\cdot,t)$ has compact support for every $t \in [t_1,t_2]$. Then*

$$\begin{aligned} &\int_{Q(t_1,t_2)} (\partial_t\phi(y,t) + A\phi(y,t))p(x,y,t)\, dy\, dt \\ = &\int_{\mathbb{R}^N} (p(x,y,t_2)\phi(y,t_2) - p(x,y,t_1)\phi(y,t_1))dy. \end{aligned}$$

Proof. Note that if $\psi \in C_c^2(\mathbb{R}^N)$ then by Proposition 1.2.2 and by Proposition 1.2.3

$$\partial_t T(t)\psi = T(t)A\psi.$$

Let $\phi(y,t)$ be as in the statement. We have

$$\partial_t(T(t)\phi(\cdot,t)) = T(t)\partial_t\phi(\cdot,t) + T(t)A\phi(\cdot,t).$$

Integrating this identity over the interval $[t_1,t_2]$ and writing $T(t)$ in terms of the kernel we obtain the claim. □

Recall that, for every $k \geq 1$, $\Gamma(k,x,a_0,b_0) = (\int_{Q(a_0,b_0)} |F(y)|^k p(x,y,t)\, dy)^{\frac{1}{k}}$.

Proposition 2.1.11. *If $\Gamma(1,x,a_0,b_0) < \infty$, then $p \in L^r(Q(a_0,b_0))$ for all $r \in [1, \frac{N+2}{N+1})$ and*

$$\|p\|_{L^r(Q(a_0,b_0))} \leq C(1 + \Gamma(1,x,a_0,b_0))$$

for some constant $C > 0$.

Proof. Consider $\phi \in C^{2,1}(Q_T)$ such that $\phi(\cdot, T) = 0$ and such that $\phi(\cdot, t)$ has compact support for all t. By Lemma 2.1.10, we deduce

$$\int_{Q(a_0,b_0)} p(\partial_t\phi + A_0\phi)dy\,dt = -\int_{Q(a_0,b_0)} pF \cdot D\phi dy\,dt \\ + \int_{\mathbb{R}^N} (p(x,y,b_0)\phi(y,b_0) - p(x,y,a_0)\phi(y,a_0))dy$$

where $A_0 = \sum_{i,j=1}^N a_{ij}D_{ij}$. Since $\int_{\mathbb{R}^N} p(x,y,t) \le 1$ for all $t \ge 0$, $x \in \mathbb{R}^N$, it follows that

$$\left|\int_{Q(a_0,b_0)} p(\partial_t\phi + A_0\phi)dy\,dt\right| \le \Gamma(1,x,a_0,b_0)\|\phi\|_{W^{1,0}_\infty(Q(a_0,b_0))} + 2\|\phi\|_\infty \\ \le (2 + \Gamma(1,x,a_0,b_0))\|\phi\|_{W^{1,0}_\infty(Q(a_0,b_0))}. \tag{2.11}$$

Fix $\psi \in C_c^\infty(Q(a_0,b_0))$ and consider the parabolic problem

$$\begin{cases} \partial_t\phi + A_0\phi = \psi & \text{in } Q_T, \\ \phi(y,T) = 0 & y \in \mathbb{R}^N. \end{cases} \tag{2.12}$$

By the Schauder theory (see Theorem A.0.10), there exists a solution $\phi \in C^{2+\alpha,1+\frac{\alpha}{2}}(Q_T)$. Fixing $r_1' > N+2$, by Theorem A.0.8, we have that $\phi \in W^{2,1}_{r_1'}(Q_T)$ and satisfies

$$\|\phi\|_{W^{2,1}_{r_1'}(Q_T)} \le C\|\psi\|_{L^{r_1'}(Q(a_0,b_0))}$$

and, by the Sobolev embedding Theorems (see Theorem A.0.9) and the previous inequality, we deduce that

$$\|\phi\|_{W^{1,0}_\infty(Q(a_0,b_0))} \le \|\phi\|_{W^{1,0}_\infty(Q_T)} \le C\|\phi\|_{W^{2,1}_{r_1'}(Q_T)} \le C\|\psi\|_{L^{r_1'}(Q(a_0,b_0))}. \tag{2.13}$$

Observe that the solution of the parabolic problem just found cannot be immediately inserted in (2.11) since in general it is not with compact support with respect to the space variable. Anyway we can approximate the solution ϕ with functions which satisfy (2.11) as follows. Let $\theta \in C_c^\infty(\mathbb{R}^N)$ such that $\theta(y) = 1$ for $|y| \le 1$ and, for each $n \in \mathbb{N}$, consider $\phi_n(y,t) = \theta(\frac{y}{n})\phi(y,t)$. Then ϕ_n satisfies (2.11) and, letting $n \to \infty$ by dominated convergence, by (2.13) we obtain

$$\left|\int_{Q(a_0,b_0)} p\psi dy\,dt\right| \le C(1 + \Gamma(1,x,a_0,b_0))\|\psi\|_{L^{r_1'}(Q(a_0,b_0))}.$$

This proves that $p \in L^{r_1}(Q(a_0,b_0))$ where $\frac{1}{r_1} + \frac{1}{r_1'} = 1$. By the arbitrarity of $r_1' > N+2$, it follows that $p \in L^r(Q(a_0,b_0))$ for all $1 \le r < \frac{N+2}{N+1}$ with

$$\|p\|_{L^r(Q(a_0,b_0))} \le C(1 + \Gamma(1,x,a_0,b_0)).$$

□

Lemma 2.1.12. *If $\Gamma(k,x,a_0,b_0)<\infty$ for some $k>1$ and $p\in L^r(Q(a_0,b_0))$ for some $1<r\le\infty$, then $p\in\mathcal{H}^{s,1}(Q(a,b))$ for $s:=\frac{rk}{r+k-1}$ if $r<\infty$ and $s=k$ if $r=\infty$.*

Proof. Let η be a smooth function such that $0\le\eta\le1$, $\eta(t)=1$ for $a\le t\le b$, $\eta(t)=0$ for $t\le a_0$ and $t\ge b_0$ and $|\eta'|\le\frac{2}{a-a_0}$. Let $\phi\in C^{2,1}(Q_T)$ such that $\phi(\cdot,t)$ has compact support for all t. Then also $\eta\phi$ has compact support for all t and by Lemma 2.1.10, setting $q=\eta p$, we obtain

$$\int_{Q_T} q(\partial_t\phi+A_0\phi)dy,\ dt=-\int_{Q_T}(qF\cdot D\phi+p\phi\partial_t\eta)dy\,dt.$$

Now we estimate the right hand side of the previous equality by using the Hölder inequality and the integrability assumption on p. We have

$$\begin{aligned}\int_{Q(a_0,b_0)}|F|^sp^s dy\,dt&=\int_{Q(a_0,b_0)}|F|^sp^{\frac{s}{k}}p^{s(1-\frac{1}{k})}dy\,dt\\&\le\left(\int_{Q(a_0,b_0)}|F|^kp\,dy\,dt\right)^{\frac{s}{k}}\left(\int_{Q(a_0,b_0)}p^{\frac{s(k-1)}{k-s}}dy\,dt\right)^{1-\frac{s}{k}}\\&=\left(\int_{Q(a_0,b_0)}|F|^kp\,dy\,dt\right)^{\frac{s}{k}}\left(\int_{Q(a_0,b_0)}p^r\,dy\,dt\right)^{1-\frac{s}{k}}\\&\le\Gamma(k,x,a_0,b_0)^s\left(\int_{Q(a_0,b_0)}p^r\,dy\,dt\right)^{1-\frac{s}{k}},\end{aligned}$$

hence we have

$$\|Fp\|_{L^s(Q(a_0,b_0))}\le C\|p\|_{L^r(Q(a_0,b_0))}^{\frac{k-1}{k}}$$

where C is a generic constant depending on k, x, a_0, b_0. Therefore

$$\left|\int_{Q_T}q(\partial_t\phi+A_0\phi)dy,\ dt\right|\le C\|p\|_{L^r(Q(a_0,b_0))}^{\frac{k-1}{k}}\|\phi\|_{W^{1,0}_{s'}(Q_T)}$$

with $\frac{1}{s}+\frac{1}{s'}=1$. Observe that we can replace ϕ by its difference quotients with respect to the variable y given by

$$\tau_{-h}\phi(y,t):=\frac{1}{|h|}(\phi(y-he_j,t)-\phi(y,t)),\quad (y,t)\in Q_T,\ 0\ne h\in\mathbb{R}.$$

In this way and recalling that $a_{ij}\in C^1_b(\mathbb{R}^N)$, we obtain

$$\left|\int_{Q_T}\tau_hq(\partial_t\phi+A_0\phi)dy,\ dt\right|\le C\|p\|_{L^r(Q(a_0,b_0))}^{\frac{k-1}{k}}\|\phi\|_{W^{1,0}_{s'}(Q_T)}\tag{2.14}$$

where C depends on k, x, a_0, b_0 and the $C^1_b(\mathbb{R}^N)$ norm of the coefficients a_{ij}. Observe that, since $q\in L^s(Q_T)$, by approximation, as in the proof of Lemma

2.1.11, the inequality (2.14) remains true for functions $\phi \in W^{2,1}_{s'}(Q_T)$. Moreover, since $q \in L^s(Q_T)$, then $|\tau_h q|^{s-2}\tau_h q \in L^{s'}(Q_T)$. By Theorem A.0.8, there exists $\phi \in W^{2,1}_{s'}(Q_T)$ such that

$$\begin{cases} \partial_t \phi + A_0 \phi = |\tau_h q|^{s-2}\tau_h q & \text{in } \; Q_T, \\ \phi(y,T) = 0 & y \in \mathbb{R}^N \end{cases} \tag{2.15}$$

and

$$\|\phi\|_{W^{2,1}_{s'}(Q_T)} \leq C \||\tau_h q|^{s-1}\|_{L^{s'}(Q_T)}.$$

By (2.14), we get

$$\int_{Q_T} |\tau_h q|^s dy\, dt \leq C\|p\|^{\frac{k-1}{k}}_{L^r(Q(a_0,b_0))} \|\tau_h q\|^{s-1}_{L^s(Q_T)},$$

By means of the properties of the differential quotients we deduce

$$\|Dq\|_{L^s(Q_T)} \leq C\|p\|^{\frac{k-1}{k}}_{L^r(Q_T)}.$$

This implies $Dq \in L^s(Q_T)$ and so $q \in W^{1,0}_s(Q_T)$ and $p \in W^{1,0}_s(Q(a,b))$. Concerning the first order time derivative, by the estimate above, integrating by parts and recalling that $a_{ij} \in C^1_b(\mathbb{R}^N)$, we have

$$\begin{aligned} \left| \int_{Q_T} q \partial_t \phi \, dy\, dt \right| &\leq \left| \int_{Q_T} q A_0 \phi \, dy\, dt \right| + C\|p\|^{\frac{k-1}{k}}_{L^r(Q(a_0,b_0))} \|\phi\|_{W^{1,0}_{s'}(Q_T)} \\ &\leq \left| \int_{Q_T} \sum_{i,j=1}^N a_{ij} D_i \phi D_j q \, dy\, dt \right| + C\|p\|^{\frac{k-1}{k}}_{L^r(Q(a_0,b_0))} \|\phi\|_{W^{1,0}_{s'}(Q_T)} \\ &\leq C\|Dq\|_{L^s(Q_T)} \|\phi\|_{W^{1,0}_{s'}(Q_T)} + C\|p\|^{\frac{k-1}{k}}_{L^r(Q(a_0,b_0))} \|\phi\|_{W^{1,0}_{s'}(Q_T)} \\ &\leq C\|p\|^{\frac{k-1}{k}}_{L^r(Q(a_0,b_0))} \|\phi\|_{W^{1,0}_{s'}(Q_T)} \end{aligned}$$

and the claim follows. □

Proposition 2.1.13. *If* $\Gamma(k,x,a_0,b_0) < \infty$ *for some* $1 < k \leq N+2$, *then* $p \in L^r(Q(a,b))$ *for all* $r \in [1, \frac{N+2}{N+2-k})$ *and* $p \in \mathcal{H}^{s,1}(Q(a,b))$ *for all* $s \in (1, \frac{N+2}{N+3-k})$.

Proof. The result follows by applying iteratively Lemma 2.1.12 and Proposition 2.1.11.
Let $r_1 < \frac{N+2}{N+1}$. Observe that $\Gamma(h,x,a_0,b_0) \leq C\Gamma(k,x,a_0,b_0)$ for $h \leq k$ and for some positive constant C. Therefore we can apply Proposition 2.1.11 and deduce $p \in L^{r_1}(Q(a_0,b_0))$. Fix a parameter m (to be chosen later) depending on k and r. Set $a_n = a_0 + \frac{n(a-a_0)}{m}$, $b_n = b_0 - \frac{n(b_0-b)}{m}$ for $n = 1, \ldots.., m$. Suppose that $p \in L^{r_n}(Q(a_0,b_0))$ and take $s_n := \frac{kr_n}{k+r_n-1}$. Then $1 < s_n < r_n$, $s_n < k$ and $r_n = \frac{s_n(k-1)}{k-s_n}$. As in the previous proof, we consider $q = \eta p$ with $\eta(t) = 1$ for

$a_{n+1} \leq t \leq b_{n+1}$ and $\eta(t) = 0$ for $t \leq a_n$, $t \geq b_n$, $|\eta'| \leq \frac{2m}{a-a_0}$. As in the proof of Lemma 2.1.12, we get

$$\left| \int_{Q_T} q \partial_t \phi \, dy \, dt \right| \leq C \|p\|^{\frac{k-1}{k}}_{L^{r_n}(Q(a_n,b_n))} \|\phi\|_{W^{1,0}_{s'_n}(Q_T)}$$

and

$$\|Dq\|_{L^{s_n}(Q_T)} \leq C \|p\|^{\frac{k-1}{k}}_{L^{r_n}(Q(a_n,b_n))}.$$

with C depending on k, x, a_0, b_0. Therefore $p \in \mathcal{H}^{s_n,1}(Q(a_{n+1}, b_{n+1}))$. By the embedding Theorem for the $\mathcal{H}^{s,1}$ spaces (see Theorem 2.1.8), we have that $p \in L^{r_{n+1}}(Q(a_{n+1}, b_{n+1}))$ where

$$\frac{1}{r_{n+1}} = \frac{1}{s_n} - \frac{1}{N+2} = \frac{k + r_n - 1}{k r_n} - \frac{1}{N+2} = \frac{1}{r_n}\left(1 - \frac{1}{k}\right) + \frac{1}{k} - \frac{1}{N+2}.$$

Since $\frac{1}{r_1} > \frac{N+1}{N+2}$, it follows that

$$\frac{1}{r_2} - \frac{1}{r_1} < -\frac{1}{k}\left(1 - \frac{1}{N+2}\right) + \frac{1}{k} - \frac{1}{N+2} = \frac{1}{N+2}\left(\frac{1}{k} - 1\right) < 0.$$

By induction, since $\dfrac{1}{r_{n+1}} = g\left(\dfrac{1}{r_n}\right)$ with g increasing function, $\left(\dfrac{1}{r_n}\right)$ is a positive and decreasing sequence which converges to $\frac{N+2-k}{N+2}$. This implies that, for any $r < \frac{N+2}{N+2-k}$, after a finite number of steps m, we get $r_n > r$ and $p \in L^r(Q(a,b))$. Finally, by Lemma 2.1.12, we handle $p \in \mathcal{H}^{s,1}(Q(a,b))$ for all $s \in (1, \frac{N+2}{N+3-k})$. $\square$

Corollary 2.1.14. *If $\Gamma(k, x, a_0, b_0) < \infty$ for some $k > N+2$, then $p \in L^\infty(Q(a,b))$.*

Proof. By assumption, $\Gamma(k, x, a_0, b_0) < \infty$ for some $k > N+2$, therefore $\Gamma(N+2, x, a_0, b_0) \leq C\Gamma(k, x, a_0, b_0) < \infty$ and, by Proposition 2.1.13, $p \in L^r(Q(a,b))$ for all $r \in [1, \infty)$. By Proposition 2.1.12, $p \in \mathcal{H}^{s,1}(Q(a,b))$ for all $1 < s < k$ and then, choosing $s > N+2$, by Theorem 2.1.8, $p \in L^\infty(Q(a,b))$. $\square$

2.1.3 Pointwise estimates of kernels

We recall that T is a fixed positive number and a_0, a, b, b_0 are such that $0 < a_0 < a < b < b_0 \leq T$. Assume that W_1, W_2 are Lyapunov functions for L, $W_1 \leq W_2$ and there exists $1 \leq \omega \in C^2(\mathbb{R}^N \times (0, \infty))$ such that for some positive constants $c_1(a_0, b_0)$, $c_2(a_0, b_0)$, $c_3(a_0, b_0)$, $c_4(a_0, b_0)$, $c_5(a_0, b_0)$ and $k > N+2$

$$\begin{aligned} &\omega \leq c_1 W_1; \quad |D\omega| \leq c_2 \omega^{\frac{k-1}{k}} W_1^{\frac{1}{k}}; \\ &|D^2\omega| \leq c_3 \omega^{\frac{k-2}{k}} W_1^{\frac{2}{k}}; \quad |\partial_t \omega| \leq c_4 \omega^{\frac{k-2}{k}} W_1^{\frac{2}{k}}; \end{aligned} \tag{2.16}$$

$$\omega |F|^k \leq c_5 W_2 \tag{2.17}$$

pointwise almost everywhere in $Q(a_0, b_0)$. Using the notation of the previous section, we write $\xi_1(x,t)$ to denote $\int_{\mathbb{R}^N} p(x,y,t)W_1(y,t)dy$ and ξ_2 for the analogous integral with W_2. Under these assumptions the following main theorem can be stated.

Theorem 2.1.15. *There exists a positive constant C such that*

$$0 < \omega(y,t)p(x,y,t) \le C\Bigg[(c_2^k + c_5 + c_3^{\frac{k}{2}} + c_2^{\frac{k}{2}} c_5^{\frac{1}{2}}) \int_{a_0}^{b_0} \xi_2 \tag{2.18}$$

$$+ \left(\frac{c_1}{(a-a_0)^{\frac{k}{2}}} + c_4^{\frac{k}{2}}\right) \int_{a_0}^{b_0} \xi_1 \Bigg] \tag{2.19}$$

for all $x,\ y \in \mathbb{R}^N$ and $a \le t \le b$.

As preliminary result we prove an estimate of the L^∞ norm of solutions of certain parabolic problems.

Theorem 2.1.16. *Let $k > N+2$, $v \in L^k(Q_T)$, $w \in L^{\frac{k}{2}}(Q_T)$ and assume that $u \in L^k(Q_T)$ satisfies*

$$\int_{Q_T} u(\partial_t \phi + A_0 \phi)\, dx\, dt = \int_{Q_T} (v \cdot D\phi + w\phi)\, dx\, dt \tag{2.20}$$

for every $\phi \in C^{2,1}(Q_T)$ such that $\phi(\cdot, t)$ has compact support for every t. Then $u \in \Theta^k(Q_T)$ and

$$\|u\|_{L^\infty(Q_T)} \le C\|u\|_{\Theta^k(Q_T)} \le C(\|v\|_{L^k(Q_T)} + \|w\|_{L^{\frac{k}{2}}(Q_T)})$$

where C is a positive constant depending on N, T, k and the C_b^1-norm of the coefficients a_{ij}.

Proof. First we prove that

$$\|u\|_{L^k(Q_T)} \le C(\|v\|_{L^k(Q_T)} + \|w\|_{L^{\frac{k}{2}}(Q_T)}). \tag{2.21}$$

As in other proofs, we observe that, since $u \in L^k(Q_T)$, by approximation, (2.20) holds for functions $\phi \in W_{k'}^{2,1}(Q_T)$. Let $\psi \in C_c^\infty(Q_T)$. By Theorem A.0.8 there exists $\phi \in W_{k'}^{2,1}(Q_T)$ such that

$$\begin{cases} \partial_t \phi + A_0 \phi = \psi & \text{in} \quad Q_T, \\ \phi(x,T) = 0, & x \in \mathbb{R}^N \end{cases}$$

and the estimate

$$\|\phi\|_{W_{k'}^{2,1}(Q_T)} \le C\|\psi\|_{L^{k'}(Q_T)}$$

holds with a constant C depending on k, T and the coefficients a_{ij}. Moreover by the Sobolev embedding theorems (see Theorem A.0.9)

$$\|\phi\|_{L^{\frac{k}{k-2}}(Q_T)} \le C\|\phi\|_{W_{k'}^{2,1}(Q_T)}.$$

By assumption (2.20), we deduce

$$\begin{aligned}\left|\int_{Q_T} u\psi\right| &\le C(\|v\|_{L^k(Q_T)}\|D\phi\|_{L^{k'}(Q_T)} + \|w\|_{L^{\frac{k}{2}}(Q_T)}\|\phi\|_{L^{\frac{k}{k-2}}(Q_T)})\\ &\le C(\|v\|_{L^k(Q_T)} + \|w\|_{L^{\frac{k}{2}}(Q_T)})\|\psi\|_{L^{k'}(Q_T)}\end{aligned}$$

and so the estimate for the $\|u\|_{L^k(Q_T)}$ follows.

Now let us prove the claim. As proved above, we have

$$\left|\int_{Q_T} u(\partial_t\phi + A_0\phi)\right| \le C\left(\|v\|_{L^k(Q_T)}\|D\phi\|_{L^{k'}(Q_T)} + \|w\|_{L^{\frac{k}{2}}(Q_T)}\|\phi\|_{L^{\frac{k}{k-2}}(Q_T)}\right)$$

for all $\phi \in W^{2,1}_{k'}(Q_T)$. Replacing ϕ by its differential quotients with respect to the space variable, we obtain

$$\begin{aligned}\left|\int_{Q_T} \tau_h u(\partial_t\phi + A_0\phi)\right| \le C\big[&\big(\|u\|_{L^k(Q_T)} + \|v\|_{L^k(Q_T)}\big)\|\phi\|_{W^{2,1}_{k'}(Q_T)}\\ &+ \|w\|_{L^{\frac{k}{2}}(Q_T)}\|D\phi\|_{L^{\frac{k}{k-2}}(Q_T)}\big].\end{aligned}$$

By Sobolev embedding Theorem (see Theorem A.0.9),

$$\|D\phi\|_{L^s(Q_T)} \le C\|\phi\|_{W^{2,1}_{\frac{k}{k-1}}(Q_T)}$$

if $\dfrac{1}{s} = 1 - \dfrac{1}{k} - \dfrac{1}{N+2}$. Since $\dfrac{k}{k-1} < \dfrac{k}{k-2} < s$ by the assumption $k > N+2$, we have

$$\|D\phi\|_{L^{\frac{k}{k-2}}(Q_T)} \le C\|\phi\|_{W^{2,1}_{\frac{k}{k-1}}(Q_T)}$$

and so

$$\left|\int_{Q_T} \tau_h u(\partial_t\phi + A_0\phi)\right| \le C\big(\|u\|_{L^k(Q_T)} + \|v\|_{L^k(Q_T)} + \|w\|_{L^{\frac{k}{2}}(Q_T)}\big)\|\phi\|_{W^{2,1}_{k'}(Q_T)}. \tag{2.22}$$

Let now $\phi \in W^{2,1}_{k'}(Q_T)$ such that

$$\begin{cases} \partial_t\phi + A_0\phi = |\tau_h u|^{k-2}\tau_h u, & \text{in} \quad Q_T\\ \phi(x,T) = 0, & x \in \mathbb{R}^N \end{cases}$$

and

$$\|\phi\|_{W^{2,1}_{k'}(Q_T)} \le \||\tau_h u|^{k-1}\|_{L^{k'}(Q_T)} = \|\tau_h u\|^{k-1}_{L^k(Q_T)}.$$

For a ϕ so done, by (2.22), we deduce $u \in W^{1,0}_k(Q_T)$ and

$$\|Du\|_{L^k(Q_T)} \le C\big(\|u\|_{L^k(Q_T)} + \|v\|_{L^k(Q_T)} + \|w\|_{L^{\frac{k}{2}}(Q_T)}\big). \tag{2.23}$$

Consider the time derivative. By assumption we have

$$\int_{Q_T} u\partial_t\phi\,dx\,dt = \int_{Q_T}\left(-\sum_{i,j=1}^{N} a_{ij}(D_{ij}\phi)u + v\cdot D\phi + w\phi\right)dx\,dt$$
$$= \int_{Q_T}\left(\sum_{i,j=1}^{N} a_{ij}D_iu\,D_j\phi + \sum_{i,j=1}^{N}(D_ia_{ij})u\,D_j\phi + v\cdot D\phi + w\phi\right)dx\,dt$$

and, as above,

$$\left|\int_{Q_T} u\partial_t\phi\,dx\,dt\right| \le C\big[(\|Du\|_{L^k(Q_T)} + \|u\|_{L^k(Q_T)} + \|v\|_{L^k(Q_T)})\|D\phi\|_{L^{k'}(Q_T)}$$
$$+ \|w\|_{L^{\frac{k}{2}}(Q_T)}\|\phi\|_{L^{\frac{k}{k-2}}(Q_T)}\big].$$

By (2.23) we obtain

$$\left|\int_{Q_T} u\partial_t\phi\,dx\,dt\right| \le C\big[(\|u\|_{L^k(Q_T)} + \|v\|_{L^k(Q_T)} + \|w\|_{L^{\frac{k}{2}}(Q_T)})\|D\phi\|_{L^{k'}(Q_T)}$$
$$+ \|w\|_{L^{\frac{k}{2}}(Q_T)}\|\phi\|_{L^{\frac{k}{k-2}}(Q_T)}\big]$$

and, by (2.21),

$$\left|\int_{Q_T} u\partial_t\phi\,dx\,dt\right| \le C\big[(\|v\|_{L^k(Q_T)} + \|w\|_{L^{\frac{k}{2}}(Q_T)})\|D\phi\|_{L^{k'}(Q_T)}$$
$$+ \|w\|_{L^{\frac{k}{2}}(Q_T)}\|\phi\|_{L^{\frac{k}{k-2}}(Q_T)}\big].$$

(2.21), (2.23) and the last inequality imply that $u \in \Theta^k(Q_T)$ with

$$\|u\|_{\Theta^k(Q_T)} = \|u\|_{W_k^{1,0}(Q_T)} + \|\partial_t u\|_{\frac{k}{2},k;Q_T} \le C(\|v\|_{L^k(Q_T)} + \|w\|_{L^{\frac{k}{2}}(Q_T)}).$$

Finally, Theorem 2.1.9 implies

$$\|u\|_{L^\infty(Q_T)} \le C_1\|u\|_{\Theta^k(Q_T)} \le C_2(\|v\|_{L^k(Q_T)} + \|w\|_{L^{\frac{k}{2}}(Q_T)}).$$

□

We can prove the main theorem.

Proof. (Theorem 2.1.15) In the first part of the proof we assume that ω is bounded.

Let $\Gamma(k,x,a_0,b_0) = (\int_{Q(a_0,b_0)}|F(y)|^k p(x,y,t)dy\,dt)^{\frac{1}{k}}$. Then, by (2.17) and Proposition 2.1.1,

$$\Gamma(k,x,a_0,b_0) \le \int_{Q(a_0,b_0)}\omega|F(y)|^k p(x,y,t)dy\,dt$$
$$\le c_5\int_{Q(a_0,b_0)} p(x,y,t)W_2(y,t) \le c_5\int_{a_0}^{b_0}\xi_2(x,t) < \infty.$$

From Corollary 2.1.14, $p \in L^\infty(Q(a,b))$. Let η be a smooth function such that $\eta(t) = 1$ for $a \le t \le b$, $\eta(t) = 0$ for $t \le a_0$, $t \ge b_0$, $|\eta'| \le \frac{2}{a-a_0}$ and let $\psi \in C^{2,1}(Q_T)$ be such that $\psi(\cdot, t)$ has compact support for every t. We set $q = \eta^{\frac{k}{2}} p$ and $\phi(y,t) = \eta^{\frac{k}{2}}(t)\omega(y,t)\psi(y,t)$. By Lemma 2.1.10, we obtain

$$\int_{Q_T} (\partial_t \phi(y,t) + A\phi(y,t)) p(x,y,t)\, dy\, dt = 0$$

and then, after some computations,

$$\begin{aligned}\int_{Q_T} \omega q(-\partial_t \psi - A_0 \psi)\, dy\, dt = \int_{Q_T} \Bigg[q\Bigg(\psi A_0 \omega + 2\sum_{i,j=1}^N a_{ij} D_i \omega D_j \psi \\ + \omega F \cdot D\psi + \psi F \cdot D\omega + \psi \partial_t \omega \Bigg) + \frac{k}{2} p\omega\psi\eta^{\frac{k-2}{2}} \partial_t \eta \Bigg] dy\, dt.\end{aligned}$$

Since ω is bounded, $\omega q \in L^1(Q_T) \cap L^\infty(Q_T)$. By Theorem (2.1.16),

$$\begin{aligned}\|\omega q\|_{L^\infty(Q_T)} \le C\big(\|qD\omega\|_{L^k(Q_T)} + \|\omega q F\|_{L^k(Q_T)} + \|qD^2\omega\|_{L^{\frac{k}{2}}(Q_T)} \\ + \|qF \cdot D\omega\|_{L^{\frac{k}{2}}(Q_T)} + \|q\partial_t \omega\|_{L^{\frac{k}{2}}(Q_T)} + \frac{1}{a-a_0} \|p\omega\eta^{\frac{k-2}{2}}\|_{L^{\frac{k}{2}}(Q_T)} \big)\end{aligned} \tag{2.24}$$

where C depends on N, k, T and the C^1_b-norm of a_{ij}. Now we estimate the right hand side in (2.24) by using (2.16) and (2.17).

$$\begin{aligned}\|\omega q F\|_{L^k(Q_T)} &= \left(\int_{Q_T} |\omega q F|^k \right)^{\frac{1}{k}} \le \left(\int_{Q_T} (q\omega)^{k-1} \omega q |F|^k \right)^{\frac{1}{k}} \\ &\le c_5(a_0,b_0)^{\frac{1}{k}} \left(\int_{Q_T} (q\omega)^{k-1} q W_2 \right)^{\frac{1}{k}} \le c_5(a_0,b_0)^{\frac{1}{k}} \|\omega q\|^{\frac{k-1}{k}}_{L^\infty(Q_T)} \left(\int_{a_0}^{b_0} \xi_2 dt \right)^{\frac{1}{k}}.\end{aligned}$$

In a similar way

$$\|p\omega\eta^{\frac{k-2}{2}}\|_{L^{\frac{k}{2}}(Q_T)} \le c_1(a_0,b_0)^{\frac{2}{k}} \|\omega q\|^{\frac{k-2}{k}}_{L^\infty} \left(\int_{a_0}^{b_0} \xi_1 dt \right)^{\frac{2}{k}};$$

$$\|qD\omega\|_{L^k(Q_T)} \le c_2(a_0,b_0) \|\omega q\|^{\frac{k-1}{k}}_{L^\infty(Q_T)} \left(\int_{a_0}^{b_0} \xi_1 dt \right)^{\frac{1}{k}};$$

$$\|qD^2\omega\|_{L^{\frac{k}{2}}(Q_T)} \le c_3(a_0,b_0) \|\omega q\|^{\frac{k-2}{k}}_{L^\infty(Q_T)} \left(\int_{a_0}^{b_0} \xi_1 dt \right)^{\frac{2}{k}};$$

$$\|q\partial_t \omega\|_{L^{\frac{k}{2}}(Q_T)} \le c_4(a_0,b_0) \|\omega q\|^{\frac{k-2}{k}}_{L^\infty(Q_T)} \left(\int_{a_0}^{b_0} \xi_1 dt \right)^{\frac{2}{k}}$$

and

$$\|qF\cdot D\omega\|_{L^{\frac{k}{2}}(Q_T)} \le c_2(a_0,b_0)c_5(a_0,b_0)^{\frac{1}{k}}\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}}\left(\int_{a_0}^{b_0}\xi_2 dt\right)^{\frac{2}{k}}.$$

Therefore, by (2.24) and the bounds above,

$$\begin{aligned}\|\omega q\|_{L^\infty(Q_T)} &\le C\Bigg[(c_2(a_0,b_0)+c_5(a_0,b_0)^{\frac{1}{k}})\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-1}{k}}\left(\int_{a_0}^{b_0}\xi_2\right)^{\frac{1}{k}}\\ &+ (c_3(a_0,b_0)+c_2(a_0,b_0)c_5(a_0,b_0)^{\frac{1}{k}})\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}}\left(\int_{a_0}^{b_0}\xi_2\right)^{\frac{2}{k}}\\ &+ \left(\frac{c_1(a_0,b_0)^{\frac{2}{k}}}{a-a_0}+c_4(a_0,b_0)\right)\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}}\left(\int_{a_0}^{b_0}\xi_1\right)^{\frac{2}{k}}\Bigg]\end{aligned}$$

and then

$$\begin{aligned}\|\omega q\|_{L^\infty(Q_T)}^{\frac{2}{k}} &\le C\Bigg[(c_2(a_0,b_0)+c_5(a_0,b_0)^{\frac{1}{k}})\|\omega q\|_{L^\infty(Q_T)}^{\frac{1}{k}}\left(\int_{a_0}^{b_0}\xi_2\right)^{\frac{1}{k}}\\ &+ (c_3(a_0,b_0)+c_2(a_0,b_0)c_5(a_0,b_0)^{\frac{1}{k}})\left(\int_{a_0}^{b_0}\xi_2\right)^{\frac{2}{k}}\\ &+ \left(\frac{c_1(a_0,b_0)^{\frac{2}{k}}}{a-a_0}+c_4(a_0,b_0)\right)\left(\int_{a_0}^{b_0}\xi_1\right)^{\frac{2}{k}}\Bigg].\end{aligned}$$

Setting

$$A=(c_2(a_0,b_0)+c_5(a_0,b_0)^{\frac{1}{k}})\left(\int_{a_0}^{b_0}\xi_2\right)^{\frac{1}{k}},$$

$$\begin{aligned}B=&(c_3(a_0,b_0)+c_2(a_0,b_0)c_5(a_0,b_0)^{\frac{1}{k}})\left(\int_{a_0}^{b_0}\xi_2\right)^{\frac{2}{k}}\\ &+\left(\frac{c_1(a_0,b_0)^{\frac{2}{k}}}{a-a_0}+c_4(a_0,b_0)\right)\left(\int_{a_0}^{b_0}\xi_1\right)^{\frac{2}{k}}\end{aligned}$$

and $X=\|\omega q\|_{L^\infty(Q_T)}^{\frac{1}{k}}$, the inequality above can be written as $X^2\le AX+B$ and so $X\le\frac{A+\sqrt{A^2+4B}}{2}$. It easily follows that

$$\begin{aligned}&0<\omega(y,t)p(x,y,t)\\ &\qquad\le C\left[(c_2^k+c_5+c_3^{\frac{k}{2}}+c_2^{\frac{k}{2}}c_5^{\frac{1}{2}})\int_{a_0}^{b_0}\xi_2+\left(\frac{c_1}{(a-a_0)^{\frac{k}{2}}}+c_4^{\frac{k}{2}}\right)\int_{a_0}^{b_0}\xi_1\right].\end{aligned}$$

If ω is not bounded, we set $\omega_\varepsilon=\dfrac{\omega}{1+\varepsilon\omega}$. Obviously ω_ε is bounded. It is easy to see that ω_ε satisfies (2.16) and (2.17) with constants $c_1,\ c_2,\ c_3,\ c_4,\ c_5$

independent of ε. Then the estimate of $\|\omega_\varepsilon q\|_{L^\infty(Q_T)}$ holds with constants in the right hand side of the previuos inequality which do not depend on ε. Letting $\varepsilon \to 0$ we deduce the claim. $\square$

Remark 2.1.17. If W is a Lyapunov function for the operator A, in particular it is a Lyapunov function for L indeed it does not depend on the time variable and so it satisfies $\partial_t W = 0$ and $LW = AW \le \lambda W$. We can therefore apply Theorem 2.1.15 to deduce upper bounds on the kernels as in [27, Theorem 4.1].

Proposition 2.1.18. *Suppose that the drift satisfies*

$$\limsup_{|x|\to\infty} |x|^{-r} F(x) \cdot \frac{x}{|x|} < -c \tag{2.25}$$

for some $r > 1$ *and* $c > 0$. *Fix* $T = 1$, *then if* $\alpha > \dfrac{r+1}{r-1}$, $\delta < \dfrac{c}{\Lambda(r+1)}$, $k > N+2$

$$p(x,y,t) \le \frac{C}{t^{\frac{\alpha k r}{r+1}-1}} \exp\{-\delta t^\alpha |y|^{r+1}\}$$

for all $x,\ y \in \mathbb{R}^N$, $0 < t \le 1$ *and for a suitable constant* C.

Proof. Let us verify assumptions (2.16) and (2.17).
Let

$$W_1(x,t) = W_2(x,t) = \exp\{t^\alpha \delta_1 |x|^{r+1}\}, \quad \omega = \exp\{t^\alpha \delta |x|^{r+1}\}$$

with $\delta < \delta_1 < \dfrac{c}{\Lambda(r+1)}$. By Proposition 2.1.2 we know that W_1 is a Lyapunov function for L. Obviously $\omega \ge 1$ and $\omega \le W_1$ with constant $c_1 = 1$. We have to find $c_2(a_0,b_0)$ such that

$$|D\omega| \le c_2(a_0,b_0)\omega^{\frac{k-1}{k}} W_1^{\frac{1}{k}}$$

that is

$$\begin{aligned}\delta t^\alpha (r+1)|x|^r &\exp\left\{t^\alpha \delta |x|^{r+1}\right\} \\ &\le c_2(a_0,b_0) \exp\left\{\frac{k-1}{k}\delta t^\alpha |x|^{r+1}\right\} \exp\left\{\frac{1}{k}\delta_1 t^\alpha |x|^{r+1}\right\}\end{aligned}$$

or, equivalently,

$$\begin{aligned}\delta t^\alpha (r+1)|x|^r &\le c_2(a_0,b_0)\exp\left\{\left(\delta\frac{k-1}{k} + \frac{\delta_1}{k} - \delta\right) t^\alpha |x|^{r+1}\right\} \\ &= c_2(a_0,b_0)\exp\left\{\frac{\delta_1 - \delta}{k} t^\alpha |x|^{r+1}\right\}.\end{aligned}$$

Observing that

$$\begin{aligned}\delta t^\alpha (r+1)|x|^r &= \frac{1}{|x|}\delta(r+1)\frac{k}{\delta_1-\delta}\frac{\delta_1-\delta}{k} t^\alpha |x|^{r+1} \\ &\le \delta(r+1)\frac{k}{\delta_1-\delta}\exp\left\{\frac{\delta_1-\delta}{k} t^\alpha |x|^{r+1}\right\}\end{aligned}$$

for $|x| \geq 1$ and

$$\delta t^\alpha (r+1)|x|^r \leq \delta(r+1)$$

for $|x| < 1$, we obtain that the desired inequality is true with

$$c_2 = \delta(r+1)\max\left\{1, \frac{k}{\delta_1 - \delta}\right\},$$

independent of a_0 and b_0.
Similarly we obtain that

$$\begin{aligned}|D^2\omega| &\leq C(\delta^2 t^{2\alpha}(r+1)^2|x|^{2r} + \delta t^\alpha (r+1)(r-1+N)|x|^{r-1}) \\ &\leq c_3 \exp\left\{\frac{2(\delta_1-\delta)}{k}t^\alpha |x|^{r+1}\right\}\end{aligned}$$

with c_3 not depending on a_0 and b_0.
Concerning $c_4(a_0, b_0)$, we have

$$\begin{aligned}|\partial_t \omega| &= \delta\alpha t^{\alpha-1}|x|^{r+1}\exp\{t^\alpha \delta |x|^{r+1}\} \\ &\leq c_4(a_0,b_0)\exp\left\{\frac{k-2}{k}t^\alpha\delta|x|^{r+1}\right\}\exp\left\{\delta_1\frac{2}{k}|x|^{r+1}\right\}\end{aligned}$$

or equivalently

$$\begin{aligned}\delta\alpha t^{\alpha-1}|x|^{r+1} &= \frac{\alpha}{t}\frac{k}{2(\delta_1-\delta)}\delta\frac{2(\delta_1-\delta)}{k}t^\alpha|x|^{r+1} \\ &\leq c_4(a_0,b_0)\exp\left\{\frac{2(\delta_1-\delta)}{k}t^\alpha\delta|x|^{r+1}\right\}\end{aligned}$$

with $c_4(a_0,b_0) = \dfrac{\alpha\delta k}{2(\delta_1-\delta)a_0}$.
Finally, we have to find $c_5(a_0,b_0)$ such that

$$\exp\{\delta t^\alpha |x|^{r+1}\}|x|^{kr} \leq c_5(a_0,b_0)\exp\{\delta_1 t^\alpha |x|^{r+1}\}.$$

The function

$$f(s) = \frac{s^{kr}}{\exp\{(\delta_2-\delta)t^\alpha s^{r+1}\}}$$

attaints its maximum for $s = \dfrac{c(k,r,\delta,\delta_1)}{t^{\frac{\alpha}{r+1}}}$. Therefore $f(s) \leq \dfrac{c}{t^{\frac{\alpha k r}{r+1}}}$ and we can set

$$c_5(a_0,b_0) = \frac{c(k,r,\delta,\delta_1)}{a_0^{\frac{\alpha k r}{r+1}}}.$$

From (2.18), choosing $a_0 = \frac{1}{2}t,\ a = t,\ b = \frac{3}{2}t,\ b_0 = 2t$ and estimating ξ_1 as in Proposition 2.1.2, we deduce

$$\begin{aligned}p(x,y,t) &\leq C\left(\frac{1}{t^{\frac{\alpha k r}{r+1}-1}} + \frac{1}{t^{\frac{\alpha k r}{2(r+1)}-1}} + \frac{1}{t^{\frac{k}{2}-1}}\right)\exp\{-\delta t^\alpha |y|^{r+1}\} \\ &\leq \frac{C}{t^{\frac{\alpha k r}{r+1}-1}}\exp\{-\delta t^\alpha |y|^{r+1}\}\end{aligned}$$

for all $x,\ y \in \mathbb{R}^N$ and $t \leq 1$. $\square$

Remark 2.1.19. The estimate of the kernel proved in Proposition 2.1.18 in particular holds when A is given by $\Delta - |x|^r \frac{x}{|x|} \cdot D$. In the unidimensional case, consider for example the operator $A = D^2 - x^3 D$. We deduce the following bound for the kernel. If $\alpha > 2$, $\delta < \frac{1}{4}$, $k > 3$

$$p(x,y,t) \leq \frac{C}{t^{\frac{3\alpha k}{4}-1}} \exp\{-\delta t^{\alpha} y^4\}$$

for some positive C and for all $x,\ y \in \mathbb{R}$, $0 < t \leq 1$.

2.2 Heat kernel bounds for Schrödinger operators

A method similar to the one applied in the first section works also for Schrödinger operators. In this section, using Lyapunov functions techniques and parabolic regularity, we prove pointwise upper bounds on the kernel p.

We will deal with the problem of finding upper bounds for the kernels of Schrödinger operators in the next chapter too. The approach will be different and sometimes will give more refined estimates. Anyway, it is interesting to complete the study started in the previous section and to prove some estimates for Schrödinger operators making use of suitable Lyapunov functions.

We consider the operator $A = -\Delta + V$ with a nonnegative potential $V \in C^{\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$, $0 < \alpha < 1$. According to the results previously obtained, the semigroup e^{-tA} generated by the operator $-A$ can be represented in the form

$$e^{-tA} f(x) = \int_{\mathbb{R}^N} p(x,y,t) f(y)dy, \quad t > 0,\ x \in \mathbb{R}^N,$$

where p is a positive $C^{2+\alpha,2+\alpha,1+\frac{\alpha}{2}}_{\mathrm{loc}}$ function, symmetric with respect to x and y which is pointwise dominated by the heat kernel of the Laplacian in $\mathbb{R}^N$, see Remark 1.3.21. More refined bounds are known when the potential V tends to ∞ at infinity in a polynomial way, see [13, Corollary 4.5.5] or [45] where also lower bounds are proved. In the case of $V(x) = |x|^{\alpha}$ we obtain estimates similar to those in [45]. However our method does not allow us to prove Davies-Simon estimate. On the other hand, it is not confined to special polynomial potentials but applies also to logarithmic or exponential growths.

As in the case of Kolmogorov operators, given a Lyapunov function ω we estimate the integral of ω against the kernel p, that is the function

$$\xi_{\omega}(x,t) = \int_{\mathbb{R}^N} p(x,y,t)\omega(y,t)\, dy.$$

Then we use parabolic regularity for Schrödinger operators with unbounded coefficients to deduce L^{∞}- bounds for ωp from the L^1-bounds. The same arguments have been applied in [28] but with Lyapunov functions independent of t,

yielding estimates in the form of Davies and Simon.
To shorten the notation we use $L = \partial_t - A = \partial_t + \Delta - V$. Observe however that the parabolic operator associated with A is $\partial_t + A$ and not L.

2.2.1 Integrability of Lyapunov functions

Since p admits Gaussian estimates, it is clear that any function with, say, an exponential growth is integrable with respect to p. Taking into account the growth of the potential V it is possible to integrate functions diverging very fast at infinity.

We say that $\omega : Q_T \to [0, +\infty)$ is a Lyapunov function for the operator L if it belongs to $C^{2,1}(Q_T)$, $\lim_{|x|\to\infty}\omega(x,t) = +\infty$ uniformly with respect to t in compact sets of $(0,T]$ and there exists $h : (0,\infty) \to [0,\infty)$ integrable in a neighborhood of 0 such that $L\omega(x,t) \leq h(t)\omega(x,t)$ for all $(x,t) \in Q_T$. Note that we do not require that $\omega(x,0)$ tends to ∞ as $|x| \to \infty$.

In the proof of the proposition below we need to approximate e^{-tA} with the semigroups generated by some Schrödinger operators with bounded potentials. To this purpose we fix $0 \leq \eta \in C_c^\infty(\mathbb{R})$ decreasing such that $\eta(s) = 1$ for $|s| \leq 1$, $\eta(s) = 0$ for $|s| \geq 2$ and define $V_n(x) = \eta\left(|\frac{x}{n}|\right) V(x)$. Let moreover e^{-tA_n} be the semigroup generated by $-A_n = \Delta - V_n$ and $p_n(x,y,t)$ its kernel. By the maximum principle one easily obtains that $p_n \geq p_{n+1}$ and that $p_n \to p$ pointwise. Note that a Lyapunov function for A always exists since $V \geq 0$ (take for example $V(x) = 1 + |x|^2$, $x \in \mathbb{R}^N$) and therefore the maximum principle holds for bounded $C^{2,1}$ solutions of the Cauchy problem associated with the Schrödinger operator.

Lemma 2.2.1. *Consider the analytic semigroup generated by* $-A_n$ *in* $C_b(\mathbb{R}^N)$. *Then, for every* $f \in C_b^{2+\alpha}(\mathbb{R}^N)$ *the function* $e^{-tA_n}f(x)$ *converges to* $e^{-tA}f(x)$ *in* $C^{2,1}(\mathbb{R}^N \times [0,T])$.

Proof. Let $f \in C_b^{2+\alpha}(\mathbb{R}^N)$. Set $u_n(x,t) = e^{-tA_n}f(x)$, $u(x,t) = e^{-tA}f(x)$. Let us fix a radius $\rho > 0$. If $n > \rho + 1$, by the Schauder estimates for the operator A (see [20, Theorem 8.1.1]) we obtain

$$\|u_n\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(B_\rho\times[0,T])} \leq C(\|u_n\|_{L^\infty(\mathbb{R}^N\times[0,\infty))} + \|f\|_{C^{2+\alpha}(\mathbb{R}^N)}).$$

By Ascoli's Theorem the sequence (u_n) converges to a function v in $C^{2,1}(\mathbb{R}^N \times [0,\infty))$. Since $\partial_t u_n + A_n u_n = 0$ in $B_\rho \times (0,T]$ for $n > \rho$ we have $\partial_t v + Av = 0$ in $\mathbb{R}^N \times (0,T]$. Moreover $v(x,0) = f(x)$ and $|v(x,t)| \leq \|f\|_\infty$. Consider now the difference $w = u - v$. Obviously $w \in C^{2,1}(\mathbb{R}^N \times [0,T])$, is bounded and satisfies

$$\begin{cases} \partial_t w + Aw = 0 & \text{in} \quad \mathbb{R}^N \times (0,T] \\ w(x,0) = 0 & \text{in} \quad \mathbb{R}^N. \end{cases}$$

By the maximum principle it follows $w = 0$ and then u_n converges to u in $C^{2,1}(\mathbb{R}^N \times [0,\infty))$.
Observe that if f is only a $C_b(\mathbb{R}^N)$ function u_n converges pointwise to u. □

We also need the following lemma.

Lemma 2.2.2. *Assume that* $V \in L^\infty(\mathbb{R}^N)$ *and let* $f \in BUC(Q_T)$*. Then the function*

$$F(x,t) = \int_{\mathbb{R}^N} p(x,y,t) f(y,t)\, dy$$

is continuous in Q_T*. Moreover, if* $f \in BUC^{2,1}(Q_T)$*, then*

$$\partial_t F(x,t) = \int_{\mathbb{R}^N} p(x,y,t) Lf(y,t)\, dt$$

with $L = \partial_t - A$*.*

Proof. Since V is bounded, the semigroup $(e^{-tA})_{t\geq 0}$ is strongly continuous in $BUC(\mathbb{R}^N)$ (the space of bounded and uniformly continuous functions on $\mathbb{R}^N$). Writing $F(\cdot,t) = e^{-tA} f(\cdot,t)$ its continuity easily follows. If $f \in BUC^{2,1}(Q_T)$, then, for every fixed t, the function $f(\cdot,t)$ belongs to the domain of the generator of $(e^{-tA})_{t\geq 0}$ in $BUC(\mathbb{R}^N)$. It follows that

$$\partial_t F(\cdot,t) = -e^{-tA} A f(\cdot,t) + e^{-tA} \partial_t f(\cdot,t)$$

and the proof follows. ☐

We refer the reader to [28, Proposition 2.5] and to [5, Lemma 2.32] for results similar to the next proposition, when the Lyapunov function is independent of t.

Proposition 2.2.3. *For each* $t \in [0,T]$*, the Lyapunov function* $\omega(\cdot,t)$ *is integrable with respect to the measure* $p(x,\cdot,t)$*. Moreover, setting*

$$\xi_\omega(x,t) = \int_{\mathbb{R}^N} p(x,y,t)\omega(y,t) dy, \tag{2.26}$$

the inequality

$$\xi_\omega(x,t) \leq e^{\int_0^t h(s)ds} \omega(x,0) \tag{2.27}$$

holds.

Proof. Let us consider, for every $\alpha \geq 0$, $\psi_\alpha \in C_b^\infty(\mathbb{R})$ such that $\psi_\alpha(s) = s$ for $s \leq \alpha$, ψ_α is constant in $[\alpha+1, \infty)$, $\psi'_\alpha \geq 0$ and $\psi''_\alpha \leq 0$. From the concavity of ψ_α it follows that

$$s\psi'_\alpha(s) \leq \psi_\alpha(s) \qquad \forall\ s \geq 0. \tag{2.28}$$

Obviously $\psi_\alpha \circ \omega \in BUC(Q_T)$ and, moreover, it belongs to $BUC^{2,1}(Q(\varepsilon,T))$ for every $\varepsilon > 0$, since is constant for $t \geq \varepsilon > 0$ and large $|x|$. According with the previous notation we set $\xi_\alpha^n(x,t) = \int_{\mathbb{R}^N} p_n(x,y,t)\psi_\alpha(\omega(y,t))dy$. Lemma 2.2.2 yields for $t \geq \varepsilon$

$$\partial_t \xi_\alpha^n(x,t) = \int_{\mathbb{R}^N} p_n(x,y,t) L_n(\psi_\alpha \circ \omega)(y,t) dy$$

where $L_n = \partial_t - A_n$. By (2.28) we obtain

$$\begin{aligned} L_n(\psi_\alpha \circ \omega)(x,t) &= \psi'_\alpha(\omega(x,t))L_n\omega(x,t) + V_n(x)[\psi'_\alpha(\omega(x,t))\omega(x,t) \\ &- \psi_\alpha(\omega(x,t))] - \psi''_\alpha(\omega(x,t))|D\omega(x,t)|^2 \\ &\leq \psi'_\alpha(\omega(x,t))L_n\omega(x,t). \end{aligned}$$

Thus, for $t \geq \varepsilon$,

$$\begin{aligned} \partial_t\xi^n_\alpha(x,t) &\leq \int_{\mathbb{R}^N} p_n(x,y,t)\psi'_\alpha(\omega(y,t))L_n\omega(y,t)dy \\ &\leq \int_{\mathbb{R}^N} p_n(x,y,t)\psi'_\alpha(\omega(y,t))L\omega(y,t)dy \end{aligned}$$

if n is sufficiently large since, for fixed α, the function $\psi'_\alpha(\omega(y,t))$ has compact support. Using the property of ω, the positivity of ψ' and (2.28) again we get

$$\partial_t\xi^n_\alpha(x,t) \leq h(t)\int_{\mathbb{R}^N} p_n(x,y,t)\psi_\alpha(\omega(y,t))dy = h(t)\xi^n_\alpha(x,t).$$

Therefore, by Gronwall's Lemma, for $t \geq \varepsilon$.

$$\xi^n_\alpha(x,t) \leq e^{\int_\varepsilon^t h(s)ds}\xi_\alpha(x,\varepsilon).$$

Since $\xi_\alpha(x,\varepsilon) \to \psi_\alpha(\omega(x,0))$ as $\varepsilon \to 0$, by Lemma 2.2.2, letting $\varepsilon \to 0$ we obtain

$$\xi^n_\alpha(x,t) \leq e^{\int_\varepsilon^t h(s)ds}\psi_\alpha(\omega(x,0)).$$

Letting $\alpha \to \infty$ in the previous inequality and using Fatou's Lemma we get

$$\int_{\mathbb{R}^N} p_n(x,y,t)\omega(y,t)dy \leq \liminf_{\alpha\to\infty}\xi^n_\alpha(x,t) \leq e^{\int_0^t h(s)ds}\omega(x,0).$$

Letting $n \to \infty$, the first member in the previous inequality tends to $\xi_\omega(x,t)$ by monotone convergence so the claim follows. ☐

2.2.2 Regularity for parabolic problems and some interpolative estimates

We prove a parabolic regularity result needed in the following subsection to deduce pointwise estimates for the kernels.

Theorem 2.2.4. *Let $1 < k < \infty$ and suppose that for every $\gamma > 0$ there exists $C_\gamma > 0$ such that $|DV| \leq \gamma V^{\frac{3}{2}} + C_\gamma$. If $u \in L^k(Q_T) \cap W^{2,1}_k(B_R \times [0,T])$ for every $R > 0$ solves*

$$\begin{cases} \partial_t u - \Delta u + Vu = g & \text{in} \quad Q_T \\ u(y,0) = 0 & y \in \mathbb{R}^N \end{cases}$$

with $g \in L^k(Q_T)$, then

$$\|u\|_{W^{2,1}_k(Q_T)} + \|Vu\|_{L^k(Q_T)} \leq C_0\|g\|_{L^k(Q_T)}$$

where C_0 depends on N, k, T and C_γ.

Proof. By [31, Proposition 6.5], there exists a function $z \in W^{2,1}_k(Q_T)$ with $Vz \in L^k(Q_T)$ which solves the problem above and satisfies the estimate

$$\|z\|_{W^{2,1}_k(Q_T)} + \|Vz\|_{L^k(Q_T)} \le C\|g\|_{L^k(Q_T)}.$$

Then we have to prove that $u = z$. The difference $w = u - z \in L^k(Q_T) \cap W^{2,1}_k(B_R \times [0,T])$ for every $R > 0$ and satisfies

$$\int_{Q_T} w(-\partial_t\phi - \Delta\phi + V\phi) = 0 \tag{2.29}$$

for every $\phi \in C^{2,1}(Q_T)$ vanishing at the time T and with support in $B_R \times [0,T]$ for some $R > 0$. By density (2.29) holds for every $\phi \in W^{2,1}_{k'}(Q_T)$ such that ϕ vanishes at the time T and $V\phi \in L^{k'}(Q_T)$. By using [31, Proposition 6.5] again, we obtain that, given $\psi \in L^{k'}(Q_T)$, there exists $\phi \in W^{2,1}_{k'}(Q_T)$ with $\phi(\cdot, T) = 0$ and $V\phi \in L^{k'}(Q_T)$ such that $-\partial_t\phi - \Delta\phi + V\phi = \psi$. Therefore

$$\int_{Q_T} w\psi = 0$$

for every $\psi \in L^{k'}(Q_T)$ and then $w = 0$ and $u = v$. □

The following interpolative estimate for the sup norm of u will be crucial in the next section.

Proposition 2.2.5. *Assume that $k > \frac{N+2}{2}$. Then there exists $C > 0$ such that for every $u \in W^{2,1}_k(Q_T)$ the estimate*

$$\|u\|_{L^\infty(Q_T)} \le C\|u\|^{1-\theta}_{L^1(Q_T)}\|u\|^{\theta}_{W^{2,1}_k(Q_T)}$$

holds with

$$\theta = \frac{N+2}{(N+2)\left(1 - \dfrac{1}{k}\right) + 2}.$$

Proof. Since there exists a linear extension operator from $W^{2,1}_k(Q_T)$ to $W^{2,1}_k(\mathbb{R}^{N+1})$ which is also continuous from $L^r(Q_T)$ to $L^r(\mathbb{R}^{N+1})$ for $1 \le r \le \infty$ we prove the claimed estimate for functions in $W^{2,1}_k(\mathbb{R}^{N+1})$. Let R be an unitary cube of $\mathbb{R}^{N+1}$. We start by proving that there exists a positive constant C such that

$$\|u\|_{L^\infty(R)} \le C(\|u\|_{L^1(R)} + \|\partial_t u\|_{L^k(R)} + \|D^2 u\|_{L^k(R)})$$

for every $u \in W^{2,1}_k(R)$. Suppose that this is not true, then for every $n \in \mathbb{N}$ there exists $u_n \in W^{2,1}_k(R)$ such that

$$\|u_n\|_{L^\infty(R)} \ge n(\|u_n\|_{L^1(R)} + \|\partial_t u_n\|_{L^k(R)} + \|D^2 u_n\|_{L^k(R)}). \tag{2.30}$$

We can also suppose $\|u_n\|_{L^\infty(R)} = 1$. Obviously we have $\|u_n\|_{L^k(R)} \le 1$ and, by (2.30), we deduce that $(u_n)_{n\in\mathbb{N}}$ is bounded in $W^{2,1}_k(R)$. Since the embedding of

$W^{2,1}_k(R)$ into $C(\overline{R})$ is compact (see Theorem A.0.9), there exists a subsequence (u_{n_k}) converging in $L^\infty(R)$ to some function $v \in C(\overline{R})$. In particular (u_{n_k}) converges to v in $L^1(R)$, but, by (2.30), $\|u_n\|_{L^1(R)} \leq \frac{1}{n}$ and then $v = 0$. This is a contraddiction since $\|u_n\|_{L^\infty(R)} = 1$. It immediately follows that there exists a positive constant C such that

$$\|u\|_{L^\infty(\mathbb{R}^{N+1})} \leq C(\|u\|_{L^1(\mathbb{R}^{N+1})} + \|\partial_t u\|_{L^k(\mathbb{R}^{N+1})} + \|D^2 u\|_{L^k(\mathbb{R}^{N+1})})$$

for every $u \in W^{2,1}_k(\mathbb{R}^{N+1})$. Let $\lambda > 0$. Choosing $v(x,t) = u(\lambda x, \lambda^2 t)$, we get

$$\begin{aligned}\|u\|_{L^\infty(\mathbb{R}^{N+1})} \leq C(&\lambda^{-(N+2)}\|u\|_{L^1(\mathbb{R}^{N+1})} \\ &+ \lambda^{(2-\frac{N+2}{k})}(\|\partial_t u\|_{L^k(\mathbb{R}^{N+1})} + \|D^2 u\|_{L^k(\mathbb{R}^{N+1})}))\end{aligned}$$

for all $\lambda > 0$ and $u \in W^{2,1}_k(\mathbb{R}^{N+1})$. It follows that the function

$$\begin{aligned}g(\lambda) = \|u\|_{L^\infty(\mathbb{R}^{N+1})} - C(&\lambda^{-(N+2)}\|u\|_{L^1(\mathbb{R}^{N+1})} \\ &+ \lambda^{(2-\frac{N+2}{k})}(\|\partial_t u\|_{L^k(\mathbb{R}^{N+1})} + \|D^2 u\|_{L^k(\mathbb{R}^{N+1})})) \leq 0\end{aligned}$$

for all $\lambda > 0$ and, in particular, minimising over λ, in correspondence of

$$\lambda = \left[\frac{N+2}{2-\frac{N+2}{k}} \frac{\|u\|_{L^1(\mathbb{R}^{N+1})}}{\|\partial_t u\|_{L^k(\mathbb{R}^{N+1})} + \|D^2 u\|_{L^k(\mathbb{R}^{N+1})}}\right]^{\frac{k}{4k+Nk-N-2}},$$

we obtain then claimed inequality. □

Finally, we state an interpolative inequality.

Proposition 2.2.6. *Let $1 \leq k \leq \infty$ and suppose that for every $\gamma > 0$ there exists $C_\gamma > 0$ such that $|DV| \leq \gamma V^{\frac{3}{2}} + C_\gamma$. Then there exists two constants m, μ_0 such that for every $u \in W^{2,1}_k(Q_T)$ with $Vu \in L^k(Q_T)$ the following estimate holds for $0 < \mu \leq \mu_0$*

$$\|V^{\frac{1}{2}} Du\|_{L^k(Q_T)} \leq \mu\|u\|_{W^{2,1}_k(Q_T)} + \frac{m}{\mu}\|Vu\|_{L^k(Q_T)}.$$

Proof. Let u be a smooth function with compact support contained in $B_R \times [0,T]$ for some $R > 0$. By [31, Proposition 2.3] there exist two positive constants m, μ_0 such that for $0 < \mu \leq \mu_0$

$$\int_{\mathbb{R}^N} V(x)^{\frac{k}{2}}|Du(x,t)|^k\,dx \leq \mu^k \int_{\mathbb{R}^N} |\Delta u(x,t)|^k\,dx + \frac{m^k}{\mu^k}\int_{\mathbb{R}^N} V(x)^k|u(x,t)|^k\,dx.$$

Integrating over $[0,T]$ with respect to t, the estimate follows for smooth and with compact support functions. By density we deduce the claim. □

2.2.3 Pointwise estimates on kernels

To prove the main result of this paper we need the following assumptions on the potential V and on the Lyapunov function ω.

(A1) $0 \leq V \in C^1(\mathbb{R}^N)$ and $\forall\, \gamma > 0$ there exists $C_\gamma > 0: \ |DV| \leq \gamma V^{\frac{3}{2}} + C_\gamma$;

(A2) $0 < \omega \in C^{2,1}(\mathbb{R}^N \times ([0,\infty))$ is a Lyapunov function satisfying

$$\frac{|\partial_t \omega|}{\omega} + \frac{|D\omega|^2}{\omega^2} + \frac{|\Delta \omega|}{\omega} \leq \gamma V + C \tag{2.31}$$

where γ, C are suitable positive constants. We denote by ξ_ω the function introduced in 2.26 and relative to ω and fix $0 < a_0 < a < b < b_0 < T$ with the property $b_0 - b \geq a - a_0$.

Theorem 2.2.7. *There exists* $\gamma_0 > 0$ *such that if assumptions (A1) and (A2) are satisfied with* $\gamma < \gamma_0$*, then*

$$\omega(y,t)p(x,y,t) \leq \frac{C}{(a-a_0)^{\frac{N+2}{2}}} \int_{a_0}^{b_0} \xi_\omega(x,t)dt$$

for $a \leq t \leq b$ *and* $x,\ y \in \mathbb{R}^N$.

Proof. In the whole proof x will be considered as a parameter and we regard the kernel as a function of the variables (y,t). Similarly, all the differential operators with respect to the space variables will act on the y variable. Observe that p satisfies $p_t = \Delta p - Vp$ for $y \in \mathbb{R}^N, t > 0$. Moreover it belongs to $L^k(Q(a,b))$ for every $1 \leq k \leq \infty$ since it admits Gaussian estimates. Let η be a smooth function such that $0 \leq \eta \leq 1$, $\eta(t) = 1$ for $a \leq t \leq b$, $\eta(t) = 0$ for $t \leq a_0$ and $t \geq b_0$, $0 \leq |\eta_t| \leq \frac{2}{a-a_0}$ and set $q = \eta^k p$. Then $q \in L^k(Q_T) \cap W^{2,1}_k(B_R \times [0,T])$ for all $R > 0$ and satisfies the parabolic problem

$$\begin{cases} \partial_t q - \Delta q + Vq = k\eta^{k-1}p\eta_t & \text{in } Q_T \\ q(y,0) = 0 & y \in \mathbb{R}^N. \end{cases}$$

From Theorem 2.2.4 it follows that, for all $1 < k < \infty$, $q \in W^{2,1}_k(Q_T)$ and $Vq \in L^k(Q_T)$. In particular, from Proposition 2.2.6, $V^{\frac{1}{2}}Dq \in L^k(Q_T)$. Let $\omega_\varepsilon = \omega/(1+\varepsilon\omega)$ for $0 < \varepsilon < 1$. We have

$$\frac{D\omega_\varepsilon}{\omega_\varepsilon} = \frac{D\omega}{\omega(1+\varepsilon\omega)}; \quad \frac{\partial_t \omega_\varepsilon}{\omega_\varepsilon} = \frac{\partial_t \omega}{\omega(1+\varepsilon\omega)};$$
$$\frac{\Delta\omega_\varepsilon}{\omega_\varepsilon} = \frac{\Delta\omega}{\omega(1+\varepsilon\omega)} - \frac{2\varepsilon}{(1+\varepsilon\omega)^2}\frac{|D\omega|^2}{\omega}.$$

Using the last equations we obtain estimates like (2.31) for ω_ε, namely

$$\frac{|\partial_t \omega_\varepsilon|}{\omega_\varepsilon} + \frac{|D\omega_\varepsilon|^2}{\omega_\varepsilon^2} + \frac{|\Delta \omega_\varepsilon|}{\omega_\varepsilon} \leq 3(\gamma V + C). \tag{2.32}$$

The function $\omega_\varepsilon q$ satisfies the parabolic equation

$$\begin{cases} \partial_t(\omega_\varepsilon q) - \Delta(\omega_\varepsilon q) + V\omega_\varepsilon q = (\partial_t \omega_\varepsilon)q + k\eta^{k-1}p\omega_\varepsilon\eta_t & \\ \qquad\qquad\qquad\qquad -q\Delta\omega_\varepsilon - 2D\omega_\varepsilon \cdot Dq & \text{in } Q_T \\ \omega_\varepsilon(y,0)q(y,0) = 0 & y \in \mathbb{R}^N. \end{cases}$$

Observe that $V\omega_\varepsilon q \in L^k(Q_T)$ since ω_ε is bounded and $Vq \in L^k(Q_T)$. In a similar way we obtain that $k\eta^{k-1}p\omega_\varepsilon\eta_t \in L^k(Q_T)$. Using (2.32) we see that the other terms in the right hand side of the previous equality are in $L^k(Q_T)$. In fact we have

$$|\partial_t\omega_\varepsilon|q \leq \gamma V\omega_\varepsilon q + C\omega_\varepsilon q \in L^k(Q_T).$$

Similarly for the remaining terms. This implies that $\omega_\varepsilon q \in W_k^{2,1}(Q_T)$. We rewrite the previous equation in the form

$$\begin{cases} \partial_t(\omega_\varepsilon q) - \Delta(\omega_\varepsilon q) + V\omega_\varepsilon q = (\partial_t\omega_\varepsilon)q + k\eta^{k-1}p\omega_\varepsilon\eta_t & \\ \qquad\qquad -2\dfrac{D\omega_\varepsilon}{\omega_\varepsilon}D(\omega_\varepsilon q) - q\Delta\omega_\varepsilon + 2\dfrac{|D\omega_\varepsilon|^2}{\omega_\varepsilon}q & \text{in } Q_T \\ \omega_\varepsilon(y,0)q(y,0) = 0 & y \in \mathbb{R}^N \end{cases}$$

and estimate the L^k-norm of the right hand side choosing k greater then $\frac{N+2}{2}$. We have

$$\begin{align} \|(\partial_t\omega_\varepsilon)q\|_{L^k(Q_T)} &\leq \gamma\|\omega_\varepsilon qV\|_{L^k(Q_T)} + C\|\omega_\varepsilon q\|_{L^k(Q_T)} \tag{2.33} \\ &\leq \gamma\|\omega_\varepsilon qV\|_{L^k(Q_T)} + C\|\omega_\varepsilon q\|^{\frac{k-1}{k}}_{L^\infty(Q_T)}\left(\int_{Q(a_0,b_0)}\omega p\right)^{\frac{1}{k}} \notag \end{align}$$

$$\|k\eta^{k-1}p\omega_\varepsilon\eta_t\|_{L^k(Q_T)} \leq \frac{2k}{a-a_0}\|\omega_\varepsilon q\|^{\frac{k-1}{k}}_{L^\infty(Q_T)}\left(\int_{Q(a_0,b_0)}\omega p\right)^{\frac{1}{k}} \tag{2.34}$$

$$\begin{align} \left\|q\left(\Delta\omega_\varepsilon - 2\frac{|D\omega_\varepsilon|^2}{\omega_\varepsilon}\right)\right\|_{L^k(Q_T)} \leq 6\Big[&\gamma\|V\omega_\varepsilon q\|_{L^k(Q_T)} \tag{2.35} \\ &+ C\|\omega_\varepsilon q\|^{\frac{k-1}{k}}_{L^\infty(Q_T)}\left(\int_{Q_T}\omega q\right)^{\frac{1}{k}}\Big] \tag{2.36} \end{align}$$

and finally, using Proposition 2.2.6 and the interpolative inequality

$$\|D(\omega_\varepsilon q)\|_{L^k(Q_T)} \leq \delta\|\omega_\varepsilon q\|_{W_k^{2,1}(Q_T)} + \frac{K}{\delta}\|\omega_\varepsilon q\|_{L^k(Q_T)},$$

for all $\delta > 0$ we obtain

$$\begin{align} \left\|\frac{D\omega_\varepsilon}{\omega_\varepsilon}D(\omega_\varepsilon q)\right\|_{L^k(Q_T)} &\leq \sqrt{3}\big\{\gamma^{\frac{1}{2}}\|V^{\frac{1}{2}}D(\omega_\varepsilon q)\|_{L^k(Q_T)} \tag{2.37} \\ &+ C^{\frac{1}{2}}\|D(\omega_\varepsilon q)\|_{L^k(Q_T)}\big\} \tag{2.38} \\ &\leq \sqrt{3}\Big\{\gamma^{\frac{1}{2}}\left(\mu\|\omega_\varepsilon q\|_{W_k^{2,1}(Q_T)} + \frac{m}{\mu}\|V\omega_\varepsilon q\|_{L^k(Q_T)}\right) \notag \\ &+ C^{\frac{1}{2}}\left(\delta\|\omega_\varepsilon q\|_{W_k^{2,1}(Q_T)} + \frac{K}{\delta}\|\omega_\varepsilon q\|_{L^k(Q_T)}\right)\Big\} \notag \end{align}$$

for all $\delta > 0$ and $\mu \le \mu_0$. Setting

$$\Lambda = \frac{2}{a-a_0}\left(\int_{Q(a_0,b_0)} \omega p\right)^{\frac{1}{k}} = \frac{2}{a-a_0}\left(\int_{a_0}^{b_0} \xi_\omega(x,t)dt\right)^{\frac{1}{k}},$$

from (2.33), (2.34), (2.35) and (2.37) and Theorem 2.2.4, we obtain

$$\begin{aligned} \|\omega_\varepsilon q\|_{W_k^{2,1}(Q_T)} + \|V\omega_\varepsilon q\|_{L^k(Q_T)} \le C_0 \Big\{ \Big(k + 7C\frac{a-a_0}{2} \\ + \quad \frac{a-a_0}{2}\sqrt{3}C^{\frac{1}{2}}\frac{K}{\delta}\Big) \|\omega_\varepsilon q\|_{L^\infty(Q_T)}^{\frac{k-1}{k}}\Lambda + \left(\sqrt{3}\gamma^{\frac{1}{2}}\mu + \sqrt{3}C^{\frac{1}{2}}\delta\right)\|\omega_\varepsilon q\|_{W_k^{2,1}(Q_T)} \\ + \quad \left(7\gamma + \sqrt{3}\gamma^{\frac{1}{2}}\frac{m}{\mu}\right)\|V\omega_\varepsilon q\|_{L^k(Q_T)}\Big\} \end{aligned}$$

for all $\delta > 0$ and $\mu \le \mu_0$. Choosing γ, δ small enough so that $\sqrt{3}C_0(\gamma^{\frac{1}{2}}\mu_0 + C^{\frac{1}{2}}\delta) < 1$ and $C_0(7\gamma + \sqrt{3}\gamma^{\frac{1}{2}}m/\mu_0) < 1$ we deduce

$$\|\omega_\varepsilon q\|_{W_k^{2,1}(Q_T)} + \|V\omega_\varepsilon q\|_{L^k(Q_T)} \le C\|\omega_\varepsilon q\|_{L^\infty(Q_T)}^{\frac{k-1}{k}}\Lambda,$$

with C independent of ε. By Proposition 2.2.5 we have

$$\|\omega_\varepsilon q\|_{L^\infty(Q_T)} \le C\|\omega_\varepsilon q\|_{L^1(Q_T)}^{1-\theta}\|\omega_\varepsilon q\|_{W_k^{2,1}(Q_T)}^{\theta}$$

with $\theta = \frac{N+2}{(N+2)(1-\frac{1}{k})+2}$ and therefore

$$\|\omega_\varepsilon q\|_{W_k^{2,1}(Q_T)} \le C\Lambda\|\omega_\varepsilon q\|_{L^1(Q_T)}^{(1-\theta)\frac{k-1}{k}}\|\omega_\varepsilon q\|_{W_k^{2,1}(Q_T)}^{\theta\frac{k-1}{k}}.$$

This yields

$$\begin{aligned} \|\omega_\varepsilon q\|_{W_k^{2,1}(Q_T)} &\le C\Lambda\|\omega_\varepsilon q\|_{L^1(Q_T)}^{\left(1-\frac{N+2}{2k}\right)(1-\frac{1}{k})} \\ &\le C\Lambda\|\omega q\|_{L^1(Q_T)}^{\left(1-\frac{N+2}{2k}\right)(1-\frac{1}{k})}. \end{aligned}$$

Using again the interpolative estimate of Proposition 2.2.5 we obtain

$$\|\omega_\varepsilon q\|_{L^\infty(Q_T)} \le C\|\omega_\varepsilon q\|_{L^1(Q_T)}^{1-\theta}\|\omega_\varepsilon q\|_{W_k^{2,1}(Q_T)}^{\theta} \le C\Lambda\|\omega q\|_{L^1(Q_T)}^{\left(1-\frac{N+2}{2k}\right)}$$

and, finally, estimating the integrals of $\omega_\varepsilon q$ trough ξ_ω,

$$\omega_\varepsilon(y,t)p(x,y,t) \le C\frac{1}{(a-a_0)^{\frac{N+2}{2}}}\int_{a_0}^{b_0}\xi_\omega(x,t)dt$$

for $a \le t \le b$ and $x,\ y \in \mathbb{R}^N$. Observing that the constant in the right hand side does not depend on ε and letting $\varepsilon \to 0$ we conclude the proof. $\square$

2.2.4 Small time estimates

In this section we apply Theorem 2.2.7 to get explicit bounds, for small times, of the heat kernels of some Schrödinger operators with unbounded potentials.

Proposition 2.2.8. *Assume that* $V(x) \geq M|x|^\alpha$ *for some* $\alpha > 2$, $M > 0$. *Then there exist* $0 < c < \frac{2\sqrt{M}}{2+\alpha}$, $C > 0$ *such that*

$$p(x,y,t) \leq \frac{C}{t^{\frac{N}{2}}} \exp\left\{-ct(|x|^{1+\frac{\alpha}{2}} + |y|^{1+\frac{\alpha}{2}})\right\}$$

for all $x,\ y \in \mathbb{R}^N$ *and* $0 < t \leq 1$.

Proof. By Remark 1.3.21 we may assume that $V(x) = M|x|^\alpha$. We define $\omega(x,t) = \exp\{ct|x|^{1+\frac{\alpha}{2}}\}$. By an easy computation we get

$$\begin{aligned} L\omega(x,t) &= \omega(x,t)\big[c|x|^{1+\frac{\alpha}{2}} + c^2(1+\frac{\alpha}{2})^2 t^2 |x|^\alpha \\ &\quad + c(1+\frac{\alpha}{2})(\frac{\alpha}{2} - 1 + N)t|x|^{\frac{\alpha}{2}-1} - V(x)\big] \\ &\leq \omega(x,t)|x|^\alpha \big[c|x|^{1-\frac{\alpha}{2}} + c^2(1+\frac{\alpha}{2})^2 t^2 \\ &\quad + c(1+\frac{\alpha}{2})(\frac{\alpha}{2} - 1 + N)t|x|^{-\frac{\alpha}{2}-1} - M\big]. \end{aligned}$$

Recalling that $t \leq 1,\ \alpha \geq 2$ and $c < \frac{2\sqrt{M}}{2+\alpha}$, we see that the last member in the previous inequality is negative for $|x|$ large. If $|x|$ is small clearly there exists a positive constant λ such that $L\omega \leq \lambda \leq \lambda\omega$. This proves that ω is a Lyapunov function with $h(t) = \lambda$ and for $0 < t \leq 1$, so, from the Proposition 2.2.3, it follows that

$$\xi_\omega(x,t) \leq e^{\lambda t}\omega(x,0) = e^{\lambda t} \leq C$$

for t small. Now we verify the hypotheses of Theorem 2.2.7. Obviously the potential V is positive, smooth and it is easy to see that V satisfies (A1). Moreover

$$\begin{aligned} \frac{|D\omega|^2}{\omega^2} + \frac{|\Delta\omega|}{\omega} &\leq c^2 t^2 \left(1+\frac{\alpha}{2}\right)^2 |x|^\alpha + c\left(1+\frac{\alpha}{2}\right)\left(\frac{\alpha}{2} - 1 + N\right) t|x|^{\frac{\alpha}{2}-1} \\ &\leq \left[c^2\left(1+\frac{\alpha}{2}\right)^2 + c\left(1+\frac{\alpha}{2}\right)\left(\frac{\alpha}{2} - 1 + N\right)\right]|x|^\alpha. \end{aligned}$$

and

$$\frac{|\partial_t \omega|}{|\omega|} = c|x|^{1+\frac{\alpha}{2}} \leq c|x|^\alpha.$$

Choosing c small enough the hypotheses of Theorem 2.2.7 are fulfilled and there exists $C > 0$ such that

$$\omega(y,t)p(x,y,t) \leq \frac{C}{(a-a_0)^{\frac{N+2}{2}}} \int_{a_0}^{b_0} \xi_\omega(x,t)dt$$

for $0 < a \le t \le b \le 1$ and $x,\ y$ in $\mathbb{R}^N$. Setting $a_0 = \frac{t}{2},\ a = t,\ b = \frac{3}{2}t,\ b_0 = 2t$ we obtain

$$p(x,y,t) \le \frac{C}{t^{\frac{N+2}{2}}}\omega(y,t)^{-1}\int_{\frac{t}{2}}^{\frac{3}{2}t} e^{\lambda s}ds \le \frac{C}{t^{\frac{N}{2}}}\omega(y,t)^{-1} = \frac{C}{t^{\frac{N}{2}}}\exp\{-ct|y|^{1+\frac{\alpha}{2}}\}.$$

Using the symmetry of p in x and y one has also

$$p(x,y,t) \le \frac{C}{t^{\frac{N}{2}}}\exp\{-ct|x|^{1+\frac{\alpha}{2}}\}.$$

Multiplying the right and the left hand side in the inequalities obtained above, we deduce

$$p(x,y,t) \le \frac{C}{t^{\frac{N}{2}}}\exp\{-\frac{c}{2}t(|x|^{1+\frac{\alpha}{2}} + |y|^{1+\frac{\alpha}{2}})\}.$$

□

Proposition 2.2.9. *Assume that* $V(x) \ge M|x|^\alpha$ *for some* $0 < \alpha \le 2,\ M > 0$. *Then there exist* $0 < c < M,\ C > 0$ *such that*

$$p(x,y,t) \le \frac{C}{t^{\frac{N}{2}}}\exp\left\{-ct[(|x|^2+1)^{\frac{\alpha}{2}} + (|y|^2+1)^{\frac{\alpha}{2}}]\right\}$$

for all $x, y \in \mathbb{R}^N$ *and* $0 < t \le 1$.

Proof. As before we assume that $V(x) = M|x|^\alpha$. Let $\omega(x,t) = \exp\{ct(|x|^2+1)^{\frac{\alpha}{2}}\}$. By an easy computation we get

$$\begin{aligned} &L\omega(x,t) = \omega(x,t)\big[c(|x|^2+1)^{\frac{\alpha}{2}} + c^2\alpha^2t^2|x|^2(|x|^2+1)^{\alpha-2} \\ +\quad & c\alpha(\alpha-2)t|x|^2(|x|^2+1)^{\frac{\alpha}{2}-2} + ct\alpha N(|x|^2+1)^{\frac{\alpha}{2}-1} - V(x)\big].\end{aligned}$$

Proceeding as in the proof of the Proposition 2.2.8 we conclude the proof. □

Proposition 2.2.10. *Assume that* $V(x) \ge M\exp\{c|x|^\alpha\}$ *for some* $\alpha > 0,\ c,$ $M > 0$. *Then there exist* $c_1,\ c_2,\ C > 0$ *such that*

$$p(x,y,t) \le \frac{C}{t^{\frac{N}{2}}}\exp\{-tc_1(\exp\{c_2|x|^\alpha\} + \exp\{c_2|y|^\alpha\})\}$$

for all $x, y \in \mathbb{R}^N$ *and* $0 < t \le 1$.

Proof. As before we assume that $V(x) = M\exp\{c|x|^\alpha\}$.
Let $\omega(x,t) = \exp\{c_1t\exp\{c_2|x|^\alpha\}\}$. By an easy computation we get

$$\begin{aligned} &L\omega(x,t) = \omega(x,t)\big[c_1\exp\{c_2|x|^\alpha\} + t^2c_1^2c_2^2\alpha^2|x|^{2\alpha-2}\exp\{2c_2|x|^\alpha\} \\ +\quad & tc_1c_2\alpha^2\exp\{c_2|x|^\alpha\}|x|^{2\alpha-2} + tc_1c_2\alpha(\alpha-2+N)\exp\{c_2|x|^\alpha\}|x|^{\alpha-2} \\ -\quad & V(x)\big] = \omega(x,t)\exp\{c|x|^\alpha\}\big[c_1\exp\{(c_2-c)|x|^\alpha\} \\ +\quad & t^2c_1^2c_2^2\alpha^2|x|^{2\alpha-2}\exp\{(2c_2-c)|x|^\alpha\} + tc_1c_2\alpha^2\exp\{(c_2-c)|x|^\alpha\}|x|^{2\alpha-2} \\ +\quad & tc_1c_2\alpha(\alpha-2+N)\exp\{(c_2-c)|x|^\alpha\}|x|^{\alpha-2} - M\big].\end{aligned}$$

Recalling that $t \leq 1$, estimating the polynomial factors with exponentials and choosing c_2 small enough, we obtain that, for $|x|$ large, the last member in the previous inequality is negative. If $|x|$ is small, by continuity there exists a positive constant λ such that $A\omega \leq \lambda \leq \lambda\omega$. This proves that ω is a Lyapunov function with $h(t) = \lambda$ and for $0 < t \leq 1$ and then Proposition 2.2.3 gives $\xi_\omega(x,t) \leq C$ for t small. The potential V satisfies assumption (A1). Moreover

$$\begin{aligned}\frac{|D\omega|^2}{\omega^2} + \frac{|\Delta\omega|}{\omega} &= 2t^2 c_1^2 c_2^2 \alpha^2 \exp\{2c_2|x|^\alpha\}|x|^{2\alpha-2} \\ &+ tc_1c_2\alpha^2 \exp\{c_2|x|^\alpha\}|x|^{2\alpha-2} \\ &+ tc_1c_2\alpha(\alpha-2+N)\exp\{c_2|x|^\alpha\}|x|^{\alpha-2}\end{aligned}$$

and

$$\frac{|\partial_t\omega|}{|\omega|} = c_1 \exp\{c_2|x|^\alpha\}.$$

Therefore (A2) is satisfied choosing c_1 and c_2 small enough and Theorem 2.2.7 yields

$$\omega(y,t)p(x,y,t) \leq \frac{C}{(a-a_0)^{\frac{N+2}{2}}} \int_{a_0}^{b_0} \xi_\omega(x,t)dt$$

for $0 < a \leq t \leq b \leq 1$ and $x,\ y$ in $\mathbb{R}^N$. As in Proposition 2.2.8 one concludes the proof. ☐

Proposition 2.2.11. *Assume $V(x) \geq M\log(1+|x|^2)$. Then there exists $C > 0$ and $\alpha < M$ such that*

$$p(x,y,t) \leq \frac{C}{t^{\frac{N}{2}}}(1+|x|^2)^{-\frac{\alpha}{2}t}(1+|y|^2)^{-\frac{\alpha}{2}t}$$

for all $x,\ y \in \mathbb{R}^N$ and $0 < t \leq 1$.

Proof. Let $\omega(x,t) = (1+|x|^2)^{\alpha t}$. Then

$$\begin{aligned}L\omega(x,t) &= \omega(x,t)\Bigg[\alpha\log(1+|x|^2) + \frac{\alpha t(\alpha t-1)4|x|^2}{(1+|x|^2)^2} + \frac{2\alpha t N}{1+|x|^2} \\ &- M\log(1+|x|^2)\Bigg] \leq 0\end{aligned}$$

for $|x|$ large since $t \leq 1$ and $\alpha < M$. Hence ω is a Lyapunov function. Moreover V satisfies (A1) and

$$\frac{|\partial_t\omega|}{\omega} = \alpha\log(1+|x|^2),$$

$$\frac{|D\omega|^2}{\omega^2} + \frac{|\Delta\omega|}{\omega} \leq 4\alpha^2\frac{|x|^2}{(1+|x|^2)^2} + 4\alpha(\alpha+1)\frac{|x|^2}{(1+|x|^2)^2} + \frac{2\alpha N}{1+|x|^2}.$$

Choosing α small enough we can apply Theorem 2.2.7 and obtain

$$\omega(y,t)p(x,y,t) \leq \frac{C}{(a-a_0)^{\frac{N+2}{2}}} \int_{a_0}^{b_0} \xi_\omega(x,t)dt.$$

for $0 < a \leq t \leq b \leq 1$ and $x,\ y$ in $\mathbb{R}^N$. Arguing as in the examples before, one concludes the proof. $\square$

Remark 2.2.12. We can easily add a Gaussian term in our estimates as follows. For example, multiplying the left and the right hand side in Proposition 2.2.8 respectively with the left and right hand side of the Gaussian bound

$$p(x,y,t) \leq \frac{C}{t^{\frac{N}{2}}} \exp\left\{-c\frac{|x-y|^2}{t}\right\},$$

we find

$$p(x,y,t) \leq \frac{C}{t^{\frac{N}{2}}} \exp\left\{-c_1 t(|x|^{1+\frac{\alpha}{2}} + |y|^{1+\frac{\alpha}{2}})\right\} \exp\left\{-c_2\frac{|x-y|^2}{t}\right\}$$

for suitable $c_1,\ c_2\ , C > 0$. The other cases are similar.

Remark 2.2.13. Finally we discuss the sharpness of the estimate proving lower bounds similar to the upper bounds obtained in the examples above with the method of [13, Theorem 4.5.10].

We start with the potential $V(x) = |x|^\alpha$, $0 < \alpha \leq 2$, considered in Proposition 2.2.9. We consider the ball $B_1(x)$ of center x and radius 1 and the Schrödinger operator A_D in $B_1(x)$ with Dirichlet boundary conditions. The maximum principle yields $e^{-tA} \geq e^{-tA_D}$ in $B_1(x)$. Since $V \leq (1+|x|)^\alpha$ in $B_1(x)$ we have $e^{-tA} \geq e^{-tA_D} \geq e^{-t(1+|x|^\alpha)}e^{-t\Delta_D}$ in $B_1(x)$, where Δ_D is the Laplacian with Dirichlet boundary conditions. Taking the inequality for the corresponding kernels and using the estimate

$$p_{\Delta_D}(x,x,t) \geq ct^{-N/2},$$

see [13, Lemma 3.3.3], we obtain

$$p_A(x,x,t) \geq e^{-t(1+|x|^\alpha)} p_{\Delta_D}(x,x,t) \geq \frac{C}{t^{\frac{N}{2}}} e^{-t(1+|x|^\alpha)}$$

for some positive constant C. This shows that Proposition 2.2.9 is sharp, concerning the exponent α appearing in the exponential. Our method does not give a precise estimate of the constant c which, however, turns out to be $1+\varepsilon$, see [45] and the next chapter.

In a similar way we obtain that, if $V(x) = \exp\{c|x|^\alpha\}$ for some $\alpha,\ c > 0$, then, as above,

$$p(x,x,t) \geq \frac{C}{t^{\frac{N}{2}}} \exp\{-t\exp\{c(1+|x|)^\alpha\}\}.$$

Therefore in the case of exponential potentials the estimate in 2.2.10 is sharp, with the exception of constants c_1, c_2.

For a logarithmic potentials $V = M\log(1+|x|^2)$ of Proposition 2.2.11, the same method gives the lower bound

$$p(x,x,t) \geq \frac{C}{t^{\frac{N}{2}}} \exp\{-t\log[1+(1+|x|)^2]\} = \frac{C}{t^{\frac{N}{2}}}(1+(1+|x|)^2)^{-Mt}.$$

Finally we consider the case of $V(x) = |x|^\alpha$ with $\alpha > 2$, see Proposition 2.2.8. As in [45] we have

$$p(x,x,t) = \sum_n e^{-\lambda_n t}\phi_n(x)^2 \geq e^{-\lambda_1 t}\phi_1(x)^2$$

where $(\phi_n), (\lambda_n)$ are the eigenfunctions and the eigenvalues of $-A$, respectively. Since

$$\phi_1(x) \geq C\exp\{-c|x|^{1+\alpha/2}\},$$

see [13, Corollary 4.5.7], we see that, for a fixed t, Proposition 2.2.8 gives the exact decay in the space variables. Also in this case we refer the reader to [45] and to the next chapter for more precise space-time estimates.

2.2.5 Large time estimates

As in [45], large time estimates are easily deduced from small time estimates.

Proposition 2.2.14. *Let λ_1 be the smallest eigenvalue of A. Then there exist positive constants C, c, δ such that for $t \geq 1, x \in \mathbb{R}^N$*

$$p(x,x,t) \leq Ce^{-\lambda_1 t}\exp\{-c|x|^{1+\frac{\alpha}{2}}\}$$

if $V(x) \geq M|x|^\alpha$ and $\alpha > 2$,

$$p(x,x,t) \leq Ce^{-\lambda_1 t}\exp\{-c(|x|^2+1)^{\frac{\alpha}{2}}\}$$

if $V(x) \geq M|x|^\alpha$ and $0 < \alpha \leq 2$,

$$p(x,x,t) \leq Ce^{-\lambda_1 t}\exp\{-c\exp\{c|x|^\alpha\}\}$$

if $V(x) \geq M\exp\{c_1|x|^\alpha\}$ and

$$p(x,x,t) \leq Ce^{-\lambda_1 t}(1+|x|^2)^{-\delta}$$

if $V(x) \geq M\log(1+|x|^2)$.

Proof. Let e^{-tA} be the semigroup generated by $-A$. We note that

$$\|e^{-tA}\|_{L^2\to L^2} = e^{-\lambda_1 t}; \tag{2.39}$$

$$e^{-tA}p(x,\cdot,s) = p(x,\cdot,s+t) \tag{2.40}$$

and

$$p(x,x,t) = \|p(x,\cdot,t/2)\|_{L^2}^2 \tag{2.41}$$

for all $t, s > 0$ and $x \in \mathbb{R}^N$. Therefore, if $t > 1$, by (2.39), (2.40) and (2.41), we have

$$\begin{aligned} p(x,x,t) &= \left\| p(x,\cdot,\frac{t}{2}) \right\|_{L^2}^2 = \|e^{-(t/2-1/2)A}p(x,\cdot,1/2)\|_{L^2}^2 \\ &\leq e^{-\lambda_1(t-1)}\|p(x,\cdot,1/2)\|_{L^2}^2 = Ce^{-\lambda_1 t}p(x,x,1). \end{aligned}$$

Estimating $p(x,x,1)$ as in Propositions 2.2.8, 2.2.9, 2.2.10 and 2.2.11, the proof follows. □

Remark 2.2.15. Off-diagonal estimates for large times can be deduced from on-diagonal bounds by the following computation

$$\begin{aligned}
|p(x,y,t)| &= \left|\int p(x,z,t/2)p(z,y,t/2)\right| dz \leq \|p(x,\cdot,t/2)\|_2 \|p(y,\cdot,t/2)\|_2 \\
&= p(x,x,t)^{\frac{1}{2}} p(y,y,t)^{\frac{1}{2}}.
\end{aligned}$$

As in Remark 2.2.12, a Gaussian factor can be added to all the estimates of this section.

Chapter 3

Kernel estimates for a class of Schrödinger semigroups

3.1 Introduction

We consider again a Schrödinger operator $A = -\Delta + V$ with a nonnegative potential $V \in L^1_{\text{loc}}(\mathbb{R}^N)$ and we look for some sharp estimates for the kernel p of the semigroup e^{-tA} generated by the operator $-A$ in $L^p(\mathbb{R}^N)$. As previously observed, the kernel is pointwise dominated by the heat kernel of the Laplacian in $\mathbb{R}^N$.

In the case $V(x) = |x|^\alpha$, $\alpha > 0$, Sikora proves precise on-diagonal bounds of the form $p(x, x, t) \leq h(x, t)$ and then he deduces off-diagonal bounds from the semigroup law, see [45]. Estimates of the same forme have been deduced in the previous chapter and will be improved here.

In Section 2 we prove Sikora-type bounds for radial increasing potentials and we treat also the case of potentials consisting of a radial part and lower order terms.

In Section 3, we report on some upper and lower bounds obtained by Sikora in suitable space-time regions to show the sharpness of our estimates.

In Section 4, we study the asymptotic distribution of eigenvalues of A using the bounds on the heat kernel of e^{-tA} and a Tauberian theorem due to Karamata. When V has a polynomial behaviour, these results have been proved by Titchmarsh (see [51] or [40, Section XIII]) using cube-decomposition methods. Our approach allows us to treat also non polynomial type potential and this seems to be new.

3.2 Pointwise estimates of kernels

Given a positive potential $V \in L^1_{loc}(\mathbb{R}^N)$, for each $s > 0$ we consider the level set

$$E_s = \{x \in \mathbb{R}^N : V(x) \le s\}.$$

We introduce a new potential V_s

$$V_s(x) = \begin{cases} s & \text{in } E_s \\ V(x) & \text{in } \mathbb{R}^N \setminus E_s \end{cases}$$

and the heat kernel p_s of the Schrödinger operator $A_s = -\Delta + V_s$.
Let us observe that $V_s \ge s$ and $V_s \ge V$. Therefore by Remark 1.3.21 it follows that

$$0 \le p_s(x,y,t) \le \frac{1}{(4\pi t)^{\frac{N}{2}}} \exp\left\{-\frac{|x-y|^2}{4t}\right\} \exp\{-ts\} \tag{3.1}$$

and

$$0 \le p(x,y,t) \le \frac{1}{(4\pi t)^{\frac{N}{2}}} \exp\left\{-\frac{|x-y|^2}{4t}\right\} \tag{3.2}$$

for all $x,\ y \in \mathbb{R}^N$ and $t > 0$. To improve the bound for p, as in [45], we estimate the difference between the kernels p and p_s and then we use the triangle inequality. Sikora used the functional calculus to estimate such a difference. Our approach, though more elementary, yields more precise bounds.

Lemma 3.2.1. *Let $p_s,\ E_s$ as above. Then there exists a positive constant $C = C(N)$ such that for all $x \in \mathbb{R}^N,\ t > 0$*

$$|p_s(x,x,t) - p(x,x,t)| \le \frac{C}{t^{\frac{N}{2}}} \int_{E_s} \frac{\exp\left\{-\frac{|x-y|^2}{4t}\right\}}{|x-y|^N} dy. \tag{3.3}$$

Remark 3.2.2. Let us observe that the integral in the right hand side above is divergent whenever $x \in E_s$. Therefore (3.3) is meaningful only if $x \notin E_s$.

Proof. Let $u,\ w$ respectively the solutions of

$$\begin{cases} u_t = \Delta u - Vu \\ u(0) = f \end{cases}$$

and

$$\begin{cases} w_t = \Delta w - V_s w \\ w(0) = f. \end{cases}$$

Then the difference $z = u - w$ satisfies $z_t = \Delta z - V_s z - (V - V_s)u,\ z(0) = 0$ and, by the variation of constants formula,

$$z(t) = -\int_0^t e^{-(t-r)A_s}(V - V_s)u(r)dr.$$

Representing the semigroup generated by $-A_s$ in the integral form through the kernel p_s we get

$$z(x,t) = -\int_0^t dr \int_{\mathbb{R}^N} p_s(x,y,t-r)(V(y)-V_s(y))u(y,r)\,dy.$$

Representing now u through the kernel p and using (3.1) and (3.2) we obtain

$$\begin{aligned}|z(x,t)| &\leq \int_0^t dr \int_{\mathbb{R}^N} dy \int_{\mathbb{R}^N} p_s(x,y,t-r)|V(y)-V_s(y)|p(y,l,r)|f(l)|\,dl \\ &\leq \frac{1}{(4\pi)^N}\int_0^t dr \int_{\mathbb{R}^N} dy \int_{\mathbb{R}^N} \frac{1}{(r(t-r))^{\frac{N}{2}}} \exp\left\{-\frac{|x-y|^2}{4(t-r)}\right\} \exp\{-(t-r)s\}\times \\ &|V(y)-V_s(y)| \exp\left\{-\frac{|y-l|^2}{4r}\right\} |f(l)|\,dl.\end{aligned}$$

By definition $V - V_s = 0$ in $\mathbb{R}^N \setminus E_s$ and $|V-V_s| \leq s$ in E_s , then

$$\begin{aligned}|z(x,t)| \leq \frac{s}{(4\pi)^N}\int_0^t dr \int_{E_s} dy \int_{\mathbb{R}^N} \frac{1}{(r(t-r))^{\frac{N}{2}}} \exp\left\{-\frac{|x-y|^2}{4(t-r)}\right\} \\ \times \exp\{-(t-r)s\} \exp\left\{-\frac{|y-l|^2}{4r}\right\} |f(l)|\,dl.\end{aligned}$$

On the other hand

$$z(x,t) = u(x,t) - w(x,t) = \int_{\mathbb{R}^N} [p(x,l,t) - p_s(x,l,t)]f(l)dl.$$

Comparing this representation and the estimate above we deduce a bound for the difference of the kernels

$$\begin{aligned}|p(x,x,t) - p_s(x,x,t)| \leq \frac{s}{(4\pi)^N}\int_{E_s} dy \int_0^t \frac{1}{(r(t-r))^{\frac{N}{2}}} \exp\left\{-\frac{|x-y|^2}{4(t-r)}\right\} \\ \times \exp\{-(t-r)s\} \exp\left\{-\frac{|x-y|^2}{4r}\right\} dr.\end{aligned}$$

We split the integral over $[0,t]$ as the sum of the integrals over $[0,t/2]$ and $[t/2,t]$. Let us consider the first one. In $[0,t/2]$, $(t-r)^{\frac{N}{2}} \geq \left(\frac{t}{2}\right)^{\frac{N}{2}}$ and $t - r \leq t$, therefore $\exp\left\{-\frac{|x-y|^2}{4(t-r)}\right\} \leq \exp\left\{-\frac{|x-y|^2}{4t}\right\}$ and

$$\begin{aligned}&\int_0^{\frac{t}{2}} \frac{1}{(r(t-r))^{\frac{N}{2}}} \exp\left\{-\frac{|x-y|^2}{4(t-r)}\right\} s\exp\{-(t-r)s\} \exp\left\{-\frac{|x-y|^2}{4r}\right\} dr \\ &\leq \left(\frac{2}{t}\right)^{\frac{N}{2}} \exp\left\{-\frac{|x-y|^2}{4t}\right\} \int_0^{\frac{t}{2}} \frac{1}{r^{\frac{N}{2}}} s\exp\{-(t-r)s\} \exp\left\{-\frac{|x-y|^2}{4r}\right\} dr.\end{aligned}$$

Similarly

$$\int_{\frac{t}{2}}^{t} \frac{1}{(r(t-r))^{\frac{N}{2}}} \exp\left\{-\frac{|x-y|^2}{4(t-r)}\right\} s\exp\{-(t-r)s\} \exp\left\{-\frac{|x-y|^2}{4r}\right\} dr$$
$$\le \left(\frac{2}{t}\right)^{\frac{N}{2}} \exp\left\{-\frac{|x-y|^2}{4t}\right\} \int_{\frac{t}{2}}^{t} \frac{1}{(t-r)^{\frac{N}{2}}} s\exp\{-(t-r)s\}$$
$$\times \exp\left\{-\frac{|x-y|^2}{4(t-r)}\right\} dr.$$

The function $g(r) = \frac{1}{r^{\frac{N}{2}}} \exp\left\{-\frac{|x-y|^2}{4r}\right\}$ attaints its maximum at

$$r = \frac{|x-y|^2}{2N},$$

so $g(r) \le \left(\frac{2N}{e}\right)^{\frac{N}{2}} \frac{1}{|x-y|^N}$. Therefore

$$\frac{s}{(4\pi)^N} \int_{E_s} dy \int_0^{\frac{t}{2}} \frac{1}{(r(t-r))^{\frac{N}{2}}} \exp\left\{-\frac{|x-y|^2}{4(t-r)}\right\} \exp\{-(t-r)s\}$$
$$\times \exp\left\{-\frac{|x-y|^2}{4r}\right\} \le \frac{1}{(4\pi)^N} \left(\frac{4N}{e}\frac{1}{t}\right)^{\frac{N}{2}} \int_{E_s} \exp\left\{-\frac{|x-y|^2}{4t}\right\}$$
$$\times \frac{1}{|x-y|^N} \int_0^{\frac{t}{2}} s\exp\{-(t-r)s\}\, dr\, dy$$
$$= C(N)\frac{1}{t^{\frac{N}{2}}} \exp\{-ts\} \left(\exp\left\{\frac{t}{2}s\right\} - 1\right) \int_{E_s} \exp\left\{-\frac{|x-y|^2}{4t}\right\} \frac{1}{|x-y|^N}\, dy$$
$$\le C(N)\frac{1}{t^{\frac{N}{2}}} \int_{E_s} \exp\left\{-\frac{|x-y|^2}{4t}\right\} \frac{1}{|x-y|^N}\, dy.$$

Similar computations yield

$$\frac{s}{(4\pi)^N} \int_{E_s} dy \int_{\frac{t}{2}}^{t} \frac{1}{(r(t-r))^{\frac{N}{2}}} \exp\left\{-\frac{|x-y|^2}{4(t-r)}\right\} \exp\{-(t-r)s\}$$
$$\exp\left\{-\frac{|x-y|^2}{4r}\right\} dr \le C(N)\frac{1}{t^{\frac{N}{2}}} \int_{E_s} \exp\left\{-\frac{|x-y|^2}{4t}\right\} \frac{1}{|x-y|^N}\, dy$$

and the proof is complete. ☐

Theorem 3.2.3. *There exists a positive constant $C = C(N)$ such that for all $s > 0$, $x \in \mathbb{R}^N$, $t > 0$*

$$p(x,x,t) \le \frac{1}{(4\pi t)^{\frac{N}{2}}} \exp\{-ts\} + \frac{C}{t^{\frac{N}{2}}} \int_{E_s} \frac{\exp\left\{-\frac{|x-y|^2}{4t}\right\}}{|x-y|^N} dy. \tag{3.4}$$

Proof. The proof easily follows from (3.1) and Lemma 3.2.1. □

Assuming that the Lebesgue measure of the level sets E_s is finite, we deduce the following result.

Corollary 3.2.4. *There exists a positive constant $C = C(N)$ such that for all $s > 0$, $x \in \mathbb{R}^N \setminus E_s$ and $t > 0$*

$$p(x,x,t) \leq \frac{1}{(4\pi t)^{\frac{N}{2}}} \exp\{-ts\} + \frac{C}{t^{\frac{N}{2}}} |E_s| \frac{\exp\left\{-\frac{d(x,E_s)^2}{4t}\right\}}{d(x,E_s)^N} dy. \tag{3.5}$$

The estimate just obtained can be more explicitly written if we ask further assumptions on the potential. In particular, for radial, increasing potentials we have the upper bound stated in the following corollary.

Corollary 3.2.5. *If V is radial and increasing ($|x| < |y|$ implies $V(x) < V(y)$), then for all $x \in \mathbb{R}^N$, $t > 0$, $0 < c < 1$*

$$p(x,x,t) \leq \frac{1}{(4\pi t)^{\frac{N}{2}}} \exp\{-tV(cx)\} + \frac{C(N)}{t^{\frac{N}{2}}} \frac{c^N \omega_N}{(1-c)^N} \exp\left\{-\frac{(1-c)^2|x|^2}{4t}\right\}.$$

Proof. Let $x \in \mathbb{R}^N$. If we choose $s = V(cx)$, from the assumptions on V we deduce that the level set E_s coincides whit the ball $B(0, c|x|)$. Moreover, since $0 < c < 1$, $x \notin E_s$. Then (3.5) holds and the bound easily follows. □

Potentials like $|x|^\alpha$, $\alpha > 0$, belong to the class of radial, increasing potentials, so from Corollary 3.2.5 we deduce the following upper bound which improves that of [45].

Example 3.2.6. Let $V(x) = M|x|^\alpha$ with $\alpha > 0$, then for all $0 < c < 1$, $x \in \mathbb{R}^N$ and $t > 0$

$$p(x,x,t) \leq \frac{1}{(4\pi t)^{\frac{N}{2}}} \exp\{-tMc^\alpha|x|^\alpha\} + \frac{C(N)}{t^{\frac{N}{2}}} \frac{c^N \omega_N}{(1-c)^N} \exp\left\{-\frac{(1-c)^2|x|^2}{4t}\right\}$$

where ω_N is the measure of the unitary ball in $\mathbb{R}^N$.

Remark 3.2.7. Similar bounds can be obtained for low-order perturbation of the potentials above, that is if $V(x) = |x|^\alpha + o(|x|^\alpha)$, as $|x| \to \infty$. In fact for every $\varepsilon > 0$ there exist C_ε, $C'_\varepsilon > 0$ such that

$$(1-\varepsilon)|x|^\alpha + C_\varepsilon \leq V(x) \leq (1+\varepsilon)|x|^\alpha + C'_\varepsilon$$

and then, by Corollary 1.3.21,

$$p(x,x,t) \leq e^{-C_\varepsilon t} p_\varepsilon(x,x,t),$$

where p_ε is the heat kernel of the Schrödinger operator with potential $(1-\varepsilon)|x|^\alpha$. By Example 3.2.6, for every $0 < c < 1$,

$$\begin{aligned} p(x,x,t) \quad &\leq \quad e^{-C_\varepsilon t} \left\{ \frac{1}{(4\pi t)^{\frac{N}{2}}} \exp\{-t(1-\varepsilon)c^\alpha|x|^\alpha\} \right. \\ &+ \quad \left. \frac{C(N)}{t^{\frac{N}{2}}} \frac{c^N \omega_N}{(1-c)^N} \exp\left\{-\frac{(1-c)^2|x|^2}{4t}\right\} \right\}. \end{aligned}$$

Therefore, given $0 < \widetilde{c} < 1$, it is sufficient to choose $\varepsilon > 0$ such that $c = \frac{\widetilde{c}}{(1-\varepsilon)^{\frac{1}{\alpha}}} < 1$ to obtain

$$\begin{aligned} p(x,x,t) \le e^{-C_\varepsilon t}\bigg\{ & \frac{1}{(4\pi t)^{\frac{N}{2}}} \exp\{-t\widetilde{c}^\alpha |x|^\alpha\} \\ & + \frac{C(N)}{t^{\frac{N}{2}}} \frac{c^N \omega_N}{(1-c)^N} \exp\left\{-\frac{(1-c)^2|x|^2}{4t}\right\}\bigg\}. \end{aligned}$$

Remark 3.2.8. Estimate for potentials going to infinity in a different way in different directions can be, sometimes, easily obtained from the previous results. For example, if $V(x,y) = x^2 + y^4$ in $\mathbb{R}^2$, then the heat kernel is the product of the heat kernels of the two one-dimensional operators $-D^2 + x^2$, $-D^2 + y^4$ which follow into the range of application of Example 3.2.6.

Remark 3.2.9. Using the semigroup law it is possible to deduce from the on-diagonal estimates just obtained some off-diagonal estimates. It is sufficient to recall that

$$p(x,y,t) = \int_{\mathbb{R}^N} p(x,z,\frac{t}{2})p(z,y,\frac{t}{2})dz. \tag{3.6}$$

In particular

$$p(x,x,t) = \|p(x,\cdot,\frac{t}{2})\|^2_{L^2}.$$

Therefore

$$p(x,y,t) \le p(x,x,t)^{\frac{1}{2}} p(y,y,t)^{\frac{1}{2}},$$

and applying the on-diagonal bounds one can estimate the right hand side.

3.3 Estimates in space-time regions

Considering suitable space-time regions, one can control the gaussian term in Theorem 3.2.3 and its corollaries with the first addendum. In what follows we consider the operator $A = -\Delta + V$ with $V(x) = |x|^\alpha$ but in a similar way bounds in regions can be obtained for other radial, increasing potentials. Moreover it is possible to prove that in these regions similar lower estimates hold and so the estimates are sharp. We refer to [45] for the next results which, however, we recall and prove here for a future discussion in the next section (see Remark 3.4.3).

In the next result, λ_1 is the first eigenvalue of A.

Proposition 3.3.1. *There exist positive constant* c_1, c_2, c_3, c_4, C_1, C_2, C_3, C_4 *such that, if* $t \le (1+|x|)^{1-\frac{\alpha}{2}}$,

$$\frac{C_1}{t^{\frac{N}{2}}} \exp\{-c_1 t|x|^\alpha\} \le p(x,x,t) \le \frac{C_2}{t^{\frac{N}{2}}} \exp\{-c_2 t|x|^\alpha\}$$

and, if $t > (1+|x|)^{1-\frac{\alpha}{2}}$,

$$C_3 e^{-\lambda_1 t} \exp\{-c_3|x|^{1+\frac{\alpha}{2}}\} \le p(x,x,t) \le C_4 e^{-\lambda_1 t} \exp\{-c_4|x|^{1+\frac{\alpha}{2}}\}.$$

Proof. Suppose first $t \leq (1+|x|)^{1-\frac{\alpha}{2}}$.
The upper bound easily follows observing that the gaussian term in Theorem 3.2.3 can be controlled with the first addendum. Indeed for $\alpha \leq 2$ we have

$$\begin{aligned} t|x|^\alpha &\leq (1+|x|)^{1-\frac{\alpha}{2}}|x|^\alpha \leq (1+|x|)^{1-\frac{\alpha}{2}}(1+|x|)^\alpha \\ &= (1+|x|)^{1-\frac{\alpha}{2}}(1+|x|)^{\alpha-2}(1+|x|)^2 \\ &= \frac{(1+|x|)^2}{(1+|x|)^{1-\frac{\alpha}{2}}} \leq \frac{2}{(1+|x|)^{1-\frac{\alpha}{2}}} + \frac{2|x|^2}{(1+|x|)^{1-\frac{\alpha}{2}}} \leq 2 + \frac{2|x|^2}{t} \end{aligned}$$

and for $\alpha > 2$

$$\begin{aligned} t|x|^\alpha = t|x|^{\alpha-2}|x|^2 &\leq (1+|x|)^{1-\frac{\alpha}{2}}|x|^{\alpha-2}|x|^2 \\ &\leq (1+|x|)^{1-\frac{\alpha}{2}}(1+|x|)^{\alpha-2}|x|^2 \leq \frac{|x|^2}{t}. \end{aligned}$$

Concerning the lower bound we refer to [45, Proposition 6.1].
If $t > (1+|x|)^{1-\frac{\alpha}{2}}$, the lower bound follows as in Remark 2.2.13 and the upper bound as in the proof of Proposition 2.2.14. □

Let us now consider small times, say $0 < t \leq 1$. We need also to distinguish between the cases $\alpha < 2$ and $\alpha \geq 2$.

Proposition 3.3.2. *If p is the heat kernel corresponding to the operator $-\Delta + |x|^\alpha$ with $\alpha < 2$ then for every $\varepsilon > 0$ there exist positive constants C_ε and C'_ε such that for $t \leq 1$*

$$\frac{C_\varepsilon}{t^{\frac{N}{2}}} \exp\{-(1+\varepsilon)t|x|^\alpha\} \leq p(x,x,t) \leq \frac{C'_\varepsilon}{t^{\frac{N}{2}}} \exp\{-(1-\varepsilon)t|x|^\alpha\}.$$

Proof. By Remark 2.2.13 we know that

$$p(x,x,t) \geq \frac{C}{t^{\frac{N}{2}}} \exp\{-t(|x|+1)^\alpha\}.$$

Observe that, given $\varepsilon > 0$, there exists $M_\varepsilon > 0$ such that

$$(|x|+1)^\alpha = |x|^\alpha + 1 + o(|x|^\alpha) \leq (1+\varepsilon)|x|^\alpha + M_\varepsilon \leq (1+\varepsilon)|x|^\alpha + \frac{M_\varepsilon}{t}$$

and so the lower bound follows. Concerning the upper bound it is sufficient to choose $c_\varepsilon = (1-\varepsilon)^{\frac{1}{\alpha}}$ in Example 3.2.6 and to observe that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\frac{(1-c_\varepsilon)^2}{4}|x|^2 \geq (1-\varepsilon)|x|^\alpha + C_\varepsilon.$$

□

On the other hand, if $\alpha \geq 2$, $1 - \frac{\alpha}{2} \leq 0$ and $(1+|x|)^{1-\frac{\alpha}{2}} \leq 1$. So, by Proposition 3.3.1, for $0 < t \leq (1+|x|)^{1-\frac{\alpha}{2}}$,

$$p(x,x,t) \leq \frac{C}{t^{\frac{N}{2}}} \exp\{-ct|x|^\alpha\}$$

and, for $(1+|x|)^{1-\frac{\alpha}{2}} < t \le 1$,

$$p(x,x,t) \le Ce^{-\lambda_1 t}\exp\{-c|x|^{1+\frac{\alpha}{2}}\}.$$

In any case, if $0 < t \le 1$, $\alpha \ge 2$, we have

$$p(x,x,t) \le \frac{C}{t^{\frac{N}{2}}}\exp\{-ct|x|^{1+\frac{\alpha}{2}}\}$$

for suitable positive constants $C,\ c$.
In the next section we will see that a similar lower bound cannot be true.

We observe that the results just proved improve the ones obtained in the previous chapter.

3.4 The asymptotic distribution of the eigenvalues

In this section we investigate the asymptotic distribution of the eigenvalues of $-\Delta+V$, when $V(x)=|x|^\alpha$ or $V(x)=\exp\{|x|^\alpha\}$. Theorem 3.4.2 and Proposition 3.4.4 can be deduced from [51, Section 17.8] or [40, Section XIII], where the proof is different. Instead of using cube decompositions or pointwise estimates on the resolvent we apply the bounds on the heat kernels obtained in the previous sections. This allows us to treat potentials having more than polynomial growth, see Proposition 3.4.5 which seems to be new.
Denote by

$$0 < \lambda_1 \le \lambda_2 \le ...$$

the eigenvalues of A and, for $\lambda > 0$, let $N(\lambda)$ be the number of λ_j such that $\lambda_j \le \lambda$. From the Spectral Theorem it follows that the eigenvalues of e^{-tA} are $e^{-\lambda_n t}$, $n \in \mathbb{N}$. The following well-known Proposition is usually obtained as a corollary of the classical Mercer's Theorem. For completeness, we provide a simple proof based on the semigroup property of the kernel.

Proposition 3.4.1. *Let $t>0$. Then*

$$\int_{\mathbb{R}^N} p(x,x,t)\,dx = \sum_{n=1}^{\infty} e^{-\lambda_n t}.$$

Proof. By the estimates in the previous sections it follows $p(x,x,t) \in L^1(\mathbb{R}^N)$. By the semigroup law and the symmetry of p

$$p(x,y,t) = \int_{\mathbb{R}^N} p(x,z,\frac{t}{2})p(y,z,\frac{t}{2})\,dz,$$

in particular

$$p(x,x,t) = \int_{\mathbb{R}^N} p(x,z,\frac{t}{2})^2\,dz$$

and

$$\int_{\mathbb{R}^N} p(x,x,t)\,dx = \int\int_{\mathbb{R}^N\times\mathbb{R}^N} p(x,z,\frac{t}{2})^2\,dx\,dz.$$

Therefore $p(\cdot,\cdot,\frac{t}{2}) \in L^2(\mathbb{R}^N\times\mathbb{R}^N)$ and the operator

$$T(\frac{t}{2})f(x) = e^{-\frac{t}{2}A}f(x) = \int_{\mathbb{R}^N} p(x,y,\frac{t}{2})f(y)\,dy$$

is a Hilbert-Schmidt operator on $L^2(\mathbb{R}^N)$. It follows that

$$\int_{\mathbb{R}^N} p(x,x,t)\,dx = \|p(\cdot,\cdot,\frac{t}{2})\|^2_{L^2(\mathbb{R}^N\times\mathbb{R}^N)} = \sum_{n=1}^{\infty} e^{-\lambda_n t}.$$

$\square$

Let us now define the discrete measure μ on $\mathbb{R}_+$ by $\mu(\lambda) = |\{n : \lambda = \lambda_n\}|$. Then $\mu([0,\lambda]) = N(\lambda)$ and

$$\hat{\mu}(t) = \int_0^{\infty} e^{-\lambda t}d\mu(\lambda) = \sum_{n=1}^{\infty} e^{-\lambda_n t} = \int_{\mathbb{R}^N} p(x,x,t)\,dx.$$

Theorem 3.4.2. *Let $V(x) = |x|^\alpha$ and $N(\lambda)$ as before. Then*

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{N(\frac{1}{2}+\frac{1}{\alpha})}} = \frac{N\omega_N}{(4\pi)^{\frac{N}{2}}} \frac{1}{\Gamma(N(\frac{1}{\alpha}+\frac{1}{2})+1)} \frac{1}{\alpha}\Gamma\left(\frac{N}{\alpha}\right).$$

Proof. By Proposition 3.4.1

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} = \int_{\mathbb{R}^N} p(x,x,t)\,dx.$$

By Example 3.2.6 there exists $C(N)$ such that for all $0 < c < 1$ and $t > 0$

$$\begin{aligned}\int_{\mathbb{R}^N} p(x,x,t)\,dx &\le \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{\mathbb{R}^N} \exp\{-tc^\alpha|x|^\alpha\}\,dx \\ &\quad + \frac{C(N)}{t^{\frac{N}{2}}} \frac{c^N}{(1-c)^N} \int_{\mathbb{R}^N} \exp\left\{-\frac{(1-c)^2|x|^2}{4t}\,dx\right\} \\ &= \frac{1}{(4\pi t)^{\frac{N}{2}}} \frac{1}{t^{\frac{N}{\alpha}}} \int_{\mathbb{R}^N} \exp\{-c^\alpha|y|^\alpha\}\,dy \\ &\quad + C(N)\frac{c^N}{(1-c)^N} \int_{\mathbb{R}^N} \exp\left\{-(1-c)^2|y|^2\,dy\right\}.\end{aligned}$$

Therefore for all $0 < c < 1$

$$\limsup_{t\to 0} t^{N(\frac{1}{2}+\frac{1}{\alpha})} \int_{\mathbb{R}^N} p(x,x,t)\,dx \le \frac{1}{(4\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} \exp\{-c^\alpha|x|^\alpha\}\,dx$$

and, letting c to 1,

$$\limsup_{t\to 0} t^{N(\frac{1}{2}+\frac{1}{\alpha})}\int_{\mathbb{R}^N} p(x,x,t)\,dx \le \frac{1}{(4\pi)^{\frac{N}{2}}}\int_{\mathbb{R}^N}\exp\{-|x|^\alpha\}\,dx. \tag{3.7}$$

In order to obtain a lower bound we proceed as in [13, Lemma 4.5.9].
If A_D is the operator obtained from A by imposing Dirichlet boundary conditions on the surface of the ball B with center x and radius r then

$$p(x,x,t) \ge p_D(x,x,t).$$

Moreover $V(x) \le (|x|+r)^\alpha$ in $B(x,r)$, so

$$p(x,x,t) \ge \exp\{-t(|x|+r)^\alpha\} p_\Delta(x,x,t)$$

where p_Δ is the heat kernel for the Laplacian on B with Dirichlet boundary conditions. By Kac's principle (see [15])

$$p_\Delta(x,x,t) \ge c(r,t) = \frac{1}{(4\pi t)^{\frac{N}{2}}}\left(1-e^{-\frac{r^2}{4t}}\right)$$

for $t \le \frac{r^2}{2N}$. Therefore

$$\begin{aligned}\int_{\mathbb{R}^N} p(x,x,t)\,dx &\ge c(r,t)\int_{\mathbb{R}^N}\exp\{-t(|x|+r)^\alpha\}\,dx\\ &= c(r,t)|S_{N-1}|\int_0^\infty \exp\{-t(\rho+r)^\alpha\}\rho^{N-1}d\rho\\ &= c(r,t)|S_{N-1}|\int_{rt^{\frac{1}{\alpha}}}^\infty \exp\{-s^\alpha\}\left(\frac{s}{t^{\frac{1}{\alpha}}}-r\right)^{N-1}\frac{ds}{t^{\frac{1}{\alpha}}}\\ &= \frac{1}{(4\pi)^{\frac{N}{2}}}\frac{1}{t^{N(\frac{1}{2}+\frac{1}{\alpha})}}\left(1-e^{-\frac{r^2}{4t}}\right)|S_{N-1}|\int_{rt^{\frac{1}{\alpha}}}^\infty \exp\{-s^\alpha\}\left(s-t^{\frac{1}{\alpha}}r\right)^{N-1}ds\end{aligned}$$

where $|S_{N-1}|$ is the measure of the unitary sphere in $\mathbb{R}^N$. Finally

$$\begin{aligned}\liminf_{t\to 0} t^{N(\frac{1}{2}+\frac{1}{\alpha})}\int_{\mathbb{R}^N} p(x,x,t)\,dx &\ge \frac{1}{(4\pi)^{\frac{N}{2}}}|S_{N-1}|\int_0^\infty \exp\{-s^\alpha\}s^{N-1}ds \qquad (3.8)\\ = \frac{1}{(4\pi)^{\frac{N}{2}}}\int_{\mathbb{R}^N}\exp\{-|x|^\alpha\}\,dx.&\end{aligned}$$

From (3.7) and (3.8) it follows that

$$\lim_{t\to 0} t^{N(\frac{1}{2}+\frac{1}{\alpha})}\int_{\mathbb{R}^N} p(x,x,t)\,dx = \frac{1}{(4\pi)^{\frac{N}{2}}}\int_{\mathbb{R}^N}\exp\{-|x|^\alpha\}\,dx$$

and so, by Karamata's Theorem (see the Appendix)

$$\lim_{\lambda\to\infty}\frac{N(\lambda)}{\lambda^{N(\frac{1}{2}+\frac{1}{\alpha})}} = \frac{1}{(4\pi)^{\frac{N}{2}}}\frac{1}{\Gamma(N(\frac{1}{\alpha}+\frac{1}{2})+1)}\int_{\mathbb{R}^N}\exp\{-|x|^\alpha\}\,dx.$$

Finally, observing that

$$\int_{\mathbb{R}^N} \exp\{-|x|^\alpha\}\,dx = \frac{N\omega_N}{\alpha}\int_0^\infty e^{-z} z^{\frac{N}{\alpha}-1}\,dz = \frac{1}{\alpha}\Gamma\Big(\frac{N}{\alpha}\Big),$$

the proof follows. □

Remark 3.4.3. The last result allows us to deduce some information on the lower bound of the heat kernel relative to the potential $V = |x|^\alpha$, for $t \le 1$ and $\alpha > 2$. We recall that, under these assumptions on t and α, the following upper bound holds

$$p(x,x,t) \le \frac{C}{t^{\frac{N}{2}}}\exp\{-ct|x|^{1+\frac{\alpha}{2}}\}.$$

If a similar lower bound were true, following the proof of Theorem 3.4.2 and applying Proposition B.0.12, we would deduce

$$\liminf_{\lambda\to\infty} \lambda^{-N(\frac{1}{2}+\frac{2}{2+\alpha})}N(\lambda) \ge C$$

for some positive constant C. Since this contradicts Theorem 3.4.2, we conclude that a similar lower bound cannot be true.

Adding a term of the form $o(|x|^\alpha)$ to the previous potential does not affect too much the asymptotic distribution of eigenvalues. In fact the following holds.

Proposition 3.4.4. *Let* $V(x) = |x|^\alpha + o(|x|^\alpha)$ *(as* $|x| \to \infty$*). Then*

$$\lim_{\lambda\to\infty}\frac{N(\lambda)}{\lambda^{N(\frac{1}{2}+\frac{1}{\alpha})}} = \frac{N\omega_n}{(4\pi)^{\frac{N}{2}}}\frac{1}{\Gamma(N(\frac{1}{\alpha}+\frac{1}{2})+1)}\frac{1}{\alpha}\Gamma\left(\frac{N}{\alpha}\right).$$

Proof. It is sufficient to observe that, given $\varepsilon > 0$, there exist $C'_\varepsilon,\ C_\varepsilon > 0$ such that

$$(1-\varepsilon)|x|^\alpha + C_\varepsilon \le V(x) \le (1+\varepsilon)|x|^\alpha + C'_\varepsilon$$

and, by the maximum principle,

$$p(x,x,t) \le e^{-C_\varepsilon t}p_\varepsilon(x,x,t)$$

where p_ε is the kernel corresponding to the potential $(1-\varepsilon)|x|^\alpha$. As in the proof of Theorem 3.4.2, it follows that for all $\varepsilon > 0$

$$\limsup_{t\to 0} t^{N(\frac{1}{2}+\frac{1}{\alpha})}\int_{\mathbb{R}^N} p(x,x,t)\,dx \le \frac{1}{(4\pi)^{\frac{N}{2}}}\int_{\mathbb{R}^N}\exp\{-(1-\varepsilon)|x|^\alpha\}\,dx$$

and, letting ε to 0,

$$\limsup_{t\to 0} t^{N(\frac{1}{2}+\frac{1}{\alpha})}\int_{\mathbb{R}^N} p(x,x,t)\,dx \le \frac{1}{(4\pi)^{\frac{N}{2}}}\int_{\mathbb{R}^N}\exp\{-|x|^\alpha\}\,dx.$$

In a similar way one obtains the bound for the lim inf and the proof follows. □

From the bound on the kernel proved in the previous section we can deduce the asymptotic behavior of $N(\lambda)$ for other radial potentials.

Proposition 3.4.5. *Let* $V(x) = \exp\{|x|^\alpha\}$ *with* $\alpha > 0$. *Then there exist* $C_1,\ C_2 > 0$ *such that*

$$\limsup_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{\frac{N}{2}}(\log\lambda)^{\frac{N}{\alpha}}} \le C_1$$

and

$$\liminf_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{\frac{N}{2}}(\log\lambda)^{\frac{N}{\alpha}}} \ge C_2.$$

Lemma 3.4.6. *Let* g *be measurable and positive in* $\mathbb{R}^N$ *and let* $E_s = \{x \in \mathbb{R}^N : g(x) \le s\}$. *Then*

$$\int_{\mathbb{R}^N} e^{-tg(x)}dx = \int_0^\infty |E_{\frac{z}{t}}|e^{-z}dz.$$

Proof. The proof easily follows by observing that

$$\int_{\mathbb{R}^N} \exp\{-tg(x)\}dx = \int_0^\infty |\{x \in \mathbb{R}^N : \exp\{-tg(x)\} > s\}|\, ds. \tag{3.9}$$

□

Proof (Proposition 3.4.5.) By Corollary 3.2.5 there exists $C = C(N)$ such that for all $0 < c < 1$ and $t > 0$

$$\begin{aligned}\int_{\mathbb{R}^N} p(x,x,t)\,dx \le \frac{1}{(4\pi t)^{\frac{N}{2}}}\int_{\mathbb{R}^N} \exp\{-t\exp\{c^\alpha|x|^\alpha\}\}\,dx \\ + C\frac{c^N\omega_N}{(1-c)^N}\int_{\mathbb{R}^N} \exp\left\{-(1-c)^2|x|^2\right\}\,dx.\end{aligned}$$

By Lemma 3.4.6

$$\begin{aligned}\int_{\mathbb{R}^N} \exp\{-t\exp\{c^\alpha|x|^\alpha\}\}\,dx &= \int_0^\infty e^{-z}|\{x : \exp\{c^\alpha|x|^\alpha\} \le \frac{z}{t}\}|dz \\ &= \frac{\omega_N}{c^N}\int_t^\infty e^{-z}(\log z - \log t)^{\frac{N}{\alpha}}dz.\end{aligned}$$

Taking the $\limsup$ as $t \to 0$ and letting $c \to 1$ we obtain

$$\limsup_{t\to 0} \frac{t^{\frac{N}{2}}}{(-\log t)^{\frac{N}{\alpha}}}\int_{\mathbb{R}^N} p(x,x,t)\,dx \le \frac{1}{(4\pi)^{\frac{N}{2}}}\omega_N\int_0^\infty e^{-z}dz = \frac{\omega_N}{(4\pi)^{\frac{N}{2}}}. \tag{3.10}$$

To prove a lower bound for the $\liminf$ of the same quantity we proceed as in the proof of Theorem 3.4.2. If A_D is the operator obtained from A by imposing Dirichlet boundary conditions on the surface of the ball B with center x and radius r then $p(x,x,t) \ge p_D(x,x,t)$. Moreover $V \le \exp\{(|x|+r)^\alpha\}$ in $B(x,r)$, so $p(x,x,t) \ge \exp\{-t\exp\{(|x|+r)^\alpha\}\}p_\Delta(x,x,t)$ where p_Δ is the heat kernel for the Laplacian on B with Dirichlet boundary conditions. By Kac's principle (see [15])

$$p_\Delta(x,x,t) \ge c(r,t) = \frac{1}{(4\pi t)^{\frac{N}{2}}}\left(1 - e^{-\frac{r^2}{4t}}\right)$$

for $t \leq \frac{r^2}{2N}$. Therefore, from Lemma 3.4.6,

$$\begin{aligned}\int_{\mathbb{R}^N} p(x,x,t)\,dx &\geq c(r,t)\int_{\mathbb{R}^N} \exp\{-t\exp\{(|x|+r)^\alpha\}\}\,dx \\ &= c(r,t)\omega_N \int_t^\infty [(\log z - \log t)^{\frac{1}{\alpha}} - r]^N e^{-z} dz.\end{aligned}$$

As above

$$\liminf_{t\to 0} \frac{t^{\frac{N}{2}}}{(-\log t)^{\frac{N}{\alpha}}} \int_{\mathbb{R}^N} p(x,x,t)\,dx \geq \frac{1}{(4\pi)^{\frac{N}{2}}}\omega_N \int_0^\infty e^{-z} dz = \frac{\omega_N}{(4\pi)^{\frac{N}{2}}}. \tag{3.11}$$

From (3.10) and (3.11) it follows that

$$\lim_{t\to 0} \frac{t^{\frac{N}{2}}}{(-\log t)^{\frac{N}{\alpha}}} \int_{\mathbb{R}^N} p(x,x,t)\,dx = \frac{\omega_N}{(4\pi)^{\frac{N}{2}}}.$$

By Proposition B.0.13, we find $C_1,\ C_2 > 0$ such that

$$\limsup_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{\frac{N}{2}}(\log\lambda)^{\frac{N}{\alpha}}} \leq C_1, \qquad \liminf_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{\frac{N}{2}}(\log\lambda)^{\frac{N}{\alpha}}} \geq C_2.$$

□

Chapter 4

Ultracontractivity of Schrödinger semigroups

In this chapter we consider again a Schrödinger operator $H = -\Delta + V$ with a nonnegative potential $V \in L^1_{\text{loc}}(\mathbb{R}^N)$. If $V(x) = |x|^\alpha$, $\alpha > 2$, an estimate of the form $p(x,y,t) \leq c(t)\psi(x)\psi(y)$ holds, where ψ is the ground state of H and $c(t)$ has an explicit behavior near 0 (see [13, Section 4.5]). We consider the Davies-Simon estimates and we obtain bounds on Schrödinger kernels using the similarity between Schrödinger and Kolmogorov operators. Even though this similarity is well-known, see [13, Section 4.7], we reverse the usual order, i.e. we deduce bounds on Schrödinger kernels from those for Kolmogorov's kernels rather than the converse and this allows us to improve the estimates obtained by Davies and Simon. It is also shown how the same technique works for other potentials, for example heat kernel bounds are obtained for $V(x) = \exp\{|x|^\alpha\}$, $\alpha > 0$.

4.1 Kernel estimates for a class of Kolmogorov operators

In this section we prove estimates of the form $p(x,y,t) \leq c(t)\omega(x)\omega(y)$ for Kolmogorov operators of the form

$$A = \Delta - \nabla\phi \cdot \nabla$$

with $\phi \in C^2(\mathbb{R}^N)$. The operator A can be easily defined, through form methods, as a self-adjoint, nonpositive operator in $L^2(\mathbb{R}^N, \mu)$, where $d\mu$ is the measure with density $\exp\{-\phi\}$. If the function $|\nabla\phi|^2 - 2\Delta\phi$ is bounded from below in $\mathbb{R}^N$, then the operator A in $L^2(\mathbb{R}^N, \mu)$ is unitarily equivalent to the Schrödinger operator $-H$ with potential $V = \frac{1}{4}|\nabla\phi|^2 - \frac{1}{2}\Delta\phi$ in $L^2(\mathbb{R}^N)$ (with respect to the Lebesgue measure), see [26, Proposition 2.2]. In particular $A = -THT^{-1}$

where T is the multiplication operator $Tu = e^{\frac{\phi}{2}}u$. Moreover $e^{tA} = Te^{-tH}T^{-1}$ and consequently for all $x,\ y \in \mathbb{R}^N$ and $t > 0$

$$p_A(x, y, t) = e^{\frac{\phi(x)}{2}} p(x, y, t) e^{-\frac{\phi(y)}{2}} \tag{4.1}$$

where p_A and p are the heat kernels corresponding to the operators A and $-H$. This equality shows that the problems of finding estimates for p_A and p are equivalent and, in [13, Section 4.7], this fact is used to deduce bounds for p_A from deep estimates on p based on log-Sobolev inequalities leading to the intrinsic ultracontractivity of the Schrödinger semigroup. We reverse the approach and show bounds on p_A based on subsolution estimates. Then we deduce bounds on p. This method has the advantage to give more precise information on the function $c(t)$ quoted at the beginning of this section and allows us to improve some kernel estimates on Schrödinger operators, as shown in the next section.

As first step we prove L^1 bounds for some Lyapunov functions (or subsolutions) for A. For all $0 < c < 1$, let $W_c = e^{c\phi}$. It is easy to check that

$$AW_c = e^{c\phi}[c\Delta\phi + (c^2 - c)|\nabla\phi|^2].$$

Under suitable assumptions on ϕ, W_c is a Lyapunov function for A that is a C^2-function $W : \mathbb{R}^N \to [0, \infty)$ such that $\lim_{|x|\to\infty} W(x) = +\infty$ and $AW \le \lambda W$ for some $\lambda > 0$.

We need some preliminary lemmas (see [30, Lemma 3.8, Lemma 3.9]).

Let W be a Lyapunov function. For $\alpha \ge 0$ set $W_\alpha = W \wedge \alpha$ and $u_\alpha(x, t) = T(t)W_\alpha(x)$.

Lemma 4.1.1. *With the notation above, the inequality*

$$\partial_t u_\alpha(x, t) \le \int_{\{W \le \alpha\}} p(x, y, t) AW(y)\, dy$$

holds for every $t \ge 0$ and $x \in \mathbb{R}^N$.

Proof. For every $\varepsilon > 0$ let $\psi_\alpha \in C^\infty(\mathbb{R})$ be such that $\psi_\varepsilon(t) = t$ for $t \le \alpha$, ψ_ε is constant in $[\alpha + \varepsilon, \infty[$, $\psi_\varepsilon' \ge 0$, $\psi_\varepsilon'' \le 0$. Observe that $\psi_\varepsilon(t) \to t \wedge \alpha$ and $\psi_\varepsilon'(t) \to \chi_{]-\infty,\alpha]}(t)$ pointwise as $\varepsilon \to 0$. Since the function $\psi_\varepsilon \circ V$ belongs to $D_{max}(A)$, we have

$$\partial_t T(t)(\psi_\varepsilon \circ W)(x) = \int_{\mathbb{R}^N} p(x, y, t) A(\psi_\varepsilon \circ W)(y)\, dy.$$

On the other hand, by the assumptions on ψ_ε,

$$\begin{aligned} A(\psi_\varepsilon \circ W)(x) &= \psi_\varepsilon'(W(x))AW(x) + \psi_\varepsilon''(W(x)) \sum_{i,j=1}^N a_{ij}(x) D_i W(x) D_j W(x) \\ &\le \psi_\varepsilon'(W(x))AW(x) \end{aligned}$$

and then

$$\begin{aligned}\partial_t T(t)(\psi_\varepsilon\circ W)(x) &\le \int_{\mathbb{R}^N} p(x,y,t)\psi'_\varepsilon(W(y))AW(y)\,dy \qquad (4.2)\\ &= \int_{0\le W\le\alpha+\varepsilon} p(x,y,t)\psi'_\varepsilon(W(y))AW(y)\,dy.\end{aligned}$$

Observe that $\psi_\varepsilon\circ W\le\alpha+1$ and $\psi_\varepsilon\circ W\to W_\alpha$ pointwise as $\varepsilon\to 0$. By Proposition 1.1.3 we deduce that $T(t)(\psi_\varepsilon\circ W)\to u_\alpha$ uniformly on compact sets of $]0,\infty[\times\mathbb{R}^N$, then by the interior Schauder estimates (see [17, Chapter 3, Section 2]) $\partial_t T(t)(\psi_\varepsilon\circ W)\to\partial_t u_\alpha$ pointwise as $\varepsilon\to 0$. Letting ε to zero in (4.2) we obtain the claim by dominated convergence. □

The next result has been partially obtained in Chapter 2 in the more general case of Lyapunov functions depending also on the variable t.

Lemma 4.1.2. *Suppose that $AW\le\lambda W$ for some positive λ. Then for every $t>0$, $x\in\mathbb{R}^N$ the functions W and $|AW|$ are integrable with respect to the measure $p(x,\cdot,t)$. If we set*

$$u(x,t)=\int_{\mathbb{R}^N} p(x,y,t)W(y)\,dy,$$

the function u belongs to $C^{1,2}(\mathbb{R}^N\times]0,\infty[)\cap C(\mathbb{R}^N\times[0,\infty[)$ and satisfies the inequalities $u(x,t)\le e^{\lambda t}W(x)$, $\partial_t u(x,t)\le\int_{\mathbb{R}^N} p(x,y,t)AW(y)\,dy$.

Proof. By Lemma 4.1.1 and by assumption we have

$$\partial_t u_\alpha(x,t)\le\int_{\{W\le\alpha\}} p(x,y,t)AW(y)\,dy\le\lambda u_\alpha(x,t). \qquad (4.3)$$

By Gronwall's lemma we deduce $u_\alpha(x,t)\le e^{\lambda t}W_\alpha(x)$. Letting α to infinity we obtain $u(x,t)\le e^{\lambda t}W(x)$ by monotone convergence. This implies that W is integrable with respect to the measure $p(x,\cdot,t)$. The inequality $0\le u_\alpha\le u$ and the interior Schauder estimates show that (u_α) is relatively compact in $C^{1,2}(\mathbb{R}^N\times(0,\infty))$. Since $u_\alpha\to u$ pointwise as $\alpha\to\infty$ it follows that $u\in C^{1,2}(\mathbb{R}^N\times(0,\infty))$. Moreover the inequality $u_\alpha(x,t)\le u(x,t)\le e^{\lambda t}W(x)$ implies that $u(\cdot,t)\to W(\cdot)$ as $t\to 0$ uniformly on compact sets. Set $E=\{x\in\mathbb{R}^N : AW(x)\ge 0\}$, clearly

$$\int_E p(x,y,t)AW(y)\,dy\le\lambda\int_E p(x,y,t)W(y)\,dy\le\lambda u(x,t)<\infty. \qquad (4.4)$$

Letting α to infinity in (4.3) we obtain

$$\partial_t u(x,t)\le\liminf_{\alpha\to\infty}\int_{\{W\le\alpha\}} p(x,y,t)AW(y)\,dy.$$

The last inequality and (4.4) imply that

$$-\int_{\{AW\le 0\}} p(x,y,t)AW(y)\,dy<\infty,$$

then $|AW|$ is integrable with respect to the measure $p(x,\cdot,t)$ and so the above $\liminf$ is a limit and the claim follows. $\square$

Proposition 4.1.3. *Let* $\phi \geq 0$ *such that* $\lim_{|x|\to\infty}\phi(x) = +\infty$ *and let* $0 < c < 1$. *Suppose that for some* $0 < \varepsilon < 1-c$ *there exists* $C_\varepsilon > 0$ *such that*

$$\Delta\phi \leq \varepsilon|\nabla\phi|^2 + C_\varepsilon \tag{4.5}$$

and suppose that

$$|\nabla\phi| \geq C_1\phi^\gamma - C_2 \tag{4.6}$$

for some positive constant C_1, C_2 *and some* $\gamma > \frac{1}{2}$. *Then the function* W_c *defined above is a Lyapunov function. Moreover, setting*

$$\xi_c(x,t) = \int_{\mathbb{R}^N} p_A(x,y,t)W_c(y)\,dy,$$

we have

$$\xi_c(x,t) \leq C_3\exp\{C_4 t^{\frac{1}{1-2\gamma}}\} \tag{4.7}$$

for some positive constants C_3, C_4.

Proof. By (4.5) and (4.6) for $|x|$ large enough

$$\begin{aligned} AW_c &= e^{c\phi}[c\Delta\phi + (c^2-c)|\nabla\phi|^2] \leq e^{c\phi}[(c\varepsilon + c^2 - c)|\nabla\phi|^2 + C_\varepsilon c] \\ &\leq e^{c\phi}(-C_1|\nabla\phi|^2 + C_2) \leq -e^{c\phi}(\widetilde{C}_1\phi^{2\gamma} - \widetilde{C}_2). \end{aligned}$$

This proves that, for $|x|$ large enough, AW_c is negative. By the regularity of W_c, for $|x|$ small $AW_c \leq \lambda \leq \lambda W_c$ for some positive λ. Therefore W_c is a Lyapunov function. Moreover, setting $g(s) = c_1 s(\log s)_+^{2\gamma} - c_2$ for suitable constants c_1 and c_2, we have

$$AW_c \leq -g(W_c)$$

for $|x|$ sufficiently large. Observe that the existence of a Lyapunov function for A implies the uniqueness for the solution of problem (1.1), hence $\mathbf{1} = T(t)\mathbf{1} = \int_{\mathbb{R}^N} p_A(x,y,t)\,dy$. Since g is convex, by Jensen's inequality

$$\int_{\mathbb{R}^N} p_A(x,y,t)g(W_c(y))\,dy \geq g(\xi_c(x,t)).$$

By Lemma 4.1.2 and the previous inequalities we have

$$\begin{aligned} \partial_t\xi_c(x,t) &\leq \int_{\mathbb{R}^N} p_A(x,y,t)AW_c(y)\,dy \leq -\int_{\mathbb{R}^N} p_A(x,y,t)g(W_c(y))\,dy \\ &\leq -g(\xi_c(x,t)) \end{aligned}$$

and then $\xi_c(x,t) \leq z(x,t)$ where z is the solution of the ordinary Cauchy problem

$$\begin{cases} z' = -g(z) \\ z(x,0) = W_c(x). \end{cases}$$

Let l be the greatest zero of g. Then $z(x,t) \le l$ if $W_c(x) \le l$. If $W_c(x) > l$, z is decreasing and satisfies

$$t = \int_{z(x,t)}^{W_c(x)} \frac{ds}{g(s)} \le \int_{z(x,t)}^{\infty} \frac{ds}{g(s)}.$$

Choosing suitable constants C_3 and C_4, we finally obtain

$$\xi_c(x,t) \le z(x,t) \le C_3 \exp\{C_4 t^{\frac{1}{1-2\gamma}}\}.$$

□

Now we are able to deduce bounds on the kernel p_A from the bound on the function ξ_{W_c} proved above.

Proposition 4.1.4. *Let ϕ as in the previous proposition and suppose moreover that*

$$\exp\{-\frac{\phi}{4}\} \in L^1(\mathbb{R}^N), \qquad |\nabla\phi| \le C\phi^{\beta}, \tag{4.8}$$

for some positive C, β. Then

$$p_A(x,y,t) \le C_1 \exp\left\{C_2 t^{\frac{1}{1-2\gamma}}\right\} \exp\{-\phi(y)\} \tag{4.9}$$

and

$$p(x,y,t) \le C_1 \exp\left\{C_2 t^{\frac{1}{1-2\gamma}}\right\} \exp\left\{-\frac{\phi(y)}{2}\right\} \exp\left\{-\frac{\phi(x)}{2}\right\} \tag{4.10}$$

for all $x,\ y \in \mathbb{R}^N$ and $0 < t \le 1$ and suitable $C_1, C_2 > 0$.

Proof. Let $\omega = W_{\frac{1}{2}}$ and $\frac{1}{2} < c < 1$. Then, if $k > N+2$, by the assumptions on ϕ it follows that

$$\begin{aligned}
&\omega \le W_c,\\
&|\nabla\omega| = \frac{1}{2} e^{\frac{\phi}{2}} |\nabla\phi| \le C\omega^{\frac{k-1}{k}} W_c^{\frac{1}{k}} = C \exp\left\{\frac{\phi}{2}\frac{k-1}{k}\right\} \exp\left\{\frac{1}{k} c\phi\right\},\\
&|D^2\omega| \le C\omega^{\frac{k-2}{k}} W_c^{\frac{2}{k}},\\
&\omega|\nabla\phi|^k \le CW_c
\end{aligned}$$

for some positive constant C. By Remark 2.1.17 or [27, Theorem 4.1] it follows that

$$\exp\left\{\frac{\phi(y)}{2}\right\} p_A(x,y,t) \le \frac{C}{t^{\frac{k}{2}}} \int_{\frac{t}{2}}^{t} \xi_c(x,s)\, ds$$

for all $x,\ y \in \mathbb{R}^N$, $0 < t \le 1$ and by (4.7)

$$p_A(x,y,t) \le C_3 \exp\left\{C_4 t^{\frac{1}{1-2\gamma}}\right\} \exp\left\{-\frac{\phi(y)}{2}\right\}$$

for suitable C_3, C_4 (we can neglect negative powers of t which can be included in the exponential changing the constant). By (4.1),

$$p(x,y,t) \leq C_3 \exp\left\{C_4 t^{\frac{1}{1-2\gamma}}\right\} \exp\left\{-\frac{\phi(x)}{2}\right\} = c(t) \exp\left\{-\frac{\phi(x)}{2}\right\}.$$

Using the symmetry of p_{-H} with respect to the variables $x,\ y$ we have

$$p(x,y,t) \leq c(t) \exp\left\{-\frac{\phi(y)}{2}\right\}.$$

Then we get

$$p(z,y,t) \leq c(t) \exp\left\{-\frac{\phi(z)}{4}\right\} \exp\left\{-\frac{\phi(y)}{4}\right\}$$

and, by the semigroup law,

$$\begin{aligned} p(x,y,t) &= \int_{\mathbb{R}^N} p(x,z,\frac{t}{2}) p(z,y,\frac{t}{2})\, dz \\ &\leq c(\frac{t}{2})^2 \exp\left\{-\frac{\phi(x)}{2}\right\} \exp\left\{-\frac{\phi(y)}{4}\right\} \int_{\mathbb{R}^N} \exp\left\{-\frac{\phi(z)}{4}\right\} dz \\ &= K_1 c(\frac{t}{2})^2 \exp\left\{-\frac{\phi(x)}{2}\right\} \exp\left\{-\frac{\phi(y)}{4}\right\}. \end{aligned}$$

As in the estimate above we deduce

$$\begin{aligned} p(x,y,t) &\leq K_1 c(\frac{t}{2}) c(\frac{t}{4})^2 \exp\left\{-\frac{\phi(y)}{2}\right\} \exp\left\{-\frac{\phi(x)}{2}\right\} \int_{\mathbb{R}^N} \exp\left\{-\frac{\phi(z)}{4}\right\} dz \\ &= c_1(t) \exp\left\{-\frac{\phi(x)}{2}\right\} \exp\left\{-\frac{\phi(y)}{2}\right\}. \end{aligned}$$

Therefore

$$p(x,y,t) \leq C_1 \exp\left\{C_2 t^{\frac{1}{1-2\gamma}}\right\} \exp\left\{-\frac{\phi(y)}{2}\right\} \exp\left\{-\frac{\phi(x)}{2}\right\}$$

and

$$p_A(x,y,t) \leq C_1 \exp\left\{C_2 t^{\frac{1}{1-2\gamma}}\right\} \exp\left\{-\phi(y)\right\}.$$

□

4.2 Intrinsic ultracontractivity for e^{-tH}

Let us consider the Schrödinger operator $H = -\Delta + V$ where $0 \leq V(x) \to \infty$ as $|x| \to \infty$. Let $E > 0$ be the first eigenvalue of H and $\psi > 0$ be the corresponding eigenfunction. Then $\Delta\psi = (V - E)\psi$. As observed in the previous section, $-H + E$ is unitarily equivalent to the Kolmogorov operator $A = \Delta + 2\frac{\nabla\psi}{\psi} \cdot \nabla$, namely $-H + E = T^{-1}AT$ where T is the multiplication operator $Tu = \psi^{-1}u$.

If $\phi = -2\log\psi$, then $A = \Delta - \nabla\phi\cdot\nabla$ and $Tu = e^{\frac{\phi}{2}}u$. If ϕ satisfies the hypotheses of the Proposition 4.1.4 then we obtain upper bounds for the kernel of the semigroup generated by $-H+E$. Let us also observe that, if p_E and p are the kernels corresponding respectively to $-H+E$ and $-H$, then $p = p_E e^{-tE} \le p_E(x,y,t)$.

We start with $V(x) = |x|^\alpha$, $\alpha > 2$ and improve [13, Corollary 4.5.5]. In what follows the knowledge of the asymptotic behavior of the first eigenfunction ψ of H will play a major role. We recall that there exist $c_1,\ c_2 > 0$ such that

$$\begin{aligned} c_1|x|^{-\frac{\alpha}{4}-\frac{N-1}{2}}\exp\left\{-\frac{2}{2+\alpha}|x|^{1+\frac{\alpha}{2}}\right\} &\le \psi(x) \qquad (4.11)\\ &\le c_2|x|^{-\frac{\alpha}{4}-\frac{N-1}{2}}\exp\left\{-\frac{2}{2+\alpha}|x|^{1+\frac{\alpha}{2}}\right\}\end{aligned}$$

for large $|x|$, see [13, Corollary 4.5.8]. Our methods, however, need also a precise asymptotic behavior of $\nabla\psi$. This can be obtained from [36, Chapter 6, Theorem 2.1] (as we shall do for other potentials) or using the following qualitative arguments for ODE's which we prefer to present in the following lemma.

Lemma 4.2.1. *Let ψ be the first eigenfunction of $-\Delta + V$ with $V(x) = |x|^\alpha$, $\alpha > 2$. Then*

$$\lim_{|x|\to\infty}\frac{|\nabla\psi|^2}{\psi^2}\cdot\frac{1}{|x|^\alpha} = 1.$$

Proof. Since the potential is radial, the first eigenfunction is radial too, so, writing the Laplacian in polar coordinates, we have

$$\psi'' + \frac{N-1}{r}\psi' = (r^\alpha - E)\psi.$$

Setting $v = -\frac{\psi'}{\psi}$, the previous differential equation becomes

$$v' = v^2 - \frac{N-1}{r}v - (r^\alpha - E).$$

The right hand side of the previous equals 0 if

$$v = \frac{N-1}{2r} \pm \frac{1}{2}\sqrt{\frac{(N-1)^2}{r^2} + 4(r^\alpha - E)}.$$

Now we prove that there exists $r_0 > 0$ such that for $r \ge r_0$

$$v \ge \frac{N-1}{2r} + \frac{1}{2}\sqrt{\frac{(N-1)^2}{r^2} + 4(r^\alpha - E)}.$$

Since

$$\frac{d}{dr}(r^{N-1}\psi') = r^{N-1}(r^\alpha - E)\psi,$$

the asymptotic behavior of ψ (see (4.11)) shows that $r^{N-1}(r^\alpha - E)\psi$ is integrable in neighborhood of $+\infty$. This implies that there exists $\lim_{r\to\infty} r^{N-1}\psi'$ and it is equal to 0, by the asymptotic behavior of ψ, again. Moreover, if $r \geq E^{\frac{1}{\alpha}}$, $\frac{d}{dr}(r^{N-1}\psi') > 0$ and

$$r^{N-1}\psi' \leq \lim_{r\to\infty} r^{N-1}\psi' = 0.$$

This means that, for r large enough, $\psi' \leq 0$ and $v = -\frac{\psi'}{\psi} > 0$. From this we deduce that for r large enough v is in the region where $v' > 0$ and

$$v \geq \frac{N-1}{2r} + \frac{1}{2}\sqrt{\frac{(N-1)^2}{r^2} + 4(r^\alpha - E)}. \tag{4.12}$$

We are now interested in the asymptotic behavior of v. Let $\delta,\ k > 0$. Suppose that there exists a sequence $(r_n)_{n\in\mathbb{N}}$ such that $r_n \to \infty$ and

$$v(r_n) \geq \frac{N-1}{2r} + \frac{1}{2}\sqrt{\frac{(N-1)^2}{r^2} + 4[(k+2\delta)^\alpha - E]}. \tag{4.13}$$

Consider the following Cauchy problem in the interval $[k, k+\delta]$:

$$\begin{cases} z' = z^2 - \frac{N-1}{k}z - [(k+\delta)^\alpha - E] \\ z(k) = \frac{N-1}{2k} + \frac{1}{2}\sqrt{\frac{(N-1)^2}{k^2} + 4[(k+2\delta)^\alpha - E]}. \end{cases}$$

In $[k, k+\delta]$,

$$v' \geq v^2 - \frac{N-1}{k}v - [(k+\delta)^\alpha - E]. \tag{4.14}$$

Let us observe that $z(k) > \frac{N-1}{2k} + \frac{1}{2}\sqrt{\frac{(N-1)^2}{k^2} + 4[(k+\delta)^\alpha - E]}$, i.e. $z(k)$ is greater than the largest zero of $z^2 - \frac{N-1}{k}z - [(k+\delta)^\alpha - E]$. Integrating the differential equation satisfied by z, we obtain

$$\int_{z(k)}^{z(r)} \frac{dw}{w^2 - \frac{N-1}{k}w - [(k+\delta)^\alpha - E]} = r - k$$

and, taking $r = k + \delta$,

$$\delta \leq \int_{z(k)}^{\infty} \frac{dw}{w^2 - \frac{N-1}{k}w - [(k+\delta)^\alpha - E]}.$$

After a simple change of variable in the integral above,

$$\delta \leq \int_0^{\infty} \frac{ds}{s^2 + 2sz(k) - \frac{N-1}{k}s + (k+2\delta)^\alpha - (k+\delta)^\alpha}.$$

The right hand side in the previous inequality goes to 0 for k tending to $+\infty$ by dominated convergence. This means that, if k is large enough, the solution z of the Cauchy problem in $[k, k+\delta]$ blows up before the point $k+\delta$. So, choosing $k = r_n$, for r_n large enough z_{r_n} blows up. By (4.13) and (4.14), $v(r) \geq z_{r_n}$ and so v blows up too. Since this is a contradiction, there exists $\overline{r}$ (depending on δ) such that, for $r \geq \overline{r}$,

$$v(r) \leq \frac{N-1}{r} + \frac{1}{2}\sqrt{\frac{(N-1)^2}{r^2} + 4[(r+2\delta)^\alpha - E]}. \tag{4.15}$$

Finally, from (4.12), (4.15) and the arbitrariness of $\delta > 0$

$$\lim_{r\to\infty} \frac{v(r)}{r^{\frac{\alpha}{2}}} = 1.$$

□

Theorem 4.2.2. *Let p be the kernel of the semigroup generated by $\Delta - V$ with $V(x) = |x|^\alpha$ for some $\alpha > 2$. Then*

$$p(x,y,t) \leq C\exp\left\{ct^{-\frac{\alpha+2}{\alpha-2}}\right\}\psi(x)\psi(y)$$

for $x, y \in \mathbb{R}^N$ and $0 < t \leq 1$.

Proof. Let $\phi = -2\log\psi$, as before. Then ϕ satisfies (4.5), (4.6) with $\gamma = \frac{\alpha}{2+\alpha}$ and (4.8).

In fact, rewriting (4.5) in terms of ψ, we can prove that for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\operatorname{div}\left(-2\frac{\nabla\psi}{\psi}\right) = -2\frac{\Delta\psi}{\psi} + 2\frac{|\nabla\psi|^2}{\psi^2} \leq 4\varepsilon\frac{|\nabla\psi|^2}{\psi^2} + C_\varepsilon$$

or, equivalently, since ψ is an eigenfunction with eigenvalue E,

$$(1-\varepsilon)\frac{|\nabla\psi|^2}{\psi^2} \leq (V-E) + C_\varepsilon.$$

This follows immediately from Lemma 4.2.1. Moreover (4.6) and (4.8) follow by Lemma 4.2.1 too. For example observe that (4.6) is equivalent to

$$\frac{|\nabla\psi|}{\psi} \geq C_1\log^\gamma\psi^{-2} - C_2$$

for some $\gamma > \frac{1}{2}$ and positive $C_1,\ C_2$. The last is true for $\gamma = \frac{\alpha}{2+\alpha}$ and in virtue of (4.11) and Lemma 4.2.1. Arguing in similar way (4.8) also follows.

At this point Proposition 4.1.4 gives

$$p(x,y,t) \leq C\exp\left\{ct^{-\frac{\alpha+2}{\alpha-2}}\right\}\exp\left\{-\frac{\phi(y)}{2}\right\}\exp\left\{-\frac{\phi(x)}{2}\right\}$$

for all $x,\ y \in \mathbb{R}^N$ and this concludes the proof. □

Comparing the last theorem with [13, Corollary 4.5.5] we conclude that the limit value $b = \frac{\alpha+2}{\alpha-2}$ is allowed.

Proceeding in a similar way we prove the following bound when the potential is $\exp\{|x|^\alpha\}$.

Theorem 4.2.3. *Let* p *the kernel of the semigroup generated by* $\Delta - V$ *with* $V(x) = \exp\{|x|^\alpha\}$ *for some positive* α. *Then for* $x, y \in \mathbb{R}^N$ *and* $0 < t \le 1$

$$p(x,y,t) \le C \exp\left\{ct^{\frac{1}{1-2\gamma}}\right\} \psi(x)\psi(y)$$

with $\gamma = 1$ *if* $\alpha \ge 1$ *and for any* $\frac{1}{2} < \gamma < 1$ *if* $\alpha < 1$. *Here* ψ *is the first eigenfunction of* $\Delta - V$ *and*

$$\psi(r) = Cr^{-\frac{N-1}{2}} \exp\left\{-\frac{r^\alpha}{4}\right\} \exp\left\{-\int_0^r \exp\left\{\frac{s^\alpha}{2}\right\} ds\right\} \{1 + \varepsilon(r)\}$$

with $\varepsilon(r) \to 0$ *for* $r \to \infty$.

Proof. Let $\psi > 0$ the first eigenfunction of the operator $-\Delta + V$ corresponding to the eigenvalue E. Since the potential is radial, the first eigenfunction is radial too, therefore, writing the Laplacian in polar coordinates, we have

$$\psi''(r) + \frac{N-1}{r}\psi'(r) = (\exp\{r^\alpha\} - E)\psi(r).$$

The function $v(r) = r^{\frac{N-1}{2}}\psi(r)$ satisfies the differential equation

$$v''(r) = v(r)\left(\exp\{r^\alpha\} - E + \frac{N-1}{2}\frac{N-3}{2}\frac{1}{r^2}\right).$$

By [36, Theorem 2.1, Chapter 6], a solution of the previous differential equation is given by

$$v(r) = \exp\left\{-\frac{r^\alpha}{4}\right\} \exp\left\{-\int_0^r \exp\left\{\frac{s^\alpha}{2}\right\} ds\right\} \{1 + \varepsilon(r)\}$$

where $\varepsilon(r)$ is a function such that $|\varepsilon(r)|$, $\frac{1}{2}\exp\{-\frac{r^\alpha}{2}\}|\varepsilon'(r)|$ goes to 0 if r goes to ∞. Then

$$\psi(r) = r^{-\frac{N-1}{2}} v(r) = r^{-\frac{N-1}{2}} \exp\left\{-\frac{r^\alpha}{4}\right\} \exp\left\{-\int_0^r \exp\left\{\frac{s^\alpha}{2}\right\} ds\right\} \{1 + \varepsilon(r)\}.$$

After simple computations we obtain

$$\psi'(r) = \psi(r)\left(-\frac{N-1}{2r} - \frac{\alpha}{4}r^{\alpha-1} - \exp\left\{\frac{r^\alpha}{2}\right\} + \frac{\varepsilon'(r)}{1+\varepsilon(r)}\right).$$

It follows that $\phi = \log \psi^{-2}$ satisfies the hypothesis in Proposition 4.1.4. In particular, choosing $\gamma = 1$ if $\alpha \ge 1$ and any $\frac{1}{2} < \gamma < 1$ if $\alpha < 1$, (4.6) is verified and the claim follows. $\square$

Chapter 5

Parabolic Schrödinger operators

In this chapter we consider the parabolic Schrödinger operator

$$\mathcal{A} = \partial_t - \Delta + V \quad \text{on} \quad \mathbb{R}^{N+1}$$

where $V = V(x,t)$ is a nonnegative potential which belongs to the parabolic Reverse Hölder class B_p for some $p > 1$. Examples of such potentials are all polynomials but also singular functions like $\max\{|x|, t^{\frac{1}{2}}\}^\alpha$ for $\alpha > -\frac{N+2}{p}$. We prove the L^p boundedness of the operators $D^2(\partial_t - \Delta + V)^{-1}$, $V(\partial_t - \Delta + V)^{-1}$ and $\partial_t(\partial_t - \Delta + V)^{-1}$, thus characterizing the domain of the operator $\mathcal{A}$ on $L^p(\mathbb{R}^{N+1})$.
The wide literature on the characterization of the domain of (elliptic) Schrödinger operator can be divided in two classes, concerning the assumptions on the potential V. The equality $D(-\Delta + V) = D(-\Delta) \cap D(V)$ holds in $L^p(\mathbb{R}^N)$, $1 < p < \infty$ either assuming an oscillation condition like $|\nabla V| \leq cV^{3/2}$, see [37], or assuming that V belongs to suitable Reverse Hölder classes. The two conditions are incomparable but one find easily examples of polynomials (which satisfy a reverse Hölder inequality) for which the oscillation condition above fails.
In [41] Shen proved the L^p boundedness of $D^2(-\Delta+V)^{-1}$ on $\mathbb{R}^N$ for $1 < p < \infty$, assuming $V \in B_p$ and under the restrictions $N \geq 3$, $p \geq \frac{N}{2}$, introducing an auxiliary function $m(x, V)$, which is well defined for $p \geq \frac{N}{2}$ and allows to estimate the fundamental solution.
In a recent work, P. Auscher and B. Ali , see [3], extended Shen's result removing the original restrictions on the space dimension and on p. In their proof they use a criterion to prove L^p boundedness of operators in absence of kernels, see [42, Theorem 3.1], [2, Theorem 3.14], and weighted mean value inequalities for nonnegative subharmonic functions, with respect to Muckenhoupt weights.
Following Shen's approach, W. Gao and Y. Jiang extended the results to the parabolic case. In [18], they consider the parabolic operator $\partial_t - \Delta + V$ where

$V \in B_p$ is a nonnegative potential depending only on the space variables and, under the assumptions $N \geq 3$ and $p > (N+2)/2$, they prove the boundedness of $V(\partial_t - \Delta + V)^{-1}$ in L^p.

We obtain the L^p boundedness of $V\mathcal{A}^{-1}$ (and consequently of $\partial_t \mathcal{A}^{-1}$ and $D^2\mathcal{A}^{-1}$) if $0 \leq V \in B_p$ for $1 < p < \infty$, without any restriction on the space dimension; moreover, our potentials may also depend on the time variable. Our approach is similar to that of [3]. We use a more general version of the boundedness criterion in absence of kernels in homogeneous spaces (see Theorem D.1.1) and the Harnack inequality for subsolutions of the heat equation. A crucial role is played by some properties of the B_p weights, originally proved in the classical case when $\mathbb{R}^N$ is equipped with the Lebesgue measure and the Euclidean distance. Since we need parabolic cylinders instead of balls of $\mathbb{R}^N$, we use the more general theory of B_p weights in homogeneous spaces, as treated in [48, Chapter I].

The chapter is organized as follows.

In Section 5.1 we introduce the reverse Hölder classes B_p and the Muckenhoupt classes A_p. We state some properties satisfied by these weights and we establish a relation between the two classes.

In Section 5.2 we define the parabolic Schrödinger operator in $L^p(\mathbb{R}^{N+1})$ and we prove some properties, in particular invertibility and consistency of the resolvent operators.

We start the last section by observing that $V\mathcal{A}^{-1}$ is always bounded in L^1. Then, using the Harnack inequality for subsolutions of the heat equation and an approximation procedure, we prove a weighted mean value inequality for positive solutions of the equation $\mathcal{A}u = 0$ with respect to B_p weights which allows us to apply Shen's interpolation theorem and deduce the boundedness of $V\mathcal{A}^{-1}$ in L^p.

For the whole chapter we fix the following notation.

Notation

Given $X_0 = (x_0^1, ..., x_0^N, t_0)$, $R > 0$, with parabolic cylinder of center $X_0 = (x_0, t_0)$ and radius R we mean the set

$$K = K(X_0, R) = \{(x^1, ..., x^N, t) \in R^{N+1} : |x^i - x_0^i| < R,\ |t - t_0| < R^2\}.$$

5.1 The parabolic reverse Hölder classes

The classical theory about Muckenhoupt and reverse Hölder classes has been originately formulated for weights in $\mathbb{R}^N$ endowed with the euclidean distance, see for example [47, Chapter V]. We will consider however potentials satisfying the "Reverse Hölder Property" with respect to cylinders rather than Euclidean balls. Many properties remain true in this setting. A theory on these classes of weights in homogeneous spaces (like $\mathbb{R}^{N+1}$ with the parabolic distance) is presented for example in [48, Chapter I] to which we refer for the proofs of the results stated in this Section and needed in what follows.

Definition 5.1.1. *Let* $1 < p \leq \infty$. *We say that* $\omega \in B_p$, *the class of the reverse Hölder weights of order* p, *if* $\omega \in L^p_{loc}$, $\omega > 0$ *a.e. and there exists a positive constant* C *such the inequality*

$$\left(\frac{1}{|K|}\int_K \omega(x,t)^p \, dx \, dt\right)^{\frac{1}{p}} \leq \frac{C}{|K|}\int_K \omega(x,t) \, dx \, dt \tag{5.1}$$

holds, for every parabolic cylinder K. *If* $p = \infty$, *the left hand side of the inequality above has to be replaced by the essential supremum of* ω *on* K. *The smallest positive constant* C *such that (5.1) holds is the* B_p *constant of* ω.

Observe that $B_q \subset B_p$ if $p < q$. An important feature of the B_p weights is the following self improvement property due to Gehring.

Proposition 5.1.2. *Assume that* $\omega \in B_p$ *for some* $p < \infty$. *Then there exists* $\varepsilon > 0$, *depending on the* B_p *constant of* ω, *such that* $\omega \in B_{p+\varepsilon}$.

The following property connects B_p weights with Muckenhoupt classes. In particular it implies that B_p weights induce doubling measures.

Definition 5.1.3. *Let* $1 < p < \infty$. *We say that* $\omega \in A_p$ *if it is nonnegative and it satisfies the inequality*

$$\frac{1}{|K|}\int_K \omega(x,t) dx \, dt \left[\frac{1}{|K|}\int_K \omega(x,t)^{-\frac{p'}{p}}\right] \leq A < \infty$$

for all K *parabolic cylinders and some positive constant* A.
The space A_1 *consists of nonnegative functions* ω *such that*

$$\frac{1}{|K|}\int_K \omega(x,t) dx \, dt \leq A\omega(x,t)$$

for almost every $(x,t) \in K$, *for all* K *parabolic cylinders and some positive constant* A.
In both cases, the smallest constant for which the inequality holds is called the A_p *bound of* ω.

Proposition 5.1.4. *If* $\omega \in B_p$ *for some* $p > 1$, *then there exists* $1 \leq t < \infty$ *and* $c > 0$, *depending on* p *and the* B_p *constant of* ω, *such that the inequality*

$$\left(\frac{1}{|K|}\int_K g\right)^t \leq \frac{c}{\omega(K)}\int_K g^t \omega \tag{5.2}$$

holds for all nonnegative functions g *and all parabolic cylinders* K. *Here* $\omega(K) = \int_K \omega$.

Remark 5.1.5. It is possible to prove that ω satisfies (5.2) is equivalent to say that $\omega \in A_t$ (see [47, Chapter V, 1.4]).

It is not hard to see that all polynomials belong to the reverse Hölder classes. The idea is that the space of all polynolmials of a fixed degree is a finite dimension space. Therefore all the norms are equivalent and the reverse Hölder inequality holds with a constant depending only on the degree of the polynomial and on N for all the cylinders with unitary radius. Up a rescaling the inequality follows for all the cylinders in $\mathbb{R}^{N+1}$. Also singular functions like $\max\{|x|, t^{\frac{1}{2}}\}^\alpha$ for $\alpha > -\frac{N+2}{p}$ belong to B_p. Here we give a proof.

Example 5.1.6. *The functions* $\max\{|x|, t^{\frac{1}{2}}\}^\alpha$ *belong to* B_p *for* $\alpha > -\frac{N+2}{p}$.

Proof. Observe that it is sufficient to prove the inequality for parabolic cylinders of unitary radius. A change of variables provides the estimate in the general case.
The hypothesis $\alpha > -\frac{N+2}{p}$ insures integrability near 0. Note that $f(x,t) = \max\{|x|, t^{\frac{1}{2}}\}^\alpha = d(x,0)^\alpha$ where d is the parabolic distance. Let $K(X_0, 1)$ be a parabolic cylinder of center X_0 and radius 1. Set

$$M = \max\left\{ \left(\int_{K(X_0,1)} f(X)^p \right)^{\frac{1}{p}} \left(\int_{K(X_0,1)} f(X) \right)^{-1}, \, X_0 : d(X_0, 0) \le 2 \right\}.$$

Suppose $d(X_0, 0) > 2$. If $X \in K(X_0, 1)$ we have

$$\frac{d(X,0)}{d(X_0,0)} \le \frac{d(X - X_0, 0)}{d(X_0,0)} + \frac{d(X_0,0)}{d(X_0,0)} \le 1 + \frac{1}{d(X_0,0)} \le \frac{3}{2}$$

and

$$\frac{d(X,0)}{d(X_0,0)} \ge \frac{d(X_0,0)}{d(X_0,0)} - \frac{d(X - X_0, 0)}{d(X_0,0)} \ge 1 - \frac{1}{2} = \frac{1}{2}.$$

Therefore if $d(X_0, 0) > 2$

$$\frac{1}{2} \le \frac{d(X,0)}{d(X_0,0)} \le \frac{3}{2}$$

and

$$\begin{aligned} \left(\int_{K(X_0,1)} f(X)^p \right)^{\frac{1}{p}} &\le \left(\frac{3}{2} d(X_0, 0) \right)^\alpha = \left(\frac{3}{2} \right)^\alpha \int_{K(X_0,1)} f(X_0) \\ &\le 3^\alpha \int_{K(X_0,1)} f(X). \end{aligned}$$

The reverse Hölder inequality is true with B_p constant given by the maximum between M and 3^α. □

5.2 Definition of the operator and some properties

In this section we assume that $0 \le V \in L^p_{loc}$ for some $1 \le p \le \infty$ and consider the parabolic operator

$$\mathcal{A} = \partial_t - \Delta + V$$

in L^p, endowed with the maximal domain

$$D_p(\mathcal{A}) = \{u \in L^p \ : \ Vu \in L^1_{loc}, \ \mathcal{A}u \in L^p\}.$$

Observe that C_c^∞ is contained in $D_p(\mathcal{A})$, since $V \in L^p_{loc}$. In some results, however, we shall only assume $0 \le V \in L^1_{loc}$.
We shall prove that $\mathcal{A}_p := (\mathcal{A}, D_p(\mathcal{A}))$ is a closed operator, that C_c^∞ is a core and that $\lambda + \mathcal{A}$ is invertible for positive λ. We follow Kato's strategy, see [19], where these results are obtained in the elliptic case.
Our main result is the following.

Theorem 5.2.1. *For every $\lambda > 0$ the operator $\lambda + \mathcal{A}_p$ is invertible and $\|(\lambda + \mathcal{A})^{-1}\|_p \le \frac{1}{\lambda}$. Moreover, if $1 \le p < \infty$, C_c^∞ is a core for $\mathcal{A}_p$*

The basic tool is a distributional inequality proved by Kato for the laplacian (see [39, Theorem X.2]). For completeness we provide here a short proof in the parabolic case.

Lemma 5.2.2 (Parabolic Kato's inequality)**.** *Let $u \in L^1_{loc}$ be such that $(\partial_t - \Delta)u \in L^1_{loc}$. Define*

$$\operatorname{sign}(u) = \begin{cases} 0 & if \quad u(x) = 0 \\ \overline{u(x)}/|u(x)| & if \quad u(x) \neq 0. \end{cases}$$

Then $|u|$ satisfies the following distributional inequality

$$(\partial_t - \Delta)|u| \le \operatorname{Re}\left[\operatorname{sign}(u)(\partial_t - \Delta)u\right].$$

Proof. We first suppose that $u \in C^\infty$. Define

$$u_\varepsilon(x) = \sqrt{|u|^2 + \varepsilon^2} \tag{5.3}$$

so that $u_\varepsilon \in C^\infty$. Since

$$u_\varepsilon \nabla u_\varepsilon = \operatorname{Re}[\overline{u}\nabla u]. \tag{5.4}$$

and $u_\varepsilon \ge |u|$, then (5.4) implies that

$$|\nabla u_\varepsilon| \le |\overline{u}||u_\varepsilon|^{-1}|\nabla u| \le |\nabla u|. \tag{5.5}$$

Taking the divergence of (5.4) we obtain

$$u_\varepsilon \Delta u_\varepsilon + |\nabla u_\varepsilon|^2 = \operatorname{Re}(\overline{u}\Delta u) + |\nabla u|^2$$

so by (5.5)

$$\Delta u_\varepsilon \ge \operatorname{Re}[\operatorname{sign}_\varepsilon(u)\Delta u], \tag{5.6}$$

where $\operatorname{sign}_\varepsilon(u) = \overline{u}/u_\varepsilon$. Differentiating (5.3) with respect to t we obtain

$$\partial_t u_\varepsilon = \operatorname{Re}[\operatorname{sign}_\varepsilon(u)\partial_t u] \tag{5.7}$$

and, combining (5.6) and (5.7),

$$(\partial_t - \Delta)u_\varepsilon \le \mathrm{Re}[\mathrm{sign}_\varepsilon(u)(\partial_t - \Delta)u]. \tag{5.8}$$

Let now $u \in L^1_{loc}$ be such that $(\Delta - \partial_t)u \in L^1_{loc}$ and let ϕ_n be an approximate identity. Since $u^n = u * \phi_n \in C^\infty$, then by (5.8)

$$(\partial_t - \Delta)(u^n)_\varepsilon \le \mathrm{Re}[\mathrm{sign}_\varepsilon(u^n)(\partial_t - \Delta)u^n]. \tag{5.9}$$

Fix $\varepsilon > 0$ and let $n \to \infty$. Then $u^n \to u$ in L^1_{loc} and a.e. (passing to a subsequence, if necessary). Thus $\mathrm{sign}_\varepsilon(u^n) \to \mathrm{sign}_\varepsilon(u)$ a.e. Since $(\partial_t - \Delta)u^n = ((\partial_t - \Delta)u) * \phi_n$ and $(\partial_t - \Delta)u \in L^1_{loc}$, then $(\partial_t - \Delta)u^n \to (\partial_t - \Delta)u$ in L^1_{loc}, too. It is now easy to see that $\mathrm{sign}_\varepsilon(u^n)(\partial_t - \Delta)u^n$ converges in the sense of distributions to $\mathrm{sign}_\varepsilon(u)(\partial_t - \Delta)u$. Thus, letting $n \to \infty$ in (5.8) we conclude that

$$(\partial_t - \Delta)u_\varepsilon \le \mathrm{Re}[\mathrm{sign}_\varepsilon(u)(\partial_t - \Delta)u].$$

Now taking $\varepsilon \to 0$ we obtain the desired inequality for u, since $\mathrm{sign}_\varepsilon(u) \to \mathrm{sign}(u)$ and $|\mathrm{sign}_\varepsilon(u)| \le 1$. ☐

Remark 5.2.3. Changing t with $-t$ one obtains that if u, $(\partial_t + \Delta)u \in L^1_{loc}$, then

$$(\partial_t + \Delta)|u| \le \mathrm{Re}[\mathrm{sign}(u)(\partial_t + \Delta)u].$$

The following results are easy consequences of Kato's inequality.

Lemma 5.2.4. *Let* $0 \le V \in L^1_{loc}$. *Assume that* $u, (\partial_t - \Delta)u, Vu \in L^1_{loc}$ *and set, for* $\lambda \ge 0$, $f = (\lambda + \mathcal{A})u$. *Then*

$$(\lambda + \partial_t - \Delta + V)|u| \le |f|. \tag{5.10}$$

Proof. The claim immediately follows by Lemma 5.2.2. Indeed

$$(\lambda + \partial_t - \Delta + V)|u| \le \mathrm{Re}[\mathrm{sign}(u)((\partial_t - \Delta)u + \lambda u + Vu)] = \mathrm{Re}[f\,\mathrm{sign}(u)] \le |f|.$$

☐

Lemma 5.2.5. *For every positive* $\lambda > 0$ *the operator* $(\lambda + \partial_t - \Delta)^{-1}$ *is a positive map of* $\mathcal{S}'$ *onto itself.*

Proof. Since $\lambda - \partial_t - \Delta$ is invertible from $\mathcal{S}$ onto $\mathcal{S}$, its adjoint operator $\lambda + \partial_t - \Delta$ is invertible from $\mathcal{S}'$ into itself. Let now $0 \le \psi \in \mathcal{S}'$ and let $\phi \in \mathcal{S}'$ be such that $0 \le \psi = (\lambda + \partial_t - \Delta)\phi$. If $0 \le u \in \mathcal{S}$, then

$$\langle \phi, u\rangle = \langle (\lambda + \partial_t - \Delta)^{-1}(\lambda + \partial_t - \Delta)\phi, u\rangle = \langle (\lambda + \partial_t - \Delta)\phi, (\lambda - \partial_t - \Delta)^{-1}u\rangle \ge 0$$

since $(\lambda - \partial_t - \Delta)^{-1}$ is positive on $\mathcal{S}$, by the maximum principle. This proves that $\phi = (\lambda + \partial_t - \Delta)^{-1}\psi$ is positive. ☐

An estimate for the resolvent operator easily follows.

Proposition 5.2.6. *Let* $1 \le p \le \infty$, $\lambda > 0$. *Then, if* $u \in D_p(\mathcal{A})$,

$$\lambda \|u\|_p \le \|(\lambda + \mathcal{A})u\|_p. \tag{5.11}$$

Proof. Let $u \in D_p(\mathcal{A})$, set $f = (\lambda + \mathcal{A})u \in L^p$. By (5.10)

$$(\lambda + \partial_t - \Delta)|u| \le (\lambda + \mathcal{A})|u| \le |f|$$

and Lemma 5.2.5 yields

$$|u| \le (\lambda + \partial_t - \Delta)^{-1}|f|. \tag{5.12}$$

Then

$$\|u\|_p \le \|(\lambda + \partial_t - \Delta)^{-1}|f|\|_p \le \frac{1}{\lambda}\|f\|_p.$$

□

The positivity of the resolvent is proved along the same way.

Proposition 5.2.7. *Let* $0 \le V \in L^1_{loc}$ *and* $\lambda > 0$. *If* $u, (\partial_t - \Delta)u, Vu \in L^1_{loc}$ *and* $f = (\lambda + \mathcal{A})u \ge 0$, *then* $u \ge 0$.

Proof. Subtracting the equality $f = (\lambda + \mathcal{A})u \ge 0$ from (5.10) we obtain $(\lambda + \partial_t - \Delta + V)(|u| - u) \le 0$, hence $(\lambda + \partial_t - \Delta)(|u| - u) \le 0$. Lemma 5.2.5 implies $|u| - u \le 0$ so that $u = |u|$. □

Proposition 5.2.8. *For every* $1 \le p \le \infty$, *the operator* $\mathcal{A}_p$ *is closed. Moreover, if* $\lambda > 0$, $\lambda + \mathcal{A}_p$ *has closed range.*

Proof. Let $(u_n) \subset D_p(\mathcal{A})$ such that

$$u_n \to u, \quad \mathcal{A}u_n = (\partial_t - \Delta)u_n + Vu_n = f_n \to f \text{ in } L^p.$$

We apply (5.10) to $u = u_n - u_m$, $f = f_n - f_m$ and $\lambda = 0$ obtaining

$$(\partial_t - \Delta + V)|u_n - u_m| \le |f_n - f_m|.$$

Then, for every $0 \le \phi \in C_c^\infty$

$$0 \le \langle V|u_n - u_m|, \phi\rangle \le \langle |f_n - f_m|, \phi\rangle + \langle |u_n - u_m|, (\Delta + \partial_t)\phi\rangle.$$

Letting n, m to infinity, the right hand side of the previous inequality tends to 0 and this shows that $Vu_n\phi$ is a Cauchy sequence in L^1. Since its limit is $Vu\phi$ we conclude (by the arbitrariness of ϕ) that $Vu \in L^1_{loc}$ and that $Vu_n \to Vu$ in L^1_{loc}. Then $f_n = (\partial_t - \Delta + V)u_n \to (\partial_t - \Delta + V)u$ in the sense of distributions. On the other hand $f_n \to f$ in L^p, therefore $u \in D_p(\mathcal{A})$ and $f = (\partial_t - \Delta + V)u \in L^p$. This proves the closedness of $\mathcal{A}$.
Finally, $\lambda + \mathcal{A}$ has closed range, by (5.11). □

Proof (Theorem 5.2.1). Assume first that $1 \le p < \infty$. Since $\mathcal{A}_p$ is closed and has closed range, we have only to prove that $(\lambda + \mathcal{A})(C_c^\infty)$ is dense in L^p.

Let $u \in L^{p'}$ such that $\int(\lambda + \partial_t - \Delta + V)\phi u = 0$ for every $\phi \in C_c^\infty$. We have to show that $u = 0$. Evidently u satisfies $\lambda u - \partial_t u - \Delta u + Vu = 0$ in the sense of distributions and, since $V \in L^p_{loc}$ and $u \in L^{p'}$, $Vu \in L^1_{loc}$. Thus $u \in D_{p'}(\mathcal{B})$ and $(\lambda + \mathcal{B})u = 0$, where $\mathcal{B} = -\partial_t - \Delta + V$. The injectivity of $\lambda + \mathcal{B}$ (that follows from Proposition 5.2.6 changing t to $-t$) implies $u = 0$ and proves the density of $(\lambda + \mathcal{A})(C_c^\infty)$ in L^p.

Next we consider the case where $p = \infty$. Let $0 \le f \in L^\infty$ and consider a sequence $f_n \in L^\infty \cap L^1$ such that $0 \le f_n \nearrow f$. By the first part of the proof, there are $u_n \in D_1(\mathcal{A})$ such that $(\lambda + \mathcal{A})u_n = f_n$. By Proposition 5.2.7 the sequence (u_n) is increasing and consists of nonnegative functions and, since $\lambda\|u_n\|_\infty \le \|f_n\|_\infty \le \|f\|_\infty$, its (pointwise) limit u belongs to L^∞. Moreover $Vu_n \to Vu$ in L^1_{loc} because $V \in L^\infty_{loc}$ and $u_n \to u$, $0 \le u_n \le u$. Hence $f_n = (\lambda + \mathcal{A})u_n \to (\lambda + \partial_t - \Delta)u + Vu$ in the sense of distributions. But $f_n \to f$ monotonically and then $(\lambda + \mathcal{A})u = f$. This means that $u \in D_\infty(\mathcal{A})$ and $(\lambda + \mathcal{A})u = f$. Since a general $f \in L^\infty$ is a linear combination of positive elements, the proof is complete. □

Finally, we prove the consistency of the resolvent operators.

Proposition 5.2.9. *Let $1 \le p \le q$ and $0 \le V \in L^q_{loc}$. If $\lambda > 0$ and $f \in L^p \cap L^q$, then $(\lambda + \mathcal{A}_p)^{-1} f = (\lambda + \mathcal{A}_q)^{-1} f$.*

Proof. Let $u = (\lambda + \mathcal{A}_p)^{-1} f$, $v = (\lambda + \mathcal{A}_q)^{-1} f$ and $w = u - v$. Then $w, Vw \in L^1_{loc}$ and $(\partial_t - \Delta)w = -(\lambda + V)w \in L^1_{loc}$. Since $(\lambda + \mathcal{A})w = 0$, by Proposition 5.2.7 we deduce that $w = 0$. □

5.3 Characterization of the domain of $\mathcal{A}$

In this section we assume that all functions are real-valued.

5.3.1 The operator $\mathcal{A}$ on L^1.

It is easy to obtain a-priori estimates for $p = 1$, leading to a (partial) description of $D_1(\mathcal{A})$. They will also play a key role in the proof of the a-priori estimates in L^p.

Lemma 5.3.1. *Assume that $0 \le V \in L^1_{loc}$. For every $u \in D_1(\mathcal{A})$ we have*

$$\|Vu\|_1 \le \|\mathcal{A}u\|_1, \quad \|(\partial_t - \Delta)u\|_1 \le 2\|\mathcal{A}u\|_1. \tag{5.13}$$

Proof. Let $h_n : \mathbb{R} \to \mathbb{R}$ be a sequence of smooth functions such that $|h_n| \le C$, $h_n'(s) \ge 0$ and $h_n(s) \to \operatorname{sign}(s)$ for $n \to \infty$ and for every $s \in \mathbb{R}$. Let H_n be such that $H_n' = h_n$ and $H_n(0) = 0$. If $u \in C_c^\infty$ then, by the Lebesgue convergence Theorem, we have

$$\int_{\mathbb{R}^{N+1}} \operatorname{sign}(u)\partial_t u = \lim_n \int_{\mathbb{R}^{N+1}} h_n(u)\partial_t u = \lim_n \int_{\mathbb{R}^{N+1}} \partial_t(H_n(u)) = 0, \tag{5.14}$$

$$-\int_{\mathbb{R}^{N+1}} \text{sign}(u)\Delta u = -\lim_n \int_{\mathbb{R}^{N+1}} h_n(u)\Delta u = \lim_n \int_{\mathbb{R}^{N+1}} |\nabla u|^2 h_n'(u) \geq 0. \tag{5.15}$$

Therefore, if $\mathcal{A}u = f$ we obtain

$$\int_{\mathbb{R}^{N+1}} V|u| \leq \int_{\mathbb{R}^{N+1}} \text{sign}(u)(\partial_t - \Delta + V)u = \int_{\mathbb{R}^{N+1}} f\,\text{sign}(u) \leq \int_{\mathbb{R}^{N+1}} |f|$$

and the first inequality is proved for $u \in C_c^\infty$. Since C_c^∞ is a core for $\mathcal{A}_1$ it is easily seen that it extends to every $u \in D_1(\mathcal{A})$.

The second inequality follows from the first, since $(\partial_t - \Delta) = \mathcal{A} - V$. □

The characterization of the domain of $\mathcal{A}_1$ is an immediate consequence of the lemma above. We refer to [50] for similar results in the elliptic case.

Proposition 5.3.2. *If* $0 \leq V \in L^1_{loc}$, *then*

$$D_1(\mathcal{A}) = \{u \in L^1 \ : \ Vu \in L^1, \ (\partial_t - \Delta)u \in L^1\}.$$

5.3.2 A priori estimates in $L^p(\mathbb{R}^{N+1})$.

We investigate when (5.13) holds for other values of p. We remark that (5.13) can fail even for $p = 2$ and in the elliptic case, see e.g. [31, Example 3.7].
The B_p property of the potential is a sufficient condition to characterize the domain of the operator. In fact we prove the following result.

Theorem 5.3.3. *Let* $1 < p < \infty$. *If* $0 \leq V \in B_p$ *then there exists a positive constant* C *depending only on* p *and the* B_p *constant of* V, *such that*

$$\|Vu\|_p \leq C\|\partial_t u - \Delta u + Vu\|_p \tag{5.16}$$

for all $u \in D_p(\mathcal{A})$. *In particular,*

$$D_p(\mathcal{A}) = \{u \in W_p^{2,1} : Vu \in L^p\}.$$

We will apply Theorem D.1.1 to the operator $T = V\mathcal{A}^{-1}|\cdot|$ with $p_0 = 1$, a suitable $q_0 > p$ and $\alpha_1 = 3$, $\alpha_2 = 4$. Therefore we have to prove that, if K is a parabolic cylinder and $f \in L_c^\infty$ has support in $\mathbb{R}^{N+1} \setminus 4K$, $u = \mathcal{A}^{-1}f$ satisfies

$$\left(\frac{1}{|K|}\int_K (V|u|)^{q_0}\right)^{\frac{1}{q_0}} \leq \frac{C}{|3K|}\int_{3K} V|u|$$

for some positive C independent of f. Observe that u satisfies the homogeneous equation

$$\mathcal{A}u = (\partial_t - \Delta + V)u = 0$$

in $4K$. As first step we prove a mean value inequality for functions u as above.

Lemma 5.3.4. *Assume that* $0 < \varepsilon \leq V \in L^p_{loc}$. *For every* $r > 0$ *there exists a positive constant* $C = C(r)$ *(hence independent of* ε*) such that*

$$\sup_K u \leq C\left(\frac{1}{|3K|}\int_{3K} u^r\right)^{\frac{1}{r}}$$

for all parabolic cylinders K, $0 \le f \in L^\infty_c(\mathbb{R}^{N+1})$ with support in $\mathbb{R}^{N+1} \setminus 4K$ and $u = \mathcal{A}^{-1}f$.

Proof. Let $K = K((x_0,t_0),R)$ a parabolic cylinder and $0 \le f \in L^\infty_c(\mathbb{R}^{N+1})$ with support in $\mathbb{R}^{N+1} \setminus 4K$. By Theorem 5.2.1 there exists $u \in D_p(\mathcal{A})$ such that

$$\mathcal{A}u = f \quad \text{in } \mathbb{R}^{N+1}.$$

By Proposition 5.2.7 $u \ge 0$. We are going to use Harnack's inequality where, however, more regularity on the solutions is required and then an approximation procedure is needed. Let $\mathcal{A}_k$ be the operators with bounded potentials $V_k = V \wedge k$. For every k let $0 \le u_k$ be such that $(\partial_t - \Delta + V_k)u_k = f$. The functions u_k are solutions of parabolic equations with bounded coefficients, then for all $k \in \mathbb{N}$ $u_k \in W^{2,1}_q(\mathbb{R}^{N+1})$ for all $1 < q < \infty$. Since f has support in $\mathbb{R}^{N+1} \setminus 4K$,

$$(\partial_t - \Delta)u_k = -V_k u_k \le 0 \quad \text{in} \quad 4K.$$

Given a parabolic cylinder $K = K((x_0,t_0),R)$ and a positive constant $c > 0$, we denote by cK the cylinder with the same center as K and radius cR and by $\widetilde{K}$ the set $K \cap \{t < t_0\}$.
Let K_1 be the cylinder of center $(x_0, t_0 + R^2)$ and radius $\sqrt{2}R$. Obviously $K \subset \widetilde{K_1}$ and $\widetilde{2K_1} \subset 2K_1 \subset 3K \subset 4K$. It follows that

$$(\partial_t - \Delta)u_k = -V_k u_k \le 0 \quad \text{in} \quad \widetilde{2K_1}.$$

By [24, Theorem 7.21] or see [35], for any $r > 0$ there exists $C = C(r) > 0$ such that

$$\sup_{\widetilde{K_1}} u_k \le C\left(\frac{1}{R^{n+2}}\int_{\widetilde{2K_1}} u_k^r\right)^{\frac{1}{r}}$$

and hence

$$\begin{aligned}\sup_K u_k \le \sup_{\widetilde{K_1}} u_k &\le C\left(\frac{1}{R^{n+2}}\int_{\widetilde{2K_1}} u_k^r\right)^{\frac{1}{r}} \le C\left(\frac{1}{R^{n+2}}\int_{3K} u_k^r\right)^{\frac{1}{r}} \qquad (5.17)\\ &= C\left(\frac{1}{|3K|}\int_{3K} u_k^r\right)^{\frac{1}{r}}.\end{aligned}$$

Let us observe that the constant C is independent of the potential V_k. This allows us to let $k \to \infty$ in the above inequality.
Let $k,\ m \in \mathbb{N}$ with $k > m$. Then

$$\partial_t(u_k - u_m) - \Delta(u_k - u_m) + V_k(u_k - u_m) = (V_m - V_k)u_m \le 0$$

and by Proposition 5.2.7 (or simply by the maximum principle) $u_k - u_m \le 0$. Therefore (u_k) is decreasing and converges pointwise to a function $w \ge 0$. Moreover, by Lemma 5.3.1, $\|V_k u_k\|_1 \le \|f\|_1$ for every $k \in \mathbb{N}$ and then, by Fatou's Lemma, $Vw \in L^1$. By Proposition 5.2.6, $\|u_k\|_q \le C\|f\|_q$ for all $1 \le$

$q \le \infty$ and, since $u_k \to w$ pointwise, $w \in L^q$ for all $1 \le q \le \infty$.
Since for every $\phi \in C_c^\infty$

$$\int_{\mathbb{R}^{N+1}} u_k(-\partial_t\phi - \Delta\phi + V_k\phi) = \int_{\mathbb{R}^{N+1}} f\phi,$$

letting k to infinity we get

$$\int_{\mathbb{R}^{N+1}} w(-\partial_t\phi - \Delta\phi + V\phi) = \int_{\mathbb{R}^{N+1}} f\phi$$

and therefore $\mathcal{A}w = f$ in the sense of distributions. This shows that w belongs to $D_p(\mathcal{A})$ and, by Theorem 5.2.1, $w = u$, that is u_k converges to u pointwise. Since u_k is decreasing, (5.17) yields

$$\sup_K u \le \sup_K u_k \le C\left(\frac{1}{|3K|}\int_{3K} (u_k)^r\right)^{\frac{1}{r}}. \tag{5.18}$$

Finally, u_k is decreasing, therefore $u_k^r \le u_1^r \in L^1$ and letting $k \to \infty$ in (5.18) we obtain the thesis by dominated convergence. □

Now we prove that Lemma 5.3.4 holds if we replace the Lebesgue measure with that induced by the density V.

Lemma 5.3.5. *Suppose* $0 < \varepsilon \le V \in B_p$ *and fix* $0 < s < \infty$ *and* u *as in Lemma 5.3.4. Then for every cylinder* K

$$\sup_K u \le \left(\frac{C}{V(3K)}\int_{3K} Vu^s\right)^{\frac{1}{s}}$$

where C *depends only on* s, p *and the* B_p *constant of* V *and*

$$V(3K) = \int_{3K} V.$$

Proof. Let $0 < s < \infty$ and K be a parabolic cylinder of $\mathbb{R}^{N+1}$. We fix t as in Proposition 5.1.4. By using Lemma 5.3.4 with $r = \frac{s}{t}$ and (5.2) we obtain

$$\sup_K u \le C\left(\frac{1}{|3K|}\int_{3K} u^{\frac{s}{t}}\right)^{\frac{t}{s}} \le C\left(\frac{1}{V(3K)}\int_{3K} Vu^s\right)^{\frac{1}{s}}.$$

□

By combining the estimate in Lemma 5.3.5 and the B_q property we deduce the following.

Corollary 5.3.6. *Let* $0 < \varepsilon \le V \in B_p$*,* $0 < s < \infty$ *and* u *as in Lemma 5.3.4. Then for every cylinder* K

$$\left(\frac{1}{|K|}\int_K (Vu^s)^p\right)^{\frac{1}{p}} \le \frac{C}{|3K|}\int_{3K} Vu^s,$$

where C *depends only on* s, p *and the* B_p *constant of* V*.*

Proof. By using the B_p property of V and Lemma 5.3.5 we obtain

$$\begin{aligned}\left(\frac{1}{|K|}\int_K (Vu^s)^p\right)^{\frac{1}{p}} &\le \left(\frac{1}{|K|}\int_K V^p\right)^{\frac{1}{p}} \sup_K u^s \le C\left(\frac{1}{|K|}\int_K V\right)\sup_K u^s \\ &\le \frac{C}{|3K|}\int_{3K} Vu^s.\end{aligned}$$

□

We can now prove our main result.

Proof (Theorem 5.3.3). Suppose first that $0 < \varepsilon \le V \in B_p$ for some ε. By Proposition 5.1.2 there exists $q_0 > p$ such that $V \in B_{q_0}$.
Let K be a parabolic cylinder in $\mathbb{R}^{N+1}$ and $f \in L_c^\infty(\mathbb{R}^{N+1})$ with support in $\mathbb{R}^{N+1} \setminus 4K$. We set $T = V\mathcal{A}^{-1}|\cdot|$. Then $Tf = Vu$ and $u \ge 0$ by Proposition 5.2.7. Note that, since $V \ge \varepsilon > 0$, Proposition 5.2.9 shows that T acts in a consistent way in the L^q scale. By Corollary 5.3.6 with $s = 1$,

$$\left(\frac{1}{|K|}\int_K (Tf)^{q_0}\right)^{\frac{1}{q_0}} = \left(\frac{1}{|K|}\int_K (Vu)^{q_0}\right)^{\frac{1}{q_0}} \le \frac{C}{|3K|}\int_{3K} Vu = \frac{C}{|3K|}\int_{3K} |Tf|.$$

By Lemma 5.3.1 T is bounded on L^1 and, by Proposition 5.2.7, it is also sublinear. Choosing $p_0 = 1$ and q_0 as above in Theorem D.1.1, we deduce that

$$\|Vu\|_p = \|Tf\|_p \le C\|f\|_p \tag{5.19}$$

for every $f \in L_c^\infty$, where C depends only on p and the B_p constant of V. Since, by Proposition 5.2.7 again, the operator $V\mathcal{A}^{-1}$ preserves positivity, we have that $|V\mathcal{A}^{-1}f| \le Tf$. Therefore by 5.19 we deduce that

$$\|V\mathcal{A}^{-1}f\|_p \le C\|f\|_p$$

for every $f \in L_c^\infty$ and finally, by approximation, for every $f \in L^p$. Then the identity

$$(\partial_t - \Delta)u = f - Vu \in L^p$$

proves, by parabolic regularity, that the distribution u belongs to $W_p^{2,1}$. Then

$$D_p(\mathcal{A}) \subset \{u \in W_p^{2,1} : Vu \in L^p\}$$

and, since the opposite inclusion is obvious, the characterization of the domain is proved. Now we prove (5.16) in the general case when $V \ge 0$. Let $u \in D_p(A)$. then for every $\varepsilon > 0$ we have

$$\|(V+\varepsilon)u\|_p \le C\|\partial_t u - \Delta u + (V+\varepsilon)u\|_p.$$

Since C depends only on p and the B_p constant of $V + \varepsilon$ which is independent of $0 < \varepsilon \le 1$, letting $\varepsilon \to 0$ the proof is complete. □

Finally we show that the results of this section hold when the time variable varies in an interval, rather than in the whole space. We fix $-\infty \le S < T \le \infty$ and consider the set

$$Q(S,T) = \mathbb{R}^N \times (S,T)$$

and the operator $\mathcal{A}$ endowed with the domain

$$D_p^{S,T} = \left\{u \in W_p^{2,1}\left(Q(S,T)\right) : Vu \in L^p\left(Q(S,T)\right), \quad u(\cdot, S) = 0\right\}.$$

Clearly the initial condition $u(\cdot, S) = 0$ makes sense only when $S > -\infty$.

Proposition 5.3.7. *If $1 < p < \infty$, $0 \le V \in B_p$ and $\lambda > 0$, then the operator $\lambda + \mathcal{A}$ is invertible from $D_p^{S,T}$ to $L^p\left(Q(S,T)\right)$.*

Proof. Given $f \in L^p\left(Q(S,T)\right)$, let $g \in L^p$ be its extension by 0 outside the time interval (S,T) and $u \in D_p(\mathcal{A})$ such that $\lambda u + \mathcal{A}u = g$ in $\mathbb{R}^{N+1}$ (hence in $Q(S,T)$). Since $\lambda u + \mathcal{A}u = 0$ for $t \le S$ (when $S > -\infty$), multiplying this identity by $u|u|^{p-2}$ and integrating by parts we get $u = 0$ for $t \le S$, hence $u(\cdot, S) = 0$ and $u \in D_p^{S,T}$. Infact we have

$$\int_{Q(-\infty,S)} (\lambda + V)|u|^p + \frac{1}{p}\int_{Q(-\infty,S)} \partial_t(|u|^p) - \int_{Q(-\infty,S)} u|u|^{p-2}\Delta u = 0,$$

which implies, since $\int_{Q(-\infty,S)} u|u|^{p-2}\Delta u \le 0$ (see Appendix C),

$$\int_{Q(-\infty,S)} (\lambda + V)|u|^p + \frac{1}{p}\int_{\mathbb{R}^N}\int_{-\infty}^{S} \partial_t(|u|^p) \le 0$$

and then $u = 0$ for $t \le S$. This proves the existence part. Concerning uniqueness, assume that $v \in D_p^{S,T}$ satisfies $\lambda v + \mathcal{A}v = 0$ in $Q_{S,T}$. Multiplying by $v|v|^{p-2}$, integrating by parts as above and using the initial condition one easily shows that $v = 0$. $\square$

As usual, if the interval (S,T) is finite, the condition $\lambda > 0$ in not needed.

Appendix A

Embedding Theorems and Solvability of Cauchy problems

In this appendix we only state some results about embeddings of parabolic Sobolev spaces and solvability of Cauchy problems in the same spaces useful to prove integrability and regularity of kernels in Section 2.1.2.
For their proofs we refer to [23, Lemma II.3.3, Theorem IV.9.1] and [20, Theorem 9.2.3].

According to notation used in [23], we introduce the norm

$$\|f\|_{r,q_T}^{loc} = \sup_{q_T} \|f\|_{L^r(q_T)}$$

where the supremum is taken over all the cylinders $q_T = \omega \times (0,T)$, the bases ω of which are some domain of unit measure, for examples cubes of $\mathbb{R}^N$. We consider the elliptic operator A given by $Au(x,t) = \sum_{i,j=1}^N a_{ij}(x)D_{ij}u(x,t) + \sum_{i=1}^N F_i(x)D_i u(x,t) - V(x)u(x,t)$ with V positive.

We recall that the parabolic distance between the points $X_1 = (x_1,t_1)$ and $X_2 = (x_2,t_2)$ is defined as

$$d(X_1,X_2) = \max\{|x^i - x_0^i|,\ 1 \le i \le N,\ |t-t_0|^{\frac{1}{2}}\}.$$

If u is a function defined on $\mathbb{R}^{N+1}$, given $\delta \in (0,1)$, we denote

$$[u]_{\delta,\frac{\delta}{2};Q_T} = \sup_{X_1 \neq X_2, X_i \in Q_T} \frac{|u(X_1)-u(X_2)|}{d(X_1,X_2)^\delta};$$

$$|u|_{\delta,\frac{\delta}{2};Q_T} = \sup \|u\|_{\infty,Q_T} + [u]_{\delta,\frac{\delta}{2};Q_T}.$$

By $C^{\delta,\frac{\delta}{2}}(Q_T)$ we denote the space of the functions u for which $|u|_{\delta,\frac{\delta}{2};Q_T}$ is finite. If u is a function depending only on the space variable we use the analogous notation for the classical Hölder spaces.

Theorem A.0.8. *Let $q > 1$. Suppose that the coefficients a_{ij} of the operator A are bounded continuous functions in Q_T, while the coefficients F_i and V have finite norms $\|F_i\|^{loc}_{r,q_T}$ and $\|V\|^{loc}_{s,q_T}$ with*

$$r = \begin{cases} \max(q, N+2) & \textit{for} \quad q \neq N+2 \\ N+2+\varepsilon & \textit{for} \quad q = N+2. \end{cases}.$$

$$s = \begin{cases} \max(q, \frac{N+2}{2}) & \textit{for} \quad q \neq \frac{N+2}{2} \\ \frac{N+2}{2}+\varepsilon & \textit{for} \quad q = \frac{N+2}{2}. \end{cases}.$$

and ε arbitrarily small positive number. Suppose moreover that the quantities $\|F_i\|^{loc}_{r,q(t,t+\tau)}$ and $\|V\|^{loc}_{s,q(t,t+\tau)}$ tend to zero for $\tau \to 0$. Then, for any $f \in L^q(Q_T)$, $\phi \in W_q^{2-\frac{2}{q}}(\mathbb{R}^N)$, the problem

$$\begin{cases} \partial_t u - Au = f & \textit{in} \quad Q_T \\ u(x,0) = \phi \end{cases}$$

has a unique solution $u \in W^{2,1}_q(Q_T)$. It satisfies the estimate

$$\|u\|_{W^{2,1}_q(Q_T)} \leq C(\|f\|_{L^q(Q_T)} + \|\phi\|_{W_q^{2-\frac{2}{q}}(\mathbb{R}^N)}).$$

Theorem A.0.9. *For any function $u \in W^{2,1}_q(Q_T)$ the inequality*

$$\|\partial_t^r D_x^s u\|_{p,Q_T} \quad \leq \quad C_1(\|\partial_t u\|_{q,Q_T} + \|D^2_x u\|_{q,Q_T}) + C_2\|u\|_{q,Q_T}$$

is valid under the condition $p \geq q$, $2-2r-s-\left(\frac{1}{q}-\frac{1}{p}\right)(N+2) \geq 0$ and for some constants C_1, C_2 depending on r, s, N, q, p. In addition, if $2-2r-s-\frac{N+2}{q} > 0$, then for any $0 \leq \delta < 2-2r-s-\frac{N+2}{q}$

$$[\partial_t^r D_x^s u]_{\delta,Q_T} \leq C_3(\|\partial_t u\|_{q,Q_T} + \|D^2_x u\|_{q,Q_T}) + C_4\|u\|_{q,Q_T}$$

for some constants C_3, C_4 depending on r, s, N, q, p.

Finally we state a solvability result in spaces of Hölder functions used in Section 2.1.2. It can be found in [20, Theorem 9.2.3].

Theorem A.0.10. *Let A be the second order elliptic operator above defined and suppose that a, F, V are Hölder continuous for some $\delta \in (0,1)$ and with $|a|_{\delta;\mathbb{R}^N}$, $|F|_{\delta,\mathbb{R}^N}$, $|V|_{\delta,\mathbb{R}^N} \leq K$. Then, for any $f \in C^{\delta,\frac{\delta}{2}}(Q_T)$, $\phi \in C^{2+\delta}(\mathbb{R}^N)$, the problem*

$$\begin{cases} \partial_t u - Au = f & \textit{in} \quad Q_T \\ u(x,0) = \phi \end{cases}$$

has a unique solution $u \in C^{2+\delta,1+\frac{\delta}{2}}(Q_T)$. *It satisfies the estimate*

$$|u|_{2+\delta,1+\frac{\delta}{2};Q_T} \le C(|f|_{\delta,\frac{\delta}{2};Q_T} + |\phi|_{2+\delta;\mathbb{R}^N}$$

for some positive constant C *depending on* $N, \delta, K,\ \lambda$ *and the ellipticity constant of* A.

Appendix B

The Karamata Theorem

In Chapter 3, to obtain the asymptotic distribution of eigenvalues, we applied the following Tauberian theorem due to Karamata. For the proof we refer to [44, Theorem 10.3].
We prove also a weaker version which we have not been able to find in the literature.

Let μ a positive Borel measure on $[0,\infty)$ such that

$$\hat{\mu}(t)=\int_0^\infty e^{-tx}d\mu(x)<\infty$$

for all $t>0$. The function $\hat{\mu}:(0,\infty)\to\mathbb{R}$ is called the Laplace Transform of μ. The theorem relates the asymptotic behavior of $\mu([0,x])$ as $x\to\infty$ to the asymptotic behavior of $\hat{\mu}(t)$ as $t\to 0$.

Theorem B.0.11. *Let $r\geq 0$, $a\in\mathbb{R}$. The following are equivalent:*

(i) $\lim_{t\to 0} t^r\hat{\mu}(t)=a$;

(ii) $\lim_{x\to\infty} x^{-r}\mu([0,x])=\dfrac{a}{\Gamma(r+1)}$

where Γ is the Euler's Gamma Function.

We have also used the following weaker version of the previous theorem which we have not been able to find in the literature. In the proposition below we fix a nonnegative, nondecreasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ such that $\exp\{-\lambda_n t\}\in l^1(\mathbb{R})$ for every $t>0$.

Proposition B.0.12. *Let $r>0$, $C_1>0$ such that*

$$\limsup_{t\to 0} t^r\sum_{n\in\mathbb{N}} e^{-\lambda_n t}\leq C_1. \tag{B.1}$$

Then

$$\limsup_{\lambda\to\infty}\lambda^{-r}N(\lambda)\leq C_1\frac{e^r}{r^r}.$$

Moreover if (B.1) holds and

$$\liminf_{t\to 0} t^r \sum_{n\in\mathbb{N}} e^{-\lambda_n t} \geq C_2 \tag{B.2}$$

for some $C_2 > 0$ *then*

$$\liminf_{\lambda\to\infty} \lambda^{-r} N(\lambda) \geq C_3$$

for some positive C_3*.*

Proof. Let us suppose that B.1 holds. Then, given $\varepsilon > 0$, there exists $\delta > 0$ such that if $t \leq \delta$

$$\sum_{n\in\mathbb{N}} e^{-\lambda_n t} \leq \frac{C_1+\varepsilon}{t^r}.$$

This implies that for $\lambda > 0$

$$N(\lambda)e^{-\lambda t} = \sum_{\lambda_n\leq\lambda} e^{-\lambda t} \leq \sum_{n\in\mathbb{N}} e^{-\lambda_n t} \leq \frac{C_1+\varepsilon}{t^r}.$$

So

$$N(\lambda) \leq (C_1+\varepsilon)\frac{e^{\lambda t}}{t^r}$$

in $[0,\delta]$. Minimizing on t in such interval it follows

$$N(\lambda) \leq (C_1+\varepsilon)\lambda^r \frac{e^r}{r^r}$$

for λ large enough.

Suppose now that (B.1) and (B.2) hold. Then, given $\varepsilon > 0$, for t small enough, we have

$$\frac{C_2-\varepsilon}{t^r} \leq \sum_{n\in\mathbb{N}} e^{-\lambda_n t} = \sum_{\lambda_n\leq\lambda} e^{-\lambda_n t} + \sum_{\lambda\leq\lambda_n\leq 2\lambda} e^{-\lambda_n t} + \ldots \leq \sum_{k=1}^{\infty} e^{-\lambda(k-1)t} N(k\lambda).$$

We have

$$sN(s\lambda) \geq \sum_{k=1}^{s} e^{-\lambda(k-1)t} N(k\lambda)$$

and, using the upper bound obtained in the first part of the proof, for λ large enough,

$$sN(s\lambda) \geq \frac{C_2-\varepsilon}{t^r} - C\lambda^r \sum_{k=s+1}^{\infty} e^{-\lambda(k-1)t} k^r.$$

Setting $t = \frac{1}{\lambda}$, then t is small when λ is large enough and one obtains

$$sN(s\lambda) \geq (C_2-\varepsilon)\lambda^r - C\lambda^r \sum_{k=s+1}^{\infty} e^{-(k-1)} k^r.$$

Choosing now s sufficiently large we obtain

$$sN(s\lambda) \geq C_3 \lambda^r$$

and the proof follows. $\square$

Arguing as in the previous proposition, it is possible to prove the following result.

Proposition B.0.13. *Let* $C_1 > 0$ *such that*

$$\limsup_{t \to 0} \frac{t^{\frac{N}{2}}}{(-\log t)^{\frac{N}{\alpha}}} \sum_{n \in \mathbb{N}} e^{-\lambda_n t} \leq C_1. \tag{B.3}$$

Then

$$\limsup_{\lambda \to \infty} \lambda^{-\frac{N}{2}} (\log \lambda)^{-\frac{N}{\alpha}} N(\lambda) \leq C_2$$

for some positive C_2*. Moreover if (B.3) holds and*

$$\liminf_{t \to 0} \frac{t^{\frac{N}{2}}}{(-\log t)^{\frac{N}{\alpha}}} \sum_{n \in \mathbb{N}} e^{-\lambda_n t} \geq C_3 \tag{B.4}$$

for some $C_3 > 0$ *then*

$$\liminf_{\lambda \to \infty} \lambda^{-\frac{N}{2}} (\log \lambda)^{-\frac{N}{\alpha}} N(\lambda) \geq C_4$$

for some positive C_4*.*

Appendix C

An inequality in Sobolev spaces

The aim of the Appendix is to study the validity of the inequality

$$\int_{\mathbb{R}^N} u|u|^{p-2}\Delta u \le 0$$

for functions $u \in W^{2,p}(\mathbb{R}^N)$, $1 < p < \infty$. Actually a more precise result can be proved, the following equality that one formally obtains integrating by parts holds

$$\int_{\mathbb{R}^N} u|u|^{p-2}\Delta u = -(p-1)\int_{\mathbb{R}^N} |u|^{p-2}|\nabla u|^2 \chi_{\{u\neq 0\}}. \tag{C.1}$$

If $p \ge 2$ and $u \in W^{2,p}(\mathbb{R}^N)$, then the function $u|u|^{p-2}$ belongs to $W^{2,p'}(\mathbb{R}^N)$, where p' is the conjugated exponent of p. Therefore integration by parts is allowed in the left hand side of (C.1) and the stated equality follows, in particular the inequality which we need is proved too. On the other hand, the situation is more complicated for $1 < p < 2$ due to the presence of the singularity of $|u|^{p-2}$ near the zeros of u. An analogous result remains true for more general elliptic operators in divergence form. Since in our proofs we need only the negativity of the right hand side, here we deduce it by elementary computations.The proof of the equality is more involved and requires a sectional characterization of Sobolev spaces, we refer to [32] for a detailed study of the subject.
We focus our attention on the case $1 < p < 2$ since, as observed, for $p \ge 2$ the equality immediately follows.

Proposition C.0.14. *Let $1 < p < 2$, $u \in C_0^2(\mathbb{R}^N)$, then u satisfies*

$$\int_{\mathbb{R}^N} u|u|^{p-2}\Delta u = -(p-1)\int_{\mathbb{R}^N} |u|^{p-2}|\nabla u|^2 \chi_{\{u\neq 0\}}.$$

PROOF. Given $\delta > 0$, set

$$u_\delta := u(u^2+\delta)^{\frac{p-2}{2}} \in C_0^2(\mathbb{R}^N).$$

We can apply the integration by parts formula to the functions u_δ to deduce

$$\begin{aligned}\int_{\mathbb{R}^N} u(u^2+\delta)^{\frac{p-2}{2}}\Delta u = \int_{\mathbb{R}^N} u_\delta \Delta u = -\int_{\mathbb{R}^N} \nabla u \nabla u_\delta \\ = \quad -\int_{\mathbb{R}^N} |\nabla u|^2 (u^2+\delta)^{\frac{p-4}{2}}((p-1)u^2+\delta). \qquad \text{(C.2)}\end{aligned}$$

Observe that, for $\delta \to 0$,

$$u_\delta \Delta u \to u|u|^{p-2}\Delta u$$

pointwise and, since $p<2$,

$$|u_\delta \Delta u| \leq |u|^{p-1}|\Delta u| \in L^1(\mathbb{R}^N).$$

Moreover

$$(u^2+\delta)^{\frac{p-4}{2}}((p-1)u^2+\delta)|\nabla u|^2 \to (p-1)|u|^{p-2}|\nabla u|^2 \chi_{\{u\neq 0\}}$$

for $\delta \to 0$ almost everywhere, since $\nabla u = 0$ almost everywhere on $\{u=0\}$ by Stampacchia's Lemma. By Fatou's Lemma and dominated convergence Theorem, we obtain

$$\begin{aligned}(p-1)\int_{\mathbb{R}^N} |\nabla u|^2 |u|^{p-2}\chi_{\{u\neq 0\}} &\leq \liminf_{\delta\to 0} -\int_{\mathbb{R}^N} u(u^2+\delta)^{\frac{p-2}{2}}\Delta u \\ &= -\int_{\mathbb{R}^N} u|u|^{p-2}\Delta u\end{aligned}$$

and then $|\nabla u|^2|u|^{p-2}\chi_{\{u\neq 0\}} \in L^1(\mathbb{R}^N)$. Recalling that $1<p<2$, we have

$$(p-1)u^2(u^2+\delta)^{\frac{p-4}{2}}|\nabla u|^2 \leq (p-1)|u|^{p-2}|\nabla u|^2\chi_{\{u\neq 0\}} \in L^1(\mathbb{R}^N);$$

$$\delta(u^2+\delta)^{\frac{p-4}{2}}|\nabla u|^2 \leq (u^2+\delta)^{\frac{p-2}{2}}|\nabla u|^2 \leq |u|^{p-2}|\nabla u|^2\chi_{\{u\neq 0\}} \in L^1(\mathbb{R}^N).$$

Applying the dominated convergence Theorem again in (C.2), the claim follows.

□

The desired inequality for functions in $W^{2,p}(\mathbb{R}^N)$ immediately follows by the last proposition.

Corollary C.0.15. *Let* $u \in W^{2,p}(\mathbb{R}^N)$, $1<p<2$. *Then*

$$(p-1)\int_{\mathbb{R}^N} |\nabla u|^2|u|^{p-2}\chi_{\{u\neq 0\}} \leq -\int_{\mathbb{R}^N} u|u|^{p-2}\Delta u < \infty$$

and, in particular,

$$\int_{\mathbb{R}^N} u|u|^{p-2}\Delta u \leq 0.$$

Proof. Let $(u_n) \subset C_0^\infty(\mathbb{R}^N)$ such that $u_n \to u$ in $W^{2,p}(\mathbb{R}^N)$, $u_n \to u$, $\nabla u_n \to \nabla u$ almost everywhere in $\mathbb{R}^N$. Therefore

$$|\nabla u_n|^2 |u_n|^{p-2} \chi_{\{u_n \neq 0\}} \chi_{\{u \neq 0\}} \to |\nabla u|^2 |u|^{p-2} \chi_{\{u \neq 0\}}$$

almost everywhere. By Fatou's Lemma, Proposition (C.0.14) and observing that $u_n|u_n|^{p-2} \to u|u|^{p-2}$ in $L^{p'}$, we deduce

$$\begin{aligned}(p-1)\int_{\mathbb{R}^N} |\nabla u|^2 |u|^{p-2} \chi_{\{u \neq 0\}} &\leq -\lim_{n\to\infty} \int_{\mathbb{R}^N} u_n |u_n|^{p-2} \Delta u_n \\ &= -\int_{\mathbb{R}^N} u|u|^{p-2} \Delta u.\end{aligned}$$

□

Appendix D

A boundedness criterion

Here we give the proof of an improved version of the L^p boundedness criterion mentioned above ([42, Theorem 3.1], Chapter 5) useful to obtain our a-priori estimates in Chapter 5. As nice application we will deduce an alternative proof of the well known a-priori estimates for the heat operator.

In this appendix, as in Chapter 5, we use the following notation.
Given $X_0 = (x_0^1, ..., x_0^N, t_0)$, $R > 0$, with parabolic cylinder of center $X_0 = (x_0, t_0)$ and radius R we mean the set

$$K = K(X_0, R) = \{(x^1, ..., x^N, t) \in R^{N+1} : |x^i - x_0^i| < R,\ |t - t_0| < R^2\}.$$

D.1 Shen's Theorem

The main result of the section is the following Theorem.

Theorem D.1.1. *Let $1 \le p_0 < q_0 \le \infty$. Suppose that T is a bounded sublinear operator on $L^{p_0}(\mathbb{R}^{N+1})$. Suppose moreover that there exist $\alpha_2 > \alpha_1 > 1,\ C > 0$ such that*

$$\left\{ \frac{1}{|K|} \int_K |Tf|^{q_0} \right\}^{\frac{1}{q_0}} \le C \Bigg\{ \left(\frac{1}{|\alpha_1 K|} \int_{\alpha_1 K} |Tf|^{p_0} \right)^{\frac{1}{p_0}} + \sup_{K' \supset K} \left(\frac{1}{|K'|} \int_{K'} |f|^{p_0} \right)^{\frac{1}{p_0}} \Bigg\}$$

for every $K \subset \mathbb{R}^{N+1}$ parabolic cylinder and every function $f \in L_c^\infty(\mathbb{R}^{N+1})$ with compact support in $\mathbb{R}^{N+1} \setminus \alpha_2 K$. Then T is bounded in $L^p(\mathbb{R}^{N+1})$ for every $p_0 \le p < q_0$.

We note that in [42, Theorem 3.1] $p_0 = 2$ and the parabolic cylinders are replaced by cubes of $\mathbb{R}^N$. We give a proof of the Theorem inspired by Shen's one.

We recall some auxiliary classical results from harmonic analysis concerning the Maximal Hardy-Littlewood function and the Lebesgue points. The proofs of the results only stated here can be found in [47] for d euclidean distance but it is possible to check that they are also true in the more general setting of the homogeneous spaces (see for example [48, Chapter I]).

Let (Ω, μ) be a measure space and $\mathcal{M}(\Omega)$ be the set of the measurable functions in Ω. Let d be a distance on Ω. Through this section, we denote with $B(x,r)$ the ball of center x and radius r for the metric induced by the distance d.
Let $f \in \mathcal{M}(\Omega)$. For every $\alpha > 0$ we set $\lambda(\alpha) = \lambda_f(\alpha) = \mu\{|f| > \alpha\}$. λ is a decreasing function in $(0, \infty)$. In the next lemma we recall an easy property of λ.

Lemma D.1.2. *Let $f \in \mathcal{M}(\Omega)$. Then*

$$\int_\Omega |f|^p \, d\mu = p \int_0^\infty \alpha^{p-1} \lambda(\alpha) \, d\alpha.$$

Let $f \in L^p(\Omega)$ with $p < \infty$, we recall the **Chebychev inequality**

$$\lambda(\alpha) = \mu\{|f| > \alpha\} \le \frac{\|f\|_p^p}{\alpha^p}. \tag{D.1}$$

Definition D.1.3. *We say that μ is a doubling measure if there exists $C_0 > 0$ such that, for every B in Ω*

$$\mu(2B) \le C_0 \mu(B)$$

where $2B$ is the ball with same center of B and double radius.

Remark D.1.4. By the previous definition it easily follows that, if μ is a doubling measure, for every $\lambda \ge 1$ there exists $C = C(C_0, \lambda)$ such that

$$\mu(\lambda B) \le C \mu(B).$$

Definition D.1.5. *Let $f \in L^1_{loc}(\Omega)$. The maximal Hardy-Littlewood function $Mf : \Omega \to \overline{\mathbb{R}}$ is so defined*

$$Mf(x) = \sup_{B \ni x, B \subseteq \Omega} \frac{1}{\mu(B)} \int_B |f| d\mu$$

for every $x \in \Omega$.

Remark D.1.6. (1) If $f,\ g \in L^1_{loc}(\Omega)$,

$$M(f+g) \le Mf + Mg.$$

(2) If $f \in L^\infty(\Omega)$, then $Mf \in L^\infty(\Omega)$ and $\|Mf\|_\infty \le \|f\|_\infty$.

For every $1 \leq p \leq \infty$ we can define the operator

$$M : L^p(\Omega) \to \mathcal{M}(\Omega), \qquad f \mapsto Mf.$$

By Remark D.1.6, M is sublinear and bounded from L^∞ in L^∞. The following theorem provides us the so called maximal Hardy-Littlewood inequality, which, with the L^∞ boundedness and the Marcinkiewicz Theorem, gives that $M : L^p(\Omega) \to L^p(\Omega)$ is bounded for every $1 < p \leq \infty$.
From now on we suppose that μ is a doubling measure.

Theorem D.1.7 (Maximal Hardy-Littlewood inequality)**.** *Let μ a doubling measure. There exists C positive constant such that for every $f \in L^1(\Omega)$ and for every $\alpha > 0$*

$$\mu(\{Mf > \alpha\}) \leq C\frac{\|f\|_1}{\alpha}. \tag{D.2}$$

Corollary D.1.8. *Let $1 < p \leq \infty$. Then there exists $A_p > 0$ such that*

$$\|Mf\|_p \leq A_p \|f\|_p$$

for every $f \in L^p(\Omega)$.

Remark D.1.9. (**Local maximal function**.) Let $Q \subseteq \Omega$, $f \in L^1(Q)$. We consider the local maximal function so defined

$$M_Q f(x) = \sup_{B \subseteq Q,\ x \in B} \frac{1}{\mu(B)} \int_B |f| d\mu$$

for every $x \in Q$. By considering the space Q equipped with the metric induced by d, we obtain the existence of a positive constant C such that for every $\alpha > 0$ and for every $f \in L^1(Q)$

$$\mu(\{M_Q f > \alpha\}) \leq C\frac{\|f\|_{L^1(Q)}}{\alpha} \tag{D.3}$$

and, by the Marcinkiewicz Theorem, it follows that, for every $1 < p \leq \infty$, there exists a positive constant A_p such that

$$\|M_Q f\|_{L^p(Q)} \leq A_p \|f\|_{L^p(Q)} \tag{D.4}$$

for every $f \in L^p(Q)$.

Definition D.1.10. *Let $f \in L^1_{loc}(\Omega)$. We say that $x \in \Omega$ is a Lebesgue point of f (we write $x \in \mathcal{L}(f)$) if*

$$\lim_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f - f(x)| \, d\mu = 0.$$

Remark D.1.11. (i) If x is a Lebesgue point of f then

$$f(x) = \lim_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f \, d\mu$$

(ii) If f is continuous in x then $x \in \mathcal{L}(f)$.

Theorem D.1.12 (Lebesgue Theorem)**.** *If* $f \in L^1(\Omega)$ *then* $|\Omega \setminus \mathcal{L}(f)| = 0$

PROOF. Given $r > 0$ we set

$$T_r f(x) = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f - f(x)| \, d\mu$$

and $Tf(x) = \limsup_{r \to 0^+} T_r f(x)$. We have to prove that $Tf = 0$ almost everywhere in Ω.
By the density of $L^1(\Omega) \cap C(\Omega)$ in $L^1(\Omega)$, given $\varepsilon > 0$ there exists $g \in L^1(\Omega) \cap C(\Omega)$ such that $\|f - g\|_1 < \varepsilon$. By Remark D.1.11(ii)

$$Tg = 0 \text{ in } \Omega. \tag{D.5}$$

Set $h = f - g$,

$$\begin{aligned} T_r h(x) &= \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |h - h(x)| \, d\mu \qquad \text{(D.6)} \\ &\leq \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |h| \, d\mu + |h(x)| \leq Mh(x) + |h(x)|, \end{aligned}$$

where Mh is the maximal Hardy-Littlewood function. Obviously T_r is sublinear, therefore $T_r f \leq T_r g + T_r h$. Taking the limsup for $r \to 0$, by (D.5) and (D.6) we deduce that

$$Tf \leq Tg + Th = Th \leq Mh + |h|.$$

By the last inequality it follows that for every $\alpha > 0$

$$\{Tf \geq \alpha\} \subset \left\{Mh \geq \frac{\alpha}{2}\right\} \cup \left\{|h| \geq \frac{\alpha}{2}\right\}$$

and then by Theorem D.1.7 and by the Chebychev inequality

$$\begin{aligned} \mu(\{Tf \geq \alpha\}) &\leq \mu\left(\left\{Mh \geq \frac{\alpha}{2}\right\}\right) + \mu\left(\left\{|h| \geq \frac{\alpha}{2}\right\}\right) \\ &\leq \frac{2\,C}{\alpha}\|h\|_1 + \frac{2}{\alpha}\|h\|_1 \\ &\leq \left(\frac{2\,C}{\alpha} + \frac{2}{\alpha}\right)\varepsilon. \end{aligned}$$

Letting ε to zero we deduce $\mu(\{Tf \geq \alpha\}) = 0$ for every $\alpha > 0$. Therefore the measure of the set $\{Tf > 0\} = \bigcup_{n \in \mathbb{N}} \{Tf > \frac{1}{n}\}$ is zero, this means that $Tf = 0$ a.e. in Ω. □

We finally state a consequence of the Lebesgue Theorem.

Definition D.1.13. *Let* $\{E_h\}_{h \geq 0}$ *a family of subsets of* Ω *and let* $x \in \Omega$. *We say that* $\{E_h\}$ *converges to* x *for* $h \to 0$ *if there exist* $\alpha > 0$ *and* $r_h \to 0$ *such that for every* $h \geq 0$

$$E_h \subset B(x, r_h) \quad \textit{and} \quad \mu(E_h) \geq \alpha \mu(B(x, r_h)).$$

Corollary D.1.14. *Let $f \in L^1_{loc}(\Omega)$, $x \in \mathcal{L}(f)$ and $\{E_h\} \to x$, then*

$$\lim_{h\to 0} \frac{1}{\mu(E_h)} \int_{E_h} |f - f(x)|\, d\mu = 0.$$

Proof. We have

$$\frac{1}{\mu(E_h)} \int_{E_h} |f - f(x)|\, d\mu \leq \frac{1}{\alpha\mu(B(x, r_h))} \int_{B(x,r_h)} |f - f(x)|\, d\mu$$

and, since x is a Lebesgue point of f, the right and side of the last inequality goes to zero for $h \to 0$. ☐

Remark D.1.15. If, given $X,\ X_0 \in \mathbb{R}^{N+1}$, we set

$$d(X, X_0) = \max\{|x^i - x^i_0|,\ 1 \leq i \leq N,\ |t - t_0|^{\frac{1}{2}}\},$$

then the ball of center X_0 and radius R is the parabolic cylinder $K(X_0, R)$. This simple remark allows us to apply the general results about the maximal Hardy-Littlewood function and the Lebesgue points stated before in the case $\Omega = \mathbb{R}^{N+1}$, μ Lebesgue measure and d parabolic distance in $\mathbb{R}^{N+1}$.

We will use the following version of the Calderón-Zygmund decomposition. The proof is similar to that in [9, Lemma 1.1] where cubes of $\mathbb{R}^N$ appear instead of parabolic cylinders.

Proposition D.1.16 (Calderón-Zygmund decomposition)**.** *Let K a parabolic cylinder of $\mathbb{R}^{N+1}$ and $A \subset K$ a measurable set satisfying*

$$0 < |A| < \delta|K| \quad \textit{for some} \quad 0 < \delta < 1.$$

Then there is a sequence of disjoint dyadic parabolic cylinders $\{K_j\}_{j\in\mathbb{N}}$ obtained from K such that

1. $|A \setminus \bigcup_{j\in\mathbb{N}} K_j| = 0$*;*
2. $|A \cap K_j| > \delta|K_j|$ *for every* $j \in \mathbb{N}$*;*
3. $|A \cap \overline{K}_j| \leq \delta|\overline{K}_j|$ *if* K_j *is a dyadic subdivision of* $\overline{K}_j$*.*

Proof. Divide K in 2^{N+2} dyadic cylinders $K_{1,1}, \ldots, K_{1,2^{N+2}}$ as follows

$$K_{1,j} = \left\{(x,t) : |x^i - x^i_{1,j}| < \frac{R}{2},\ |t - t_{1,j}| < \frac{R^2}{4}\right\}.$$

Choose those for which $|K_{1,j} \cap A| > \delta|K_{1,j}|$. Divide each cylinder that has not been chosen in 2^{N+2} dyadic cylinders $\{K_{2,j}\}$ and repeat the process above iteratively. In this way we obtain a sequence of disjoint dyadic cylinders which we denote $\{K_j\}$. If $X \notin \bigcup_j K_j$, there exists a sequence of cylinders $C_h = K(X_h, R_h)$ containing X with diameter going to zero for $h \to \infty$ and such that

$$|C_h(X) \cap A| \leq \delta|C_h(X)| < |C_h(X)|. \tag{D.7}$$

Observe that $C_h(X) = K(X_h, R_h) \subset K(X, 2R_h)$ indeed if $Y \in C_h(X) = K(X_h, R_h)$ we have $d(Y, X_h) < R_h$, on the other hand, since $X \in C_h$, we have $d(X, X_h) < R_h$, therefore

$$d(Y, X) < d(Y, X_h) + d(X_h, X) < 2R_h.$$

Moreover

$$|C_h(X)| = R_h^{N+2} = \frac{1}{2^{N+2}}(2R_h)^{N+2} = \frac{1}{2^{N+2}}|K(X, 2R_h)|.$$

Apply Corollary D.1.14 to the family $\{C_h\}$ and $f = \chi_A \in L^1(\mathbb{R}^{N+1})$. By (D.7) we obtain that, if X is a Lebesgue point for χ_A,

$$\chi_A(X) = \lim_{h\to\infty} \frac{1}{|C_h|}\int_{C_h} \chi_A(Y)dY = \frac{|C_h(X) \cap A|}{C_h(X)} < 1.$$

This means that $\chi_A(X) = 0$, that is $X \notin A$. By the Lebesgue Theorem it follows that almost everywhere if $X \notin \cup_j K_j$ then $X \in K \setminus A$. This proves (1) and concludes the proof. □

PROOF (Theorem D.1.1). Let $p_0 < p < q_0$. Let $f \in L_c^\infty(\mathbb{R}^{N+1})$. For $\lambda > 0$, we consider the set

$$E(\lambda) = \{(x,t) \in \mathbb{R}^{N+1} : M(|Tf|^{p_0})(x,t) > \lambda\}$$

where M is the maximal operator. Since $Tf \in L^{p_0}$, by the maximal inequality

$$|E(\lambda)| \le C\frac{\|Tf\|_{p_0}^{p_0}}{\lambda} < \infty. \tag{D.8}$$

Let $A = 1/(2\delta^{\frac{p_0}{p}})$ with $0 < \delta < 1/2^{\frac{p}{p_0}}$ small constant to be determined. Observe that $A > 1$. Divide $\mathbb{R}^{N+1}$ in parabolic cylinders $\{K_h\}$ big enough such that

$$|K_h \cap E(A\lambda)| < \delta|K_h|$$

and apply the Calderón-Zygmund decomposition to each K_h. For every $h \in \mathbb{N}$ we obtain a family of parabolic cylinders $\{K_{h,j}\}$ such that

$|(K_h \cap E(A\lambda)) \setminus \bigcup_j K_{h,j}| = 0;$

$|(K_h \cap E(A\lambda)) \cap K_{h,j}| > \delta|K_{h,j}|;$

$|(K_h \cap E(A\lambda)) \cap \overline{K}_{h,j}| \le \delta|\overline{K}_{h,j}|.$

Consider the family of cylinders $\{K_{h,j}\}$ obtained for h and j running in $\mathbb{N}$ and call it $\{K_j\}$ again. In this way we have a family of cylinders $\{K_j\}$ satisfying

1. $|E(A\lambda) \setminus \bigcup_j K_j| = 0;$

2. $|E(A\lambda) \cap K_j| > \delta |K_j|$;
3. $|E(A\lambda)) \cap \overline{K}_j| \leq \delta |\overline{K}_j|$.

We split the proof in three steps.

Step 1

There exist $0 < \delta < 1/2^{\frac{p}{p_0}}$, $0 < \gamma < 1$ such that if

$$\overline{K}_j \cap \{(x,t) \in \mathbb{R}^{n+1} : M(|f|^{p_0})(x,t) \leq \gamma\lambda\} \neq \emptyset$$

then $\overline{K}_j \subseteq E(\lambda)$.

PROOF (*Step 1*). Suppose by contradiction that for every $0 < \gamma < 1$, $0 < \delta < 1/2^{\frac{p}{p_0}}$ there exists $\overline{K}_j$ such that $\overline{K}_j \cap \{(x,t) \in \mathbb{R}^{n+1} : M(|f|^{p_0})(x,t) \leq \gamma\lambda\} \neq \emptyset$ and $\overline{K}_j \not\subseteq E(\lambda)$. In particular the previous property holds for δ small enough such that $A \geq 5^{n+2}$. Fixed γ and δ, let $\overline{K}_j$ the corresponding cylinder as above and let $\overline{X} \in \overline{K}_j \cap \{(x,t) \in \mathbb{R}^{n+1} : M(|f|^{p_0})(x,t) \leq \gamma\lambda\}$ and $X_0 \in \overline{K}_j \setminus E(\lambda)$. Then

$$M(|Tf|^{p_0})(X_0) = \sup_{K \ni X_0} \frac{1}{|K|} \int_K |Tf|^{p_0}(Y)dY \leq \lambda$$

and

$$M(|f|^{p_0})(\overline{X}) = \sup_{K \ni \overline{X}} \frac{1}{|K|} \int_K |f|^{p_0}(Y)dY \leq \gamma\lambda.$$

In particular, if $K \supseteq \overline{K}_j$, then X_0, $\overline{X} \in K$ and, consequently,

$$\frac{1}{|K|} \int_K |Tf|^{p_0} \leq \lambda \quad \text{and} \quad \frac{1}{|K|} \int_K |f|^{p_0} \leq \gamma\lambda. \tag{D.9}$$

Let K_j a parabolic cylinder obtained by the dyadic division of $\overline{K}_j$ and prove that if $X \in K_j$

$$M(|Tf|^{p_0})(X) \leq \max\{M_{2\overline{K}_j}(|Tf|^{p_0})(X),\ 5^{n+2}\lambda\} \tag{D.10}$$

where $M_{2\overline{K}_j}$ is the local maximal function so defined:

$$M_{2\overline{K}_j}(|Tf|^{p_0})(X) = \sup_{K' \ni X,\ K' \subset 2\overline{K}_j} \frac{1}{|K'|} \int_{K'} |Tf|^{p_0}$$

for $X \in 2\overline{K}_j$.

Let $X \in K_j$ and K a parabolic cylinder containing X. If $K \subset 2\overline{K}_j$

$$\frac{1}{|K|} \int_K |Tf|^{p_0} \leq M_{2\overline{K}_j}(|Tf|^{p_0})(X)$$

and (D.10) holds. Suppose now $K \not\subseteq 2\overline{K}_j$ and let $(\overline{Z}, r)$ and (Z_0, R) center and radius respectively of K and $\overline{K}_j$. We have $r \geq \frac{R}{2}$ indeed, if $r < \frac{R}{2}$ and $Y \in K$, we have

$$\begin{aligned} d(Y, Z_0) &\leq d(Y, \overline{Z}) + d(\overline{Z}, Z_0) < r + d(\overline{Z}, X) + d(X, Z_0) \\ &< r + r + R < \frac{R}{2} + \frac{R}{2} + R = 2R \end{aligned}$$

and then $K \subseteq 2\overline{K}_j$ which is a contradiction. It is easy to check that $\widetilde{K}(\overline{Z}, 5r) \supseteq \overline{K}_j(Z_0, R)$. In fact, let $Y \in \overline{K}_j$, then

$$\begin{array}{rcl} d(Y,Z) & \leq & d(Y,X) + d(X,\overline{Z}) \leq d(Y,Z_0) + d(Z_0,X) + d(X,\overline{Z}) \\ & < & R + R + r < 5r, \end{array}$$

therefore $Y \in \widetilde{K}(\overline{Z}, 5r)$. By (D.9) we have

$$\frac{1}{|\widetilde{K}|} \int_{\widetilde{K}} |Tf|^{p_0} \leq \lambda$$

and, since $(5r)^{n+2} = |\widetilde{K}| = 5^{n+2}|K|$,

$$\frac{1}{|K|} \int_K |Tf|^{p_0} \leq \frac{5^{n+2}}{|\widetilde{K}|} \int_{\widetilde{K}} |Tf|^{p_0} \leq 5^{n+2}\lambda$$

which ends the proof of (D.10).
Let now $X \in K_j \cap E(A\lambda)$, then

$$\max\{M_{2\overline{K}_j}(|Tf|^{p_0})(X),\ 5^{n+2}\lambda\} = M_{2\overline{K}_j}(|Tf|^{p_0})(X)$$

because if not, since $A \geq 5^{n+2}$, by (D.10) we have

$$5^{n+2}\lambda = \max\{M_{2\overline{K}_j}(|Tf|^{p_0})(X),\ 5^{n+2}\lambda\} \geq M(|Tf|^{p_0})(X) > A\lambda \geq 5^{n+2}\lambda$$

and this is a contradiction. Then $M_{2\overline{K}_j}(|Tf|^{p_0}) = M(|Tf|^{p_0})$ in $K_j \cap E(A\lambda)$ and

$$\begin{array}{rcl} |K_j \cap E(A\lambda)| & = & |\{X \in K_j : M(|Tf|^{p_0})(X) > A\lambda\}| \\ & = & |\{X \in K_j : M_{2\overline{K}_j}(|Tf|^{p_0})(X) > A\lambda\}|. \end{array}$$

Let $\eta \in C_c^\infty(\mathbb{R}^{n+1})$ such that $0 \leq \eta \leq 1$, $\eta = 1$ in $2\alpha_2\overline{K}_j$ e $\eta = 0$ in $\mathbb{R}^{n+1} \backslash 3\alpha_2\overline{K}_j$. Split f as follows:

$$f = \eta f + (1-\eta)f.$$

The support of $(1-\eta)f$ is contained in $\mathbb{R}^{n+1} \setminus 2\alpha_2\overline{K}_j$. Since T is sublinear,

$$|Tf|^{p_0} \leq 2^{p_0-1}\left(|T(\eta f)|^{p_0} + |T((1-\eta)f)|^{p_0}\right)$$

and, since the maximal operator is sublinear,

$$M_{2\overline{K}_j}(|Tf|^{p_0}) \leq 2^{p_0-1} M_{2\overline{K}_j}(|T(\eta f)|^{p_0}) + 2M_{2\overline{K}_j}(|T((1-\eta)f)|^{p_0}).$$

It follows

$$
\begin{aligned}
&|K_j \cap E(A\lambda)| = |\{X \in K_j : M_{2\overline{K}_j}(|Tf|^{p_0})(X) > A\lambda\}| \\
\leq\ & |\{X \in K_j : M_{2\overline{K}_j}(|T(\eta f)|^{p_0}) + M_{2\overline{K}_j}(|T((1-\eta)f)|^{p_0} > \frac{A\lambda}{2^{p_0-1}}\}| \\
\leq\ & |\{X \in K_j : M_{2\overline{K}_j}(|T(\eta f)|^{p_0}) > \frac{A\lambda}{2^{p_0}}\}| \\
+\ & |\{X \in K_j : M_{2\overline{K}_j}(|T((1-\eta)f)|^{p_0}) > \frac{A\lambda}{2^{p_0}}\}| \\
\leq\ & \frac{C}{A\lambda}\int_{2\overline{K}_j} |T(\eta f)|^{p_0}) + \frac{C}{(A\lambda)^{\frac{q_0}{p_0}}}\int_{2\overline{K}_j} |M_{2\overline{K}_j}(|T((1-\eta)f)|^{p_0})|^{\frac{q_0}{p_0}} \\
\leq\ & \frac{C}{A\lambda}\int_{2\overline{K}_j} |T(\eta f)|^{p_0} + \frac{C}{(A\lambda)^{\frac{q_0}{p_0}}}\int_{2\overline{K}_j} |T((1-\eta)f)|^{q_0}
\end{aligned}
$$

with C depending on $n,\ p_0,\ q_0$. The last two addenda have been obtained estimating the previous ones using respectively the local maximal Hardy-Littlewood inequality (D.3) and the Chebychev inequality. Moreover the second addendum has been estimated using the boundedness of the local maximal operator (see (D.4)).

By the boundedness in L^{p_0}, the sublinearity of T and the hypothesis we obtain

$$
\begin{aligned}
&|K_j \cap E(A\lambda)| \\
&\quad \leq \frac{C}{A\lambda}\int_{3\alpha_2\overline{K}_j} |f|^{p_0} + \frac{C|2\overline{K}_j|}{(A\lambda)^{\frac{q_0}{p_0}}} N^{q_0}\Bigg\{ \left(\frac{1}{|\alpha_1 2\overline{K}_j|}\int_{2\alpha_1\overline{K}_j} |T((1-\eta)f)|^{p_0} \right)^{\frac{1}{p_0}} \\
&\quad + \sup_{K' \supset 2\overline{K}_j} \left(\frac{1}{|K'|}\int_{K'} |(1-\eta)f|^{p_0} \right)^{\frac{1}{p_0}} \Bigg\}^{q_0} \leq \frac{C}{A\lambda}\int_{3\alpha_2\overline{K}_j} |f|^{p_0} \\
&\quad + \frac{C|2\overline{K}_j|}{(A\lambda)^{\frac{q_0}{p_0}}} N^{q_0}\Bigg\{ \left(\frac{1}{|\alpha_1 2\overline{K}_j|}\int_{2\alpha_1\overline{K}_j} (|Tf|^{p_0} + |T(\eta f)|^{p_0}) \right)^{\frac{1}{p_0}} \\
&\quad + \sup_{K' \supset 2\overline{K}_j} \left(\frac{1}{|K'|}\int_{K'} |(1-\eta)f|^{p_0} \right)^{\frac{1}{p_0}} \Bigg\}^{q_0} \leq \frac{C}{A\lambda}\frac{|3\alpha_2\overline{K}_j|}{|3\alpha_2\overline{K}_j|}\int_{3\alpha_2\overline{K}_j} |f|^{p_0} \\
&\quad + \frac{C|2\overline{K}_j|}{(A\lambda)^{\frac{q_0}{p_0}}} N^{q_0}\Bigg\{ \left(\frac{1}{|3\alpha_2\overline{K}_j|}\int_{3\alpha_2\overline{K}_j} |f|^{p_0} + \frac{1}{|\alpha_1 2\overline{K}_j|}\int_{2\alpha_1\overline{K}_j} |Tf|^{p_0} \right)^{\frac{1}{p_0}} \\
&\quad + \sup_{K' \supset 2\overline{K}_j} \left(\frac{1}{|K'|}\int_{K'} |f|^{p_0} \right)^{\frac{1}{p_0}} \Bigg\}^{q_0}.
\end{aligned}
$$

Observe that, since $\alpha_i > 1$, $\alpha_i \overline{K}_j \supset \overline{K}_j$, then by (D.9)

$$|K_j \cap E(A\lambda)| \leq C|\overline{K}_j| \left\{ \frac{\gamma\lambda}{A\lambda} + \left(\frac{\gamma\lambda + \lambda}{A\lambda} \right)^{\frac{q_0}{p_0}} \right\} \leq C|\overline{K}_j| \left\{ \frac{\gamma}{A} + \left(\frac{1}{A} \right)^{\frac{q_0}{p_0}} \right\}$$
$$= C|\overline{K}_j| \left\{ 2\gamma\delta^{\frac{p_0}{p}} + \left(2\delta^{\frac{p_0}{p}} \right)^{\frac{q_0}{p_0}} \right\} = \delta|K_j| C \left\{ 2\gamma\delta^{\frac{p_0}{p}-1} + 2^{\frac{q_0}{p_0}}\delta^{\frac{q_0}{p}-1} \right\}$$

where $C = C(n, p_0, q_0, \alpha_1, \alpha_2)$. If we choose δ small enough such that

$$C 2^{\frac{q_0}{p_0}} \delta^{\frac{q_0}{p}-1} \leq \frac{1}{2}$$

(this is possible since $\dfrac{q_0}{p} > 1$) and $A = \dfrac{1}{2\delta^{\frac{p_0}{p}}} \geq 5^{n+2}$ and γ such that

$$2C\gamma\delta^{\frac{q_0}{p}-1} \leq \frac{1}{p_0}$$

we obtain

$$|K_j \cap E(A\lambda)| \leq \delta|K_j|.$$

This contradicts the properties of the Calderón-Zygmund decomposition and proves the assertion in Step 1.

Step 2

There exist $0 < \gamma < 1$, $0 < \delta < 1/2^{\frac{p}{p_0}}$ such that

$$|E(A\lambda)| \leq \delta|E(\lambda)| + |\{(x,t) \in \mathbb{R}^{n+1} : M(|f|^{p_0})(x,t) > \gamma\lambda\}| \qquad \text{(D.11)}$$

for every $\lambda > 0$.

Proof (*Step 2*). Let $\{\overline{K}_j\}$ a disjoint subcover of $E(A\lambda) \cap \{(x,t) \in \mathbb{R}^{n+1} : M(|f|^{p_0})(x,t) \leq \gamma\lambda\}$ with the property that

$$\overline{K}_j \cap \{(x,t) \in \mathbb{R}^{n+1} : M(|f|^{p_0})(x,t) \leq \gamma\lambda\} \neq \emptyset.$$

A such subcover exists in fact by property (1) of the Calderön-Zygmund decomposition there exists a family K_j of disjoint cylinders such that tale che

$$|E(A\lambda) \setminus \cup_j K_j| = 0$$

and each K_j is obtained by the dyadic division of a cylinder $\overline{K}_j$. Therefore we can cover $E(A\lambda)$ with the dyadic parents of each K_j. In order to have disjoint cylinders $\overline{K}_j$, if K_r, K_s have the same parent, we include it only one time, if $\overline{K}_r \subset \overline{K}_s$ we take $\overline{K}_s$. Reject finally all the cylinders that don't intersect $\{(x,t) \in \mathbb{R}^{n+1} : M(|f|^{p_0})(x,t) \leq \gamma\lambda\}$.

By Step 1,

$$\begin{aligned} |E(A\lambda) \cap \{(x,t) \in \mathbb{R}^{n+1} : M(|f|^{p_0})(x,t)| \leq \gamma\lambda\} &\leq \sum_j |E(A\lambda) \cap \overline{K}_j| \\ &\leq \delta \sum_j |\overline{K}_j| \leq \delta|E(\lambda)|. \end{aligned}$$

Hence

$$\begin{aligned}|E(A\lambda)| &\le |E(A\lambda)\cap\{(x,t)\in\mathbb{R}^{n+1}: M(|f|^{p_0})(x,t)| \le \gamma\lambda\}|\\ &+ |E(A\lambda)\cap\{(x,t)\in\mathbb{R}^{n+1}: M(|f|^{p_0})(x,t)| > \gamma\lambda\}|\\ &\le \delta|E(\lambda)| + |E(A\lambda)\cap\{(x,t)\in\mathbb{R}^{n+1}: M(|f|^{p_0})(x,t)| > \gamma\lambda\}|\end{aligned}$$

and the statement in Step 2 is proved.

Step 3

We finally deduce the L^p boundedness of T from the results proved in the previous steps.

For every $\lambda_0 > 0$

$$\begin{aligned}\int_0^{A\lambda_0} \lambda^{\frac{p}{p_0}-1}|E(\lambda)|d\lambda &\le \int_0^{A\lambda_0} \lambda^{\frac{p}{p_0}-1}\big[\delta\big|E\big(\frac{\lambda}{A}\big)\big|\\ &+ \big|\big\{(x,t)\in\mathbb{R}^{n+1}: M(|f|^{p_0})(x,t) > \frac{\gamma\lambda}{A}\big\}\big|d\lambda\big]\\ &= \delta\int_0^{A\lambda_0} \lambda^{\frac{p}{p_0}-1}\big|E\big(\frac{\lambda}{A}\big)\big|d\lambda\\ &+ \int_0^{A\lambda_0} \lambda^{\frac{p}{p_0}-1}\big|\big\{(x,t)\in\mathbb{R}^{n+1}: M(|f|^{p_0})(x,t) > \frac{\gamma\lambda}{A}\big\}\big|d\lambda\\ &= \delta A^{\frac{p}{p_0}}\int_0^{\lambda_0} \lambda^{\frac{p}{p_0}-1}|E(\lambda)|d\lambda\\ &+ \left(\frac{A}{\gamma}\right)^{\frac{p}{p_0}}\int_0^{\lambda_0\gamma} \lambda^{\frac{p}{p_0}-1}\big|\big\{(x,t)\in\mathbb{R}^{n+1}: M(|f|^{p_0})(x,t) > \lambda\big\}\big|d\lambda\\ &\le \delta A^{\frac{p}{p_0}}\int_0^{\lambda_0} \lambda^{\frac{p}{p_0}-1}|E(\lambda)|d\lambda\\ &+ \left(\frac{A}{\gamma}\right)^{\frac{p}{p_0}}\int_0^{\infty} \lambda^{\frac{p}{p_0}-1}\big|\big\{(x,t)\in\mathbb{R}^{n+1}: M(|f|^{p_0})(x,t) > \lambda\big\}\big|d\lambda\\ &= \delta A^{\frac{p}{p_0}}\int_0^{\lambda_0} \lambda^{\frac{p}{p_0}-1}|E(\lambda)|d\lambda + C(\gamma,\delta)\int_{\mathbb{R}^{n+1}} |M(|f|^{p_0})|^{\frac{p}{p_0}}\\ &\le \delta A^{\frac{p}{p_0}}\int_0^{\lambda_0} \lambda^{\frac{p}{p_0}-1}|E(\lambda)|d\lambda + C(\gamma,\delta)\int_{\mathbb{R}^{n+1}} |f|^p\end{aligned}$$

where we used (D.11), Lemma D.1.2 and Corollary D.1.8 (observe that $\frac{p}{p_0} > 1$). Recall that $A = \dfrac{1}{2\delta^{\frac{p_0}{p}}} > 1$ and $\delta A^{\frac{p}{p_0}} = \dfrac{1}{2^{\frac{p}{p_0}}} < 1$. By the inequalities above

$$\int_0^{\lambda_0} \lambda^{\frac{p}{p_0}-1}|E(\lambda)|d\lambda \le \frac{1}{2^{\frac{p}{p_0}}}\int_0^{\lambda_0} \lambda^{\frac{p}{p_0}-1}|E(\lambda)|d\lambda + C(\gamma,\delta)\int_{\mathbb{R}^{n+1}} |f|^p$$

which implies

$$\left(1-\frac{1}{2^{\frac{p}{p_0}}}\right)\int_0^{\lambda_0} \lambda^{\frac{p}{p_0}-1}|E(\lambda)|d\lambda \le C(\gamma,\delta)\int_{\mathbb{R}^{n+1}} |f|^p$$

and, changing the constant C,

$$\int_0^{\lambda_0} \lambda^{\frac{p}{p_0}-1}|E(\lambda)|d\lambda \leq C(\gamma,\delta)\int_{\mathbb{R}^{n+1}}|f|^p.$$

Almost everywhere it holds

$$|Tf|^{p_0}(x,t) > \lambda \Rightarrow M(|Tf|^{p_0})(x,t) > \lambda$$

because

$$\begin{aligned} M(|Tf|^{p_0})(x,t) &= \sup_{K\ni(x,t)=X}\frac{1}{|K|}\int_K |Tf|^{p_0}(Y)dY \\ &\geq \frac{1}{|K(X,R)|}\int_K |Tf|^{p_0}(Y)dY \end{aligned}$$

for every $R > 0$ and

$$\frac{1}{|K(X,R)|}\int_K |Tf|^{p_0}(Y)dY \to |Tf|^{p_0}(X)$$

almost everywhere by the Lebesgue Theorem. Therefore we have

$$\int_0^{\lambda_0} \lambda^{\frac{p}{p_0}-1}|\{|Tf|^{p_0} > \lambda\}|d\lambda \leq \int_0^{\lambda_0} \lambda^{\frac{p}{p_0}-1}|E(\lambda)|d\lambda \leq C(\gamma,\delta)\int_{\mathbb{R}^{n+1}}|f|^p. \tag{D.12}$$

Moreover $\int_0^{\lambda_0} \lambda^{\frac{p}{p_0}-1}|E(\lambda)|d\lambda$ is finite indeed, by the maximal Hardy-Littlewood inequality, $B = \sup_{\lambda>0}\lambda|E(\lambda)| < \infty$, this implies $\lambda^{\frac{p}{p_0}-1}|E(\lambda)| \leq B\lambda^{\frac{p}{p_0}-2}$ which is integrable near zero for $2 - \dfrac{p}{p_0} < 1 \Leftrightarrow p > p_0$. Letting λ_0 to $+\infty$ in (D.12) we obtain

$$\int_0^{\infty} \lambda^{\frac{p}{p_0}-1}|\{|Tf|_0^p > \lambda\}|d\lambda \leq C(\gamma,\delta)\int_{\mathbb{R}^{n+1}}|f|^p$$

and, by Lemma D.1.2,

$$\int_{\mathbb{R}^{n+1}}|Tf|^p \leq C\int_{\mathbb{R}^{n+1}}|f|^p.$$

$\square$

Remark D.1.17. By the proof, it follows that it is sufficient to require that the inequality in the assumption of Theorem D.1.1 is verified for all $f \in C_c^\infty(\mathbb{R}^{N+1})$ with compact support in $\mathbb{R}^{N+1} \setminus \alpha_2 K$.

D.2 An application of Shen's Theorem

The boundeness result for operators just proved allows us to give an alternative proof of the classical a-priori estimates for the operator $\partial_t - \Delta$.
In this Section we will denote by X the space $(\partial_t - \Delta)C_c^\infty(\mathbb{R}^{N+1})$.

Proposition D.2.1. *Let* $1 < p < \infty$*. There exist* $C_1,\ C_2 > 0$ *such that*

$$\|D_{ij}(\partial_t - \Delta)^{-1} g\|_p \leq C_1 \|g\|_p$$

and

$$\|\partial_t(\partial_t - \Delta)^{-1} g\|_p \leq C_2 \|g\|_p$$

for all $1 \leq i, j \leq N$ *and for all* $g \in X$*.*

Theorem D.2.2. *Let* $1 < p < \infty$*. Then there exists* $C > 0$ *such that*

$$\|D^2 u\|_p + \|\partial_t u\|_p \leq C\|\partial_t u - \Delta u\|_p \tag{D.13}$$

for all $u \in W^{2,1}_p(\mathbb{R}^{N+1})$*.*

Proof. Let $u \in C_c^\infty(\mathbb{R}^{N+1})$, then $u = (\partial_t - \Delta)^{-1}(\partial_t - \Delta)u$ and $g = (\partial_t - \Delta)u \in X$. By proposition D.2.1 we obtain the claimed inequality for test functions. By density the estimate follows for the functions in $W^{2,1}_p(\mathbb{R}^{N+1})$. □

Lemma D.2.3. *The space* X *is dense in* $L^2(\mathbb{R}^{N+1})$*.*

Proof. Denote by $\mathcal{S}(\mathbb{R}^{N+1})$ the Schwartz space and by $\widehat{g}$ the Fourier transform of a function g. First let us prove that $(\partial_t - \Delta)\mathcal{S}(\mathbb{R}^{N+1})$ is dense in $L^2(\mathbb{R}^{N+1})$. Let $v \in L^2(\mathbb{R}^{N+1})$ orthogonal to $(\partial_t - \Delta)u$ for all u in $\mathcal{S}(\mathbb{R}^{N+1})$. We claim that $v \equiv 0$. We have

$$\int_{\mathbb{R}^{N+1}} \widehat{v}(\xi,\tau)(i\tau + |\xi|^2)\widehat{u}(\xi,\tau) = 0$$

for all $u \in \mathcal{S}(\mathbb{R}^{N+1})$ and then

$$\int_{\mathbb{R}^{N+1}} \widehat{v}(\xi,\tau)\frac{i\tau + |\xi|^2}{1 + i\tau + |\xi|^2}(1 + i\tau + |\xi|^2)\widehat{u}(\xi,\tau) = 0$$

for all $u \in \mathcal{S}(\mathbb{R}^{N+1})$. The operator $I + \partial_t - \Delta : \mathcal{S}(\mathbb{R}^{N+1}) \to \mathcal{S}(\mathbb{R}^{N+1})$ is surjective, therefore by the previous equality we deduce

$$\int_{\mathbb{R}^{N+1}} \widehat{v}(\xi,\tau)\frac{i\tau + |\xi|^2}{1 + i\tau + |\xi|^2} w(\xi,\tau) = 0$$

for all $w \in \mathcal{S}(\mathbb{R}^{N+1})$ and then

$$\widehat{v}(\xi,\tau)\frac{i\tau + |\xi|^2}{1 + i\tau + |\xi|^2} \equiv 0$$

almost everywhere in $\mathbb{R}^{N+1}$. This implies $v \equiv 0$. Observe now that X is dense in $(\partial_t - \Delta)\mathcal{S}(\mathbb{R}^{N+1})$ indeed if $f = \partial_t u - \Delta u$ with $u \in \mathcal{S}(\mathbb{R}^{N+1})$ then it can be approximated in the L^2 norm by the sequence $(\partial_t(\eta_n u) - \Delta(\eta_n u))$ where $\eta_n(x,t) = \eta\left(\frac{x}{n}, \frac{t}{n}\right)$ with $\eta \in C_c^\infty(\mathbb{R}^{N+1})$, $0 \leq \eta \leq 1$, $\eta = 1$ if $|(x,t)| \leq 1$ and $\eta = 0$ if $|(x,t)| \geq 2$. □

Proof (Proposition D.2.1). Let $1 \leq i,j \leq N$. Consider the operators $T_1 = D_{ij}(\partial_t - \Delta)^{-1}$ and $T_2 = \partial_t(\partial_t - \Delta)^{-1}$ from X to $C_c^\infty(\mathbb{R}^{N+1})$. By Lemma D.2.3, T_1 and T_2 extend by density to $L^2(\mathbb{R}^{N+1})$ and in particular they are defined on $C_c^\infty(\mathbb{R}^{N+1})$. By Shen's Theorem, applied in correspondence of $p_0 = 2$, we will deduce the boundedness of these operators in L^p, for $2 \leq p < \infty$ and then, by duality, the boundedness for $1 < p \leq 2$.
Let us prove now the boundedness in L^2 of T_1 and T_2. Let $f \in X$. We have

$$\widehat{T_1 f} = -\frac{\xi_i \xi_j}{i\tau + |\xi|^2}\widehat{f}$$

and then

$$\|T_1 f\|_2 = \|\widehat{T_1 f}\|_2 \leq \|\widehat{f}\|_2 = \|f\|_2.$$

Similarly the T_2 boundedness in L^2 follows. Prove now the inequality in the assumptions of Shen's Theorem.
Let $\alpha_2 > \alpha_1 > 1$, $K \subset \mathbb{R}^{N+1}$ parabolic cylinder and $f \in C_c^\infty(\mathbb{R}^{N+1})$ with compact support in $\mathbb{R}^{N+1} \setminus \alpha_2 K$. We have

$$\widehat{T_1 f} = -\frac{\xi_i \xi_j}{i\tau + |\xi|^2}\widehat{f}.$$

Set $v = T_1 f$. Since $f \in C_c^\infty(\mathbb{R}^{n+1})$, f and $\widehat{f} \in \mathcal{S}(\mathbb{R}^{N+1})$, it follows that

$$-(1 + |(\xi,\tau)|^2)^k \frac{\xi_i \xi_j}{i\tau + |\xi|^2}\widehat{f} = (1 + |(\xi,\tau)|^2)^k \widehat{v} \in L^2(\mathbb{R}^{N+1})$$

for all $k \in \mathbb{N}$ and then $v \in H^k(\mathbb{R}^{N+1})$ for all $k \in \mathbb{N}$. This proves that $v \in C^\infty(\mathbb{R}^{N+1})$. Moreover $\partial_t v - \Delta v = D_{ij} f$ and $\partial_t v - \Delta v = 0$ in $\alpha_2 K$ since $f = 0$ in $\alpha_2 K$. In the same way one can prove that $T_2 f$ satisfies the same equation.
Let K be a parabolic cylinder with center (x_0, t_0) and radius R. We will prove that, for all $p \geq 2$, there exists $C > 0$ such that , if $v \in C^\infty$ solves $\partial_t v - \Delta v = 0$ in $\alpha_2 K$, then

$$\left(\frac{1}{|K|}\int_K |v|^p\right)^{\frac{1}{p}} \leq C\left(\frac{1}{|\alpha_1 K|}\int_{\alpha_1 K} |v|^2\right)^{\frac{1}{2}}.$$

Observe that it is sufficient to prove

$$\left(\int_{K_1} |w|^p\right)^{\frac{1}{p}} \leq C\left(\int_{\alpha_1 K_1} |w|^2\right)^{\frac{1}{2}}$$

for w smooth solution of $\partial_t w - \Delta w = 0$ in $\alpha_2 K_1$ with $K_1 = K_1((x_0, t_0), 1)$ cylinder with unitary radius. Infact let v such that $\partial_t v - \Delta v = 0$ in $\alpha_2 K$ and set $w(x,t) = v(Rx - (R-1)x_0, R^2 t - (R^2 - 1)t_0)$. Then $\partial_t w - \Delta w = 0$ in $\alpha_2 K_1$. Moreover

$$\left(\int_{K_1} |w(x,t)|^p\right)^{\frac{1}{p}} \leq C\left(\int_{\alpha_1 K_1} |w(x,t)|^2\right)^{\frac{1}{2}}$$

implies

$$\left(\int_{K_1} |v(Rx-(R-1)x_0,R^2t-(R^2-1)t_0)|^p\right)^{\frac{1}{p}} \le$$
$$C\left(\int_{\alpha_1 K_1} |v(Rx-(R-1)x_0,R^2t-(R^2-1)t_0)|^2\right)^{\frac{1}{2}}$$

and, setting $\tau = R^2t-(R^2-1)t_0$, $\xi = Rx-(R-1)x_0$,

$$\left(\frac{1}{R^{n+2}}\int_K |v|^p\right)^{\frac{1}{p}} \le C\left(\frac{1}{R^{n+2}}\int_{\alpha_1 K}|v|^2\right)^{\frac{1}{2}}$$

which is the estimate for general cylinders.
Let K be a parabolic cylinder of radius 1, w such that $\partial_t w - \Delta w = 0$ in $\alpha_2 K$ and $1 \le a < b \le \alpha_1 < \alpha_2$. Let $0 \le \eta \le 1$ be a smooth function such that $\eta = 1$ in aK and $\eta = 0$ in $\mathbb{R}^{N+1} \setminus bK$. We write K as $Q \times I$ where Q is the cube in the space $\mathbb{R}^N$ and I the time interval, we multiply the equation satisfied by w times $\eta^2 w$ and we integrate both members with respect to the space variable x on bQ. We obtain

$$\int_{bQ} w_t\eta^2 w + \int_{bQ}\eta^2|\nabla w|^2 + 2\int_{bQ} w(\nabla w)\eta\nabla\eta = 0$$

and, writing the first integral in different way,

$$\frac{1}{2}\frac{d}{dt}\int_{bQ}\eta^2 w^2 - \int_{bQ} w^2\eta\eta_t + \int_{bQ}\eta^2|\nabla w|^2 + 2\int_{bQ} w(\nabla w)\eta\nabla\eta = 0.$$

Integrate now with respect to the time variable on I. For all $\varepsilon > 0$, we hawe

$$\begin{aligned}\int_{bK}\eta^2|\nabla w|^2 &\le \int_{bK}|w^2\eta\eta_t| + 2\left(\int_{bK}\eta^2|\nabla w|^2\right)^{\frac{1}{2}}\left(\int_{bK} w^2|\nabla\eta|^2\right)^{\frac{1}{2}}\\ &\le C\int_{bK}|w|^2 + \varepsilon^2\int_{bK}\eta^2|\nabla w|^2 + \frac{1}{\varepsilon^2}\int_{bK} w^2|\nabla\eta|^2.\end{aligned}$$

Choosing ε small enough,

$$\int_{bK}\eta^2|\nabla w|^2 \le C\int_{bK}|w|^2$$

and, since $\eta = 1$ on aK,

$$\int_{aK}|\nabla w|^2 \le C\int_{bK}|w|^2.$$

Note that, for every β multi-index,

$$\partial_t(D^\beta w) - \Delta(D^\beta w) = 0$$

in $\alpha_2 K$ and, by the previous computations,

$$\int_{aK} |D^\gamma w|^2 \leq C \int_{bK} |D^\beta w|^2 \tag{D.14}$$

for γ multi-index of lenght $|\gamma| = |\beta| + 1$ (with D^γ we mean the derivatives of order γ with respect to the space variable). Choose α multi-index of lenght $m = |\alpha| > N + 1$ and divide the interval $[1, \alpha_1]$ in m intervals $[a_i, b_i]$ with $1 = a_1 < b_1 < a_2 < \ldots < a_m < b_m = \alpha_1$. Applying (D.14) iteratively to $[a_i, b_i]$, we obtain

$$\int_K |D^\alpha w|^2 \leq C \int_{\alpha_1 K} |w|^2$$

and

$$\int_K |D^\mu w|^2 \leq C \int_{\alpha_1 K} |w|^2$$

for all μ multi-index of lenght less than m. Moreover, since

$$\partial_t^{\frac{\alpha}{2}} w = \Delta^\alpha w,$$

$$\int_K |\partial_t^\alpha w|^2 \leq C \int_{\alpha_1 K} |w|^2.$$

We obtained

$$\|w\|_{W_2^{\frac{N+1}{2}}(K)} \leq \|w\|_{L^2(\alpha_1 K)}.$$

By the Sobolev embedding Theorem, $W_2^{\frac{N+1}{2}}(K) \subset L^\infty(K)$, it follows that

$$\|w\|_{L^\infty(K)} \leq \|w\|_{L^2(\alpha_1 K)}$$

and

$$\|w\|_{L^p(K)} \leq \|w\|_{L^\infty(K)} \leq \|w\|_{L^2(\alpha_1 K)}$$

for all $1 \leq p \leq \infty$. By Theorem D.1.1, T_1 and T_2 are bounded in $L^p(\mathbb{R}^{N+1})$ for all $2 \leq p < \infty$.

Let $1 < p \leq 2$ and p' such that $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. Consider

$$T_1 : L^2(\mathbb{R}^{N+1}) \to L^2(\mathbb{R}^{N+1})$$

so defined

$$\widehat{T_1 f} = -\frac{\xi_i \xi_j}{i\tau + |\xi|^2} \widehat{f}.$$

$T_1 = \mathcal{F}^{-1} M_q \mathcal{F}$ where M_q is the multiplication operator with

$$q(\xi, \tau) = -\frac{\xi_i \xi_j}{i\tau + |\xi|^2}$$

and $\mathcal{F}$ is the unitary operator that to $f \in L^2(\mathbb{R}^{N+1})$ associates its Fourier transform. Denoted by T_1^* the adjoint operator of T_1, we have

$$T_1^* = \mathcal{F}^{-1} M_{\overline{q}} \mathcal{F}$$

with $M_{\overline{q}}$ multiplication operator and $\overline{q}(\xi, \tau) = -\dfrac{\xi_i \xi_j}{-i\tau + |\xi|^2}$. Observe that, if $f \in X$, $T_1^* f = D_{ij}(-\partial_t - \Delta)^{-1} f$ and, since we are considering the heat operator all over $\mathbb{R}^{N+1}$, T_1^* enjoies the same properties of T_1. Let $f, g \in C_c^\infty(\mathbb{R}^{N+1})$. Obvioulsy $2 \le p' < \infty$. By the first part of the proof, there exists $C > 0$ such that

$$\left| \int_{\mathbb{R}^{N+1}} (T_1 f) g \right| = \left| \int_{\mathbb{R}^{N+1}} f (T_1^* g) \right| \le C \|f\|_p \|g\|_{p'}.$$

It follows that $\|T_1 f\|_p \le \|f\|_p$. In similar way one can prove the same result for T_2. ☐

If u does not depend on the time variable, the following elliptic version of the Calderón- Zygmund Theorem immediately follows.

Theorem D.2.4. *Let $1 < p < \infty$. There exists C positive constant such that*

$$\|D^2 u\|_p \le C \|\Delta u\|_p$$

for all $u \in W^{2,p}(\mathbb{R}^N)$.

Anyway, by means of the mean value Theorem for harmonic functions, an alternative direct proof gives the same result.

Proposition D.2.5. *Let $1 < p < \infty$. There exists $C > 0$ such that*

$$\|D_{ij}(\Delta)^{-1} g\|_p \le C \|g\|_p$$

for all $1 \le i, j \le N$ and for all $g \in \Delta(C_c^\infty(\mathbb{R}^N))$.

As before, the following lemma can be proved.

Lemma D.2.6. *The space $\Delta(C_c^\infty(\mathbb{R}^N))$ is dense in $L^2(\mathbb{R}^N)$.*

PROOF (Proposition D.2.5). Let $1 \le i, j \le N$. Consider the operator $T = D_{ij}(\Delta)^{-1}$ from $\Delta(C_c^\infty(\mathbb{R}^N))$ to $C_c^\infty(\mathbb{R}^N)$. By Lemma D.2.6, T extends by density to all $L^2(\mathbb{R}^N)$.
As in the parabolic case the L^2 boundedness follows by using the Fourier transform. Let us prove the assumption in Shen's Theorem.
Choose $\alpha_2 = 4$, $\alpha_1 = 2$. Let $Q \subset \mathbb{R}^N$ and $f \in C_c^\infty(\mathbb{R}^N)$ with compact support in $\mathbb{R}^N \setminus 4Q$. Set $v = Tf$. As in the parabolic case we have $v \in C^\infty(\mathbb{R}^N)$ and $\Delta v = D_{ij} f$. Since $f = 0$ in $4Q$, $\Delta v = 0$ in $4Q$. Suppose $Q = Q(y, R)$, consider

the ball $B(y,R)$. Obviously $B(y,R) \subset Q(y,R)$ and $\Delta v = 0$ in $4B(y,R)$. By the mean value Theorem for harmonic functions

$$v(x) = \frac{1}{|B(x,r)|}\int_{B(x,r)} v(z)dz$$

for all $x \in 4B(y,R),\ r > 0$ such that $B(x,r) \subset 4B(y,R)$. Note that if $x \in B(y,R)$ then $B(x,R) \subset B(y,2R)$ and

$$\begin{aligned} v(x) &= \frac{1}{|B(x,R)|}\int_{B(x,R)} v(z)dz \le \frac{C}{|B_R|^{\frac{1}{2}}}\left(\int_{B(x,R)} |v|^2\right)^{\frac{1}{2}} \\ &\le \frac{C}{|B_R|^{\frac{1}{2}}}\left(\int_{B(y,2R)} |v|^2\right)^{\frac{1}{2}}. \end{aligned}$$

Let $p > 2$. By taking the p-power and integrating over $B(y,R)$,

$$\frac{1}{|B_R|}\int_{B(y,R)} |v|^p \le \frac{C}{|B_R|^{\frac{p}{2}}}\left(\int_{B(y,2R)} |v|^2\right)^{\frac{p}{2}}.$$

By Theorem D.1.1 the boundedness of T in L^p for $2 \le p < \infty$ follows and then by duality we deduce the boundedness in L^p for $1 < p \le 2$. □

List of symbols

Let $1 \leq k \leq \infty$, $N \in \mathbb{N}$, $0 < \alpha < 1$, $T > 0$, $a < b$, u real valued function.

$\mathbb{R}^N$	euclidean N-dimensional space
$Q(a,b)$	$\mathbb{R}^N \times (a,b)$
Q_T	$Q(0,T)$
(X,d)	a metric space X endowed with the distance d
$(\cdot\vert\cdot)$	scalar product or, in general, duality
$\vert x\vert$	euclidean norm of $x \in \mathbb{R}^N$
$B_\rho(x)$	open ball for the euclidean distance with centre x and radius ρ
$\vert E\vert$	Lebesgue measure of a given set E
χ_E	characteristic function of e set E
$\mathrm{supp\,u}$	support of a given function u
$D_i u$	partial derivative with respect to x_i
$\partial_t u$	partial derivative with respect to t
$D_{ij} u$	$D_i D_j u$
Du	$(D_1 u, \ldots\ldots, D_N u)$
$D^2 u$	hessian matrix $(D_{ij} u)_{i,j=1,\ldots,N}$
$\vert Du\vert^2$	$\sum_{j=1}^N \vert D_i u\vert^2$
$\vert D^2 u\vert^2$	$\sum_{i,j=1}^N \vert D_{ij} u\vert^2$
$f^+,\ f^-$	positive part $f \vee 0$ and negative part $-(f \wedge 0)$ of f
$\mathbf{1}$	function identically equal to 1 everywhere
$\mathcal{L}(X)$	space of bounded linear operators from X to X
$C_b(\mathbb{R}^N)$	space of bounded continuous functions in $\mathbb{R}^N$
$C_b^j(\mathbb{R}^N)$	space of real functions with derivatives up to the order j in $C_b(\mathbb{R}^N)$
$C^\alpha(\mathbb{R}^N)$	space of Hölder continuous functions
$C^\alpha_{loc}(\mathbb{R}^N)$	space of Hölder continuous functions in Ω for all bounded open set $\Omega \subset \mathbb{R}^N$
$C^{k+\alpha}(\mathbb{R}^N)$	space of functions such that the derivatives of order k are α-Hölder continuous
$C_c^\infty(\mathbb{R}^N)$	space of test functions
$L^p(\mathbb{R}^N)$	usual Lebesgue space
$L_c^\infty(\mathbb{R}^N)$	space of all bounded measurable functions on $\mathbb{R}^N$ having compact support
$\mathcal{S}(\mathbb{R}^N)$	Schwartz space
$\mathcal{S}'(\mathbb{R}^N)$	space of tempered distributions
$B_b(\mathbb{R}^N)$	space of bounded Borel functions

$C_0(R^N)$	space of continuous functions tending to 0 for $\lvert x\rvert$ tending to $+\infty$
$C_0(B_\rho)$	space of continuous functions in B_ρ vanishing on the boundary
$BUC(Q(a.b))$	space of bounded and uniformly continuous functions in $Q(a.b)$
$C^{2,1}(Q(a,b))$	space of functions continuous with their indicated derivatives
$C_b^{2,1}(Q(a,b))$	space of functions having bounded time derivative and bounded space derivatives up to the second order
$BUC^{2,1}(Q(a,b))$	subspace of $C_b^{2,1}(Q(a,b))$ consisting of all functions for which u_t and $D_x^\alpha u$, $\lvert\alpha\rvert = 2$ are uniformly continuous in $Q(a,b)$
$C^{2+\alpha,1+\frac{\alpha}{2}}(Q(a,b))$	space of functions such that $\partial_t u$ and $D_{ij}u$ are α Hölder continuous with respect to the parabolic distance
$W_k^j(\mathbb{R}^N)$	space of functions $u \in L^k(\mathbb{R}^N)$ having weak space derivatives up to the order j in $L^k(\mathbb{R}^N)$
$W_k^{2,1}(Q(a,b))$	space of functions $u \in L^k(Q(a,b))$ having weak space derivatives $D^\alpha u \in L^k(Q(a,b))$ for $\lvert\alpha\rvert \leq 2$ and weak time derivative $\partial_t u \in L^k(Q(a,b))$
$\lVert u\rVert_{W_k^{2,1}(Q(a,b))}$	$\lVert u\rVert_{L^k(Q(a,b))} + \lVert\partial_t u\rVert_{L^k(Q(a,b))} + \sum_{1\leq\lvert\alpha\rvert\leq 2} \lVert D^\alpha u\rVert_{L^k(Q(a,b))}$
$[u]_{\alpha,\frac{\alpha}{2};Q_T}$	$\sup_{(x,y)\in\mathbb{R}^N, t\in(0,T)} \frac{\lvert u(x,t)-u(y,t)\rvert}{\lvert x-y\rvert^\alpha} + \sup_{s\neq t, x\in\mathbb{R}^N} \frac{\lvert u(x,t)-u(s,x)\rvert}{\lvert t-s\rvert^{\frac{\alpha}{2}}}$
$\lvert u\rvert_{\alpha,\frac{\alpha}{2};Q_T}$	$\lVert u\rVert_\infty + [u]_{\alpha,\frac{\alpha}{2};Q_T}$
$\lvert u\rvert_{2+\alpha,1+\frac{\alpha}{2};Q_T}$	$\lVert u\rVert_\infty + [\partial_t u]_{\alpha,\frac{\alpha}{2};Q_T} + [D^2 u]_{\alpha,\frac{\alpha}{2};Q_T}$
$W \hookrightarrow H$	the space W is continuously embedded in H.
$l^1(\mathbb{R})$	space of sequences $(\lambda_n)_{n\in\mathbb{N}}$ such that $\sum_{n\in\mathbb{N}} \lvert\lambda_n\rvert < \infty$.

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