Chapter 5

Invariant measures: main properties and some applications

In this last chapter we collect some known facts concerning invariant measures, most of which have been already used. Here we provide the relative proofs and we also show some other results which complete the exposition and make it clearer. Even though the subject has a certain relevance from a probabilistic point of view and can be treated by making use of probabilistic tools, our approach is purely analytic.

We start by introducing Feller semigroups in $C_b(\mathbb{R}^N)$. These are semigroups of positive contractions that are not strongly continuous in general, but continuous only with respect to the pointwise convergence. In our framework, we also assume that each operator of a Feller semigroup admits an integral representation and that it can be extended to the bounded Borel functions in $\mathbb{R}^N$. Then we give the definition of an invariant measure μ for a Feller semigroup (P_t). If one considers the underlying stochastic process $\{\xi_t\}$, μ can be interpreted as a stationary distribution for $\{\xi_t\}$. A quite general result concerning existence of invariant measures is given by Krylov and Bogoliubov (see Theorem 5.1.6). The main tool to prove it is a weak* compactness result for probability measures, which is due to Prokhorov. As a consequence, we infer that the semigroup (P_t) extends to a strongly continuous contractions semigroup in $L^p(\mathbb{R}^N, \mu)$, for all $1 \leq p < +\infty$. In order to deal with uniqueness, we have to require some regularity properties to (P_t), namely irreducibility and strong Feller property. Under these further assumptions, if an invariant measure exists, it is unique. To prove such a result we make use of some known facts concerning ergodic means of linear operators in Hilbert spaces and in particular the Von Neumann Theorem. Ergodicity of invariant measures concludes the first section.

In the second section we show how Feller semigroups arise naturally when one deals with a second order partial differential operator in $\mathbb{R}^N$ of the form

$$A = \sum_{i,j=1}^{N} q_{ij} D_{ij} + \sum_{i=1}^{N} F_i D_i.$$

The absence of a zero order term is a necessary condition for the existence of an invariant measure for the associated semigroup $T(t)$ (see Remark 5.2.12). The construction of $T(t)$ is based on an approximation argument which consists of finding a bounded classical solution u to the Cauchy problem

$$\begin{cases} u_t - Au = 0 & \text{in } (0,\infty) \times \mathbb{R}^N \\ u(0,x) = f(x) & x \in \mathbb{R}^N \end{cases}$$

as limit of solutions of parabolic problems in cylinders $(0,\infty) \times B_\rho$. The main tools to carry out this procedure are the classical maximum principle and interior Schauder estimates. Then one sets $u(t,x) = T(t)f(x)$. It turns out that $T(t)$ is a Feller semigroup in $C_b(\mathbb{R}^N)$, which is represented by a strictly positive integral kernel. Even though $T(t)$ is not strongly continuous we can associate a "weak" generator, which enjoys several classical properties of generators of strongly continuous semigroups. We show that assuming the existence of a Liapunov function, the weak generator coincides with the operator A endowed with the maximal domain in $C_b(\mathbb{R}^N)$ (see Proposition 5.2.3). Under the same assumption the semigroup $T(t)$ yields the *unique* bounded classical solution to the problem above. Concerning invariant measures, we establish two existence criteria, whose assumptions are expressed in terms of the coefficients of the operator A. The first is due to Khas'minskii and uses the existence of suitable supersolutions of the equation $\lambda u - Au = 0$ to apply the Krylov-Bogoliubov Theorem. The second is due to Varadhan and show directly the existence of an invariant measure for an operator of the form $\Delta - \langle D\Phi + G, D\rangle$, given by $\mu(dx) = e^{-\Phi}dx$.

The last section is devoted to the characterization of the domain of a class of elliptic operators in $L^p(\mathbb{R}^N, \mu)$. The main tools are the results of Chapter 1, where the same problem has been studied for differential operators in $L^p(\mathbb{R}^N)$. In fact, we show that the given operator on $L^p(\mathbb{R}^N, \mu)$ is similar to an operator in the unweighted space $L^p(\mathbb{R}^N)$ which satisfy the generation results of Chapter 1 that provide also an explicit description of the domain.

5.1 Existence and uniqueness of invariant measures for Feller semigroups

Throughout this section $(P_t)_{t\geq 0}$ is a family of linear operators in $C_b(\mathbb{R}^N)$, the space of all continuous and bounded functions in $\mathbb{R}^N$, satisfying the following properties:

(i) $P_0 = I, P_{t+s} = P_tP_s$, for all $t,s \geq 0$;

(ii) $P_tf \geq 0$ for all $t \geq 0$ and $f \in C_b(\mathbb{R}^N)$ with $f \geq 0$;

(iii) $\lim_{t\to 0} P_tf(x) = f(x)$, for all $x \in \mathbb{R}^N$ and $f \in C_b(\mathbb{R}^N)$;

(iv) $P_t\mathbb{1} = \mathbb{1}$, for all $t \geq 0$,

where $\mathbb{1}$ denotes the function with constant value 1. From (ii) and (iv) it follows that each operator P_t is a contraction. Indeed, for all $f \in C_b(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$

$$|P_tf(x)| \leq P_t|f|(x) \leq \|f\|_\infty P_t\mathbb{1} = \|f\|_\infty,$$

hence $\|P_tf\|_\infty \leq \|f\|_\infty$. Under a probabilistic point of view (P_t) is a *Feller semigroup* and condition (iii) represents the *stochastic continuity* of (P_t).

It is useful to make the following additional assumption:

(I) for all $t > 0$ and $x \in \mathbb{R}^N$ there exists a positive Borel measure $p_t(x,\cdot)$ such that $p_t(x,\mathbb{R}^N) = 1$ and

$$(P_tf)(x) = \int_{\mathbb{R}^N} f(y)p_t(x,dy), \tag{5.1.1}$$

for all $f \in C_b(\mathbb{R}^N)$.

We set $p_0(x,\cdot) = \delta_x$, the Dirac measure concentrated at x.

We note that (5.1.1) makes sense also for bounded Borel functions. In particular, if Γ is a Borel set of $\mathbb{R}^N$ and χ_Γ is the corresponding characteristic function, then

$$(P_t\chi_\Gamma)(x) = p_t(x,\Gamma), \qquad x \in \mathbb{R}^N,\ t \geq 0. \tag{5.1.2}$$

Then we also assume that

(II) for every bounded Borel function f and for every $t \geq 0$ the function $P_t f$ is still Borel measurable.

In general such a semigroup is not strongly continuous in $C_b(\mathbb{R}^N)$, a simple counterexample being the heat semigroup.

Definition 5.1.1 *A probability Borel measure μ is said to be invariant for (P_t) if*

$$\int_{\mathbb{R}^N} (P_t f)(x)\mu(dx) = \int_{\mathbb{R}^N} f(x)\mu(dx) \tag{5.1.3}$$

for all $t \geq 0$ and for every bounded Borel function f.

It is readily seen that μ is invariant if and only if

$$\mu(\Gamma) = \int_{\mathbb{R}^N} p_t(x,\Gamma)\mu(dx) \tag{5.1.4}$$

for any borelian set Γ. Indeed, if (5.1.3) holds, then (5.1.4) easily follows by taking $f = \chi_\Gamma$. Conversely, assume that (5.1.4) is true. This means that (5.1.3) is satisfied by any characteristic function. By linearity, one has the same formula also for simple functions. If f is a bounded nonnegative Borel function, then let (s_n) be an increasing sequence of simple functions such that $s_n(x)$ converges to $f(x)$, for every $x \in \mathbb{R}^N$. Writing (5.1.3) for each s_n and letting $n \to \infty$, by monotone convergence we get the identity for f. In the general case, it is sufficient to write $f = f^+ - f^-$.

From a probabilistic point of view, let us consider the stochastic process $\{\xi_t\}$ having $p_t(x,\Gamma)$ as transition functions. This means that $p_t(x,\Gamma)$ represents the probability that the process reaches Γ at the time t starting from x at $t = 0$. In order to determine completely the process, that is the probability that the process is in Γ at the time t, for any Γ and $t > 0$, it is sufficient to know the law $p_t(x,\Gamma)$ and the initial distribution σ, since, applying the formula of total probability, it holds

$$P(\xi_t \in \Gamma) = \int_{\mathbb{R}^N} p_t(x,\Gamma)\sigma(dx).$$

In this context, an invariant measure is a stationary distribution for the process, since

$$P(\xi_t \in \Gamma) = \int_{\mathbb{R}^N} p_t(x,\Gamma)\mu(dx) = \mu(\Gamma) = P(\xi_0 \in \Gamma),$$

for all $t \geq 0$.

A first basic result is the following.

Proposition 5.1.2 *Assume that μ is an invariant measure for (P_t). Then for all $p \in [1,+\infty[$, (P_t) can be extended uniquely to a strongly continuous contractions semigroup in $L^p(\mathbb{R}^N,\mu)$, still denoted by (P_t). Moreover, if $(A_p, D(A_p))$ is the generator of such a semigroup, then (5.1.3) is equivalent to have $\int_{\mathbb{R}^N}(A_p f)(x)\mu(dx) = 0$, for all $f \in D(A_p)$.*

Proof. Let $\varphi \in C_b(\mathbb{R}^N)$. From (5.1.1) and Hölder's inequality it follows that

$$|P_t\varphi(x)|^p \leq \int_{\mathbb{R}^N} |\varphi(y)|^p p_t(x,dy) = P_t(|\varphi|^p)(x).$$

Integrating with respect to μ, we get

$$\int_{\mathbb{R}^N} |P_t\varphi(x)|^p\mu(dx) \leq \int_{\mathbb{R}^N} P_t(|\varphi|^p)(x)\mu(dx) = \int_{\mathbb{R}^N} |\varphi(x)|^p\mu(dx).$$

Since $C_b(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N,\mu)$, P_t has a unique continuous extension to $L^p(\mathbb{R}^N,\mu)$, still denoted by P_t, such that $\|P_t\| \le 1$. The strong continuity of $P_t f$ in $L^p(\mathbb{R}^N,\mu)$ for $f \in C_b(\mathbb{R}^N)$ follows easily from property *(iii)* of (P_t) and the dominated convergence theorem. The general case can be treated by a standard density argument.

Let us prove the last assertion. If $f \in D(A_p)$ then $P_t f \in D(A_p)$, the map $t \to P_t f$ is of class $C^1([0,+\infty[; L^p(\mathbb{R}^N,\mu))$ and $\frac{d}{dt}P_t f = A_p P_t f = P_t A_p f$. Differentiating with respect to t the identity (5.1.3) we have

$$\begin{aligned} 0 &= \frac{d}{dt}\int_{\mathbb{R}^N}(P_t f)(x)\mu(dx) = \int_{\mathbb{R}^N}\frac{d}{dt}(P_t f)(x)\mu(dx) = \int_{\mathbb{R}^N} P_t(A_p f)(x)\mu(dx) \\ &= \int_{\mathbb{R}^N}(A_p f)(x)\mu(dx). \end{aligned}$$

Conversely, if $\int_{\mathbb{R}^N}(A_p f)(x)\mu(dx) = 0$, for all $f \in D(A_p)$, then

$$\frac{d}{dt}\int_{\mathbb{R}^N}(P_t f)(x)\mu(dx) = \int_{\mathbb{R}^N} A_p(P_t f)(x)\mu(dx) = 0$$

and (5.1.3) holds in $D(A_p)$. Since $D(A_p)$ is dense in $L^p(\mathbb{R}^N,\mu)$, (5.1.3) is also true for $f \in L^p(\mathbb{R}^N,\mu)$. □

Now, our aim is to prove a quite general result on existence of invariant measures due to Krylov and Bogoliubov. Before stating it, we need to introduce some basic notions from measure theory.

We denote by $\mathcal{M}(\mathbb{R}^N)$ the set of all Borel probability measures on $\mathbb{R}^N$.

Definition 5.1.3 *A subset Λ of $\mathcal{M}(\mathbb{R}^N)$ is said to be relatively weakly compact if for any sequence (μ_n) in Λ there exist a subsequence (μ_{n_k}) and $\mu \in \mathcal{M}(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} f(x)\mu_{n_k}(dx) \to \int_{\mathbb{R}^N} f(x)\mu(dx)$, for all $f \in C_b(\mathbb{R}^N)$. In this case, we say that μ_{n_k} weakly converges to μ.*

The set Λ is said to be tight if for all $\varepsilon > 0$ there is a compact set K_ε such that $\mu(K_\varepsilon) \ge 1-\varepsilon$, for all $\mu \in \Lambda$.

The Prokhorov theorem, proved below, shows that in fact the previous two notions are equivalent. Even though it holds in a general separable complete metric space, we state and prove it in $\mathbb{R}^N$, since this case is closer to our interests. We first need a lemma.

Lemma 5.1.4 *Let $\mu_n, \mu \in \mathcal{M}(\mathbb{R}^N)$ be such that μ_n converges weakly to μ. Then one has $\limsup_{n\to\infty}\mu_n(F) \le \mu(F)$, for every closed set F of $\mathbb{R}^N$ or, equivalently, $\liminf_{n\to\infty}\mu_n(G) \ge \mu(G)$, for every open set G of $\mathbb{R}^N$.*

Proof. Let F be a closed set and consider $F_\delta = \{x \in \mathbb{R}^N \mid \mathrm{dist}(x,F) < \delta\}$. Since F_δ is decreasing with respect to δ and $\cap_{\delta>0}F_\delta = F$, we have that $\lim_{\delta\to 0}\mu(F_\delta) = \mu(F)$. Therefore, given a positive ε there exists $\delta > 0$ such that $\mu(F_\delta) < \mu(F) + \varepsilon$. Let

$$\varphi(t) = \begin{cases} 1 & \text{if} \quad t \le 0, \\ 1-t & \text{if} \quad 0 \le t \le 1, \\ 0 & \text{if} \quad t \ge 1, \end{cases}$$

and define $f(x) = \varphi(\delta^{-1}\mathrm{dist}(x,F))$. Since f is nonnegative and assumes the value 1 on F, we have

$$\mu_n(F) = \int_F f(x)\mu_n(dx) \le \int_{\mathbb{R}^N} f(x)\mu_n(dx).$$

Since f vanishes outside F_δ and never exceeds 1

$$\int_{\mathbb{R}^N} f(x)\mu(dx) = \int_{F_\delta} f(x)\mu(dx) \le \mu(F_\delta).$$

Finally, since μ_n converges weakly to μ and $f \in C_b(\mathbb{R}^N)$ we deduce

$$\limsup_{n\to\infty} \mu_n(F) \le \lim_{n\to\infty} \int_{\mathbb{R}^N} f(x)\mu_n(dx) = \int_{\mathbb{R}^N} f(x)\mu(dx) \le \mu(F_\delta) < \mu(F) + \varepsilon.$$

Since ε was arbitrary, the thesis follows. A simple complementation argument proves the last assertion. $\square$

Theorem 5.1.5 (Prokhorov) *A subset Λ in $\mathcal{M}(\mathbb{R}^N)$ is relatively weakly compact if and only if it is tight.*

Proof. Let $\overline{B}_n$ be the closed ball in $\mathbb{R}^N$ with radius $n \in \mathbb{N}$ and centered at zero. Assume first that Λ is tight and consider a sequence (μ_k) in Λ. We have to show that it is possible to extract a weakly convergent subsequence. Consider the restrictions $\big(\mu_{k|\overline{B}_1}\big)$. Since $C(\overline{B}_1)$ is separable, the weak* topology of the unit ball of the dual space (of all finite Borel measures) is metrizable. Hence, there exists a subsequence of $\big(\mu_{k|\overline{B}_1}\big)$ which converges weakly in $C(\overline{B}_1)^*$. By a diagonal procedure, since $(\overline{B}_n)$ is increasing, we can construct a subsequence (μ_{n_k}) such that $\int_{\overline{B}_n} f(x)\mu_{n_k}(dx)$ converges to $\int_{\overline{B}_n} f(x)\mu(dx)$ for all $f \in C(\overline{B}_n)$, and $n \in \mathbb{N}$ and for some positive Borel measure μ with $\mu(\mathbb{R}^N) \le 1$. Now, let $\varepsilon > 0$ be fixed. Since Λ is tight, there exists $r \in \mathbb{N}$ such that $\mu_{n_k}(\mathbb{R}^N \setminus \overline{B}_r) < \varepsilon$, for all $k \in \mathbb{N}$. If $n > r$, let $g \in C(\mathbb{R}^N)$ be such that $0 \le g \le 1$, $g \equiv 1$ in $\overline{B}_n \setminus B_{r+1}$ and supp $g \subset B_{n+1} \setminus \overline{B}_r \subset B_{n+1}$. Then

$$\mu(\overline{B}_n \setminus B_{r+1}) \le \int_{\mathbb{R}^N} g(x)\mu(dx) = \lim_{k\to+\infty} \int_{\mathbb{R}^N} g(x)\mu_{n_k}(dx) \le \limsup_{k\to+\infty} \mu_{n_k}(B_{n+1} \setminus \overline{B}_r) \le \varepsilon.$$

Letting $n \to +\infty$ we find that $\mu(\mathbb{R}^N \setminus\ B_{r+1}) \le \varepsilon$. Now, we can conclude. Indeed, if $f \in C_b(\mathbb{R}^N)$ then

$$\begin{aligned}\left|\int_{\mathbb{R}^N} f(x)\mu(dx) - \int_{\mathbb{R}^N} f(x)\mu_{n_k}(dx)\right| \le\; & \left|\int_{\overline{B}_{r+1}} f(x)\mu(dx) - \int_{\overline{B}_{r+1}} f(x)\mu_{n_k}(dx)\right| \\ & + \int_{\mathbb{R}^N\setminus\overline{B}_{r+1}} |f(x)|\mu(dx) \\ & + \int_{\mathbb{R}^N\setminus\overline{B}_{r+1}} |f(x)|\mu_{n_k}(dx).\end{aligned}$$

If $\varepsilon > 0$ is given, we first choose $r \in \mathbb{N}$ sufficiently large in such a way that $\mu(\mathbb{R}^N \setminus \overline{B}_{r+1}), \mu_{n_k}(\mathbb{R}^N \setminus \overline{B}_{r+1}) \le \varepsilon$ for all $k \in \mathbb{N}$. Then we choose $k \in \mathbb{N}$ large enough to make the first term in the right hand side smaller than ε. At the end we find

$$\left|\int_{\mathbb{R}^N} f(x)\mu(dx) - \int_{\mathbb{R}^N} f(x)\mu_{n_k}(dx)\right| \le \varepsilon + 2\varepsilon\|f\|_\infty$$

for k large. Thus the statement follows. In particular, taking $f = \mathbf{1}$, we have that μ is a probability measure, i.e. $\mu \in \mathcal{M}(\mathbb{R}^N)$.

Conversely, let us show that a relatively weakly compact set Λ must be tight. Consider the open ball B_n of $\mathbb{R}^N$ centered at zero and with radius $n \in \mathbb{N}$. For each $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $\nu(B_n) > 1 - \varepsilon$ for all $\nu \in \Lambda$. Otherwise, for each n we have $\nu_n(B_n) \le 1 - \varepsilon$, for some $\nu_n \in \Lambda$. By weakly compactness, there exist a subsequence (ν_{n_k}) and $\nu_0 \in \mathcal{M}(\mathbb{R}^N)$ such that ν_{n_k} converges to ν_0 weakly. From Lemma 5.1.4 it follows that $\nu_0(B_n) \le \liminf_{k\to\infty} \nu_{n_k}(B_n) \le \liminf_{k\to\infty} \nu_{n_k}(B_{n_k}) \le 1 - \varepsilon$, which is impossible, since $B_n \uparrow \mathbb{R}^N$. Thus, the closure of B_n is a compact set of $\mathbb{R}^N$ such that $\nu(\overline{B}_n) > 1 - \varepsilon$, for all $\nu \in \Lambda$. $\square$

Now, we are ready to prove the announced result of existence of an invariant measure for the semigroup (P_t).

Theorem 5.1.6 (Krylov-Bogoliubov) *Assume that for some $T_0 > 0$ and $x_0 \in \mathbb{R}^N$ the set $\{\mu_T\}_{T>T_0}$, where*

$$\mu_T = \frac{1}{T}\int_0^T p_t(x_0,\cdot)dt,$$

is tight. Then there is an invariant measure μ for (P_t).

Proof. From Theorem 5.1.5 it follows that there exist a sequence (T_n) going to $+\infty$ and a probability measure μ such that $\lim_{n\to\infty}\int_{\mathbb{R}^N} f(x)\mu_{T_n}(dx) = \int_{\mathbb{R}^N} f(x)\mu(dx)$, for all $f \in C_b(\mathbb{R}^N)$. Taking into account (5.1.1), this is equivalent to

$$\lim_{n\to\infty}\frac{1}{T_n}\int_0^{T_n}(P_t f)(x_0)dt = \int_{\mathbb{R}^N} f(x)\mu(dx). \tag{5.1.5}$$

Setting $f = P_s g$ we have

$$\lim_{n\to\infty}\frac{1}{T_n}\int_0^{T_n}(P_{t+s} g)(x_0)dt = \int_{\mathbb{R}^N}(P_s g)(x)\mu(dx),$$

for all $g \in C_b(\mathbb{R}^N)$. Now, we show that the limit at the left hand side above is equal to $\int_{\mathbb{R}^N} g(x)\mu(dx)$. We have in fact

$$\begin{aligned}
\frac{1}{T_n}\int_0^{T_n}(P_{t+s} g)(x_0)dt &= \frac{1}{T_n}\int_s^{T_n+s}(P_t g)(x_0)dt \\
&= \frac{1}{T_n}\int_0^{T_n}(P_t g)(x_0)dt + \frac{1}{T_n}\int_{T_n}^{T_n+s}(P_t g)(x_0)dt \\
&\quad -\frac{1}{T_n}\int_0^{s}(P_t g)(x_0)dt.
\end{aligned}$$

Since the last two terms above are infinitesimal and condition (5.1.5) holds, we find that (5.1.3) holds for $g \in C_b(\mathbb{R}^N)$. If g is a bounded Borel function in $\mathbb{R}^N$, then $g \in L^1(\mathbb{R}^N,\mu)$, hence, by density, there exists a sequence (g_n) in $C_b(\mathbb{R}^N)$ converging to g in $L^1(\mathbb{R}^N,\mu)$. By continuity, $P_t g_n$ converges to $P_t g$ in $L^1(\mathbb{R}^n,\mu)$ as well. Now, the thesis follows easily writing (5.1.3) for g_n and letting $n \to \infty$. □

In the next section we will see an application of this general result in the case of semigroups associated with differential operators.

Once that an invariant measure exists, one can ask whether it is unique or not. Such a problem requires more attention and suitable regularity properties for the semigroup (P_t) that we introduce below.

Definition 5.1.7 - *(P_t) is irreducible if for any ball $B(z,\varepsilon)$ one has $P_t\chi_{B(z,\varepsilon)}(x) > 0$ or, equivalently, $p_t(x, B(z,\varepsilon)) > 0$ for all $t > 0$ and $x \in \mathbb{R}^N$.*

- *(P_t) has the strong Feller property if for any bounded Borel function f and $t > 0$ we have $P_t f \in C_b(\mathbb{R}^N)$.*

- *P_t is called regular if all the probabilities $p_t(x,\cdot)$, $t > 0, x \in \mathbb{R}^N$, are equivalent, i.e. they are mutually absolutely continuous.*

It is clear that if (P_t) is irreducible, then it is *positivity improving*, in the sense that given a bounded Borel nonnegative function φ on $\mathbb{R}^N$ such that φ is strictly positive on some ball, then $P_t\varphi(x) > 0$, for all $t > 0$ and $x \in \mathbb{R}^N$. In this way, irreducibility says that a strong maximum principle holds. From a probabilistic point of view, this means that the underlying Markov process diffuses with infinite speed.

The main result concerning uniqueness is the following.

Theorem 5.1.8 *If (P_t) is regular then it has at most one invariant measure μ. Moreover, μ is equivalent to $p_t(x,\cdot)$, for all $t > 0, x \in \mathbb{R}^N$.*

Before proving the above theorem, we show an important tool to have regularity due to Khas'minskii.

Proposition 5.1.9 *If (P_t) is strong Feller and irreducible, then it is regular.*

PROOF. It is sufficient to prove that all the probabilities $p_t(x,\cdot)$, $t > 0, x \in \mathbb{R}^N$, have the same null sets. This means that if Γ is a Borel set, then

(i) either $p_t(x,\Gamma) = 0$, for all $t > 0, x \in \mathbb{R}^N$,

(ii) or $p_t(x,\Gamma) > 0$, for all $t > 0, x \in \mathbb{R}^N$.

Assume that (i) does not hold. Then, there exist $x_0 \in \mathbb{R}^N$ and $t_0 > 0$ such that $P_{t_0}\chi_\Gamma(x_0) > 0$. By the strong Feller property, $P_{t_0}\chi_\Gamma \in C_b(\mathbb{R}^N)$, hence $P_{t_0}\chi_\Gamma(x) > 0$ for $x \in B(x_0,\delta)$. From the irreducibility and the semigroup law it follows $P_t\chi_\Gamma(x) > 0$ for all $x \in \mathbb{R}^N$, $t > t_0$, respectively. We claim that this holds for $t \le t_0$, too. If $t_1 < t \le t_0$ then there exists $x_1 \in \mathbb{R}^N$ such that $P_{t_1}\chi_\Gamma(x_1) > 0$ (otherwise $P_{t_0}\chi_\Gamma$ would be identically zero). By the same argument as before, we have $P_t\chi_\Gamma(x) > 0$ for all $x \in \mathbb{R}^N$ and the proof is concluded. □

In order to prove Theorem 5.1.8, we need some results about ergodic means of linear operators, in particular the Von Neumann Theorem. Let T be a linear bounded operator on a Hilbert space H and set

$$M_n = \frac{1}{n}\sum_{k=0}^{n-1} T^k, \quad n \in \mathbb{N}.$$

Proposition 5.1.10 *Assume that there is a positive constant K such that $\|T^n\| \le K$, for all $n \in \mathbb{N}$. Then, the limit*

$$\lim_{n\to\infty} M_n x =: M_\infty x \tag{5.1.6}$$

exists for every $x \in H$. Moreover, $M_\infty^2 = M_\infty$, $M_\infty(H) = \ker(I-T)$, that is M_∞ is a projection on $\ker(I-T)$.

PROOF. The stated limit trivially exists when $x \in \ker(I-T)$ or $x \in (I-T)(H)$. Indeed, in the first case we have $T^k x = x$ for all $k \in \mathbb{N}$, hence $M_n x = x$ for all $n \in \mathbb{N}$. In the second case, if $x = (I-T)y$, for some $y \in H$, taking into account the identity

$$M_n(I-T) = (I-T)M_n = \frac{1}{n}(I-T^n), \tag{5.1.7}$$

we have

$$\|M_n x\| = \left\|\frac{1}{n}(y - T^n y)\right\| \le \frac{1}{n}(\|y\| + K\|y\|),$$

and consequently $\lim_{n\to\infty} M_n x = 0$. Since $\|M_n x\| \le K\|x\|$, it follows that

$$\lim_{n\to\infty} M_n x = 0, \quad x \in \overline{(I-T)(H)}. \tag{5.1.8}$$

Now, let $x \in H$ be fixed. Then, there exist $y \in H$ and a subsequence $M_{n_k}x$ weakly convergent to y. Since T is bounded, $TM_{n_k}x$ converges weakly to Ty. On the other hand, from (5.1.7) it follows that $TM_n x = M_n x - \frac{1}{n}x + \frac{1}{n}T^n x$, hence $TM_{n_k}x$ converges weakly also to y. By uniqueness, $Ty = y$, i.e. $y \in \ker(I - T)$. Now we claim that $M_n x$ converges to y. Since $y \in \ker(I-T)$, we have $M_n y = y$ and consequently

$$M_n x = M_n y + M_n(x-y) = y + M_n(x-y),$$

so that it is sufficient to show that $M_n(x-y)$ converges to zero. To this aim, recalling (5.1.8), we prove that $x - y \in \overline{(I-T)(H)}$. We have in fact $x - M_{n_k}x \in (I-T)(H)$, because

$$x - M_{n_k}x = \frac{1}{n_k}\sum_{j=0}^{n_k-1}(I - T^j)x = \frac{1}{n_k}(I-T)\sum_{j=0}^{n_k-1}(I + T + \cdots + T^{j-1})x$$

and $x - M_{n_k}x$ converges weakly to $x - y$. Since $(I-T)(H)$ is convex, its strong and weak closures coincide, hence $x - y \in \overline{(I-T)(H)}$. Therefore (5.1.6) is proved. As far as the last part of the statement is concerned, since $(I-T)M_n = M_n(I-T)$ converges to zero in the strong topology, we have $M_\infty = TM_\infty$ and therefore $M_\infty = T^k M_\infty$, for every $k \in \mathbb{N}$. This implies that $M_\infty = M_n M_\infty$, which yields, as $n \to \infty$, $M_\infty = M_\infty^2$, as required. □

Now we use this general result in our framework. More precisely, let μ be an invariant measure for the semigroup (P_t) and consider the Hilbert space $L^2(\mathbb{R}^N, \mu)$. Proposition 5.1.2 ensures that each P_t extends to a linear bounded operator in $L^2(\mathbb{R}^N, \mu)$ with $\|P_t\| \le 1$. Consider the ergodic mean

$$M(T)\varphi = \frac{1}{T}\int_0^T P_s\varphi ds, \quad \varphi \in L^2(\mathbb{R}^N, \mu),\ T > 0. \tag{5.1.9}$$

Clearly, $M(T)$ is a linear operator and, by the Minkowski inequality, it is bounded in $L^2(\mathbb{R}^N, \mu)$:

$$\|M(T)\varphi\|_{L^2(\mathbb{R}^N,\mu)} \le \frac{1}{T}\int_0^T \|P_s\varphi\|_{L^2(\mathbb{R}^N,\mu)}ds \le \|\varphi\|_{L^2(\mathbb{R}^N,\mu)}.$$

Theorem 5.1.11 (Von Neumann) *For every $\varphi \in L^2(\mathbb{R}^N, \mu)$, the limit*

$$\lim_{T\to\infty} M(T)\varphi =: M_\infty\varphi$$

exists in $L^2(\mathbb{R}^N, \mu)$. Moreover $M_\infty = M_\infty^2$ and $M_\infty(L^2(\mathbb{R}^N, \mu)) = \Sigma$, where Σ is the set of all the stationary points of (P_t), i.e.

$$\Sigma = \{\varphi \in L^2(\mathbb{R}^N, \mu) \mid P_t\varphi = \varphi,\ \mu \text{ a.e.},\ \forall t \ge 0\}. \tag{5.1.10}$$

Finally

$$\int_{\mathbb{R}^N} M_\infty\varphi(x)\mu(dx) = \int_{\mathbb{R}^N}\varphi(x)\mu(dx).$$

Proof. For all $T > 0$, let $n_T \in \mathbb{N}\cup\{0\}$ and $r_T \in [0,1[$ be the integer and fractional part of T, respectively. If $\varphi \in L^2(\mathbb{R}^N, \mu)$, then

$$\begin{aligned} M(T)\varphi &= \frac{1}{T}\sum_{k=0}^{n_T-1}\int_k^{k+1} P_s\varphi ds + \frac{1}{T}\int_{n_T}^T P_s\varphi ds = \frac{1}{T}\sum_{k=0}^{n_T-1}\int_0^1 P_{s+k}\varphi ds \\ &\quad + \frac{1}{T}\int_0^{r_T} P_{s+n_T}\varphi ds \\ &= \frac{n_T}{T}\frac{1}{n_T}\sum_{k=0}^{n_T-1} P_1^k(M(1)\varphi) + \frac{r_T}{T}P_1^{n_T}(M(r_T)\varphi). \end{aligned}$$

Since

$$\lim_{T\to\infty}\frac{n_T}{T}=1, \quad \lim_{T\to\infty}\frac{r_T}{T}=0,$$

letting $T\to\infty$ and recalling Proposition 5.1.10, we get that $M(T)\varphi$ has limit in $L^2(\mathbb{R}^N,\mu)$, say $M_\infty\varphi$. Let us prove that

$$M_\infty P_t = P_t M_\infty = M_\infty. \tag{5.1.11}$$

Given $t\geq 0$ we have

$$\begin{aligned} M_\infty P_t\varphi &= \lim_{T\to\infty}\frac{1}{T}\int_0^T P_{t+s}\varphi ds = \lim_{T\to\infty}\frac{1}{T}\int_t^{t+T} P_s\varphi ds \\ &= \lim_{T\to\infty}\frac{1}{T}\Big(\int_0^T P_s\varphi ds + \int_T^{t+T} P_s\varphi ds - \int_0^t P_s\varphi ds\Big) = M_\infty\varphi. \end{aligned}$$

In a similar way, one can check that $P_t M_\infty\varphi = M_\infty\varphi$, so (5.1.11) is completely proved.

For all $\varphi\in L^2(\mathbb{R}^N,\mu)$, (5.1.11) implies that $M_\infty\varphi\in\Sigma$. Conversely, if $\varphi\in\Sigma$, then $M(T)\varphi=\varphi$ and consequently, taking the limit as $T\to\infty$, $M_\infty\varphi=\varphi\in M_\infty(L^2(\mathbb{R}^N,\mu))$. Since $P_tM_\infty = M_\infty P_t = M_\infty$, it follows that $M_\infty M(T) = M(T)M_\infty = M_\infty$, that yields $M_\infty = M_\infty^2$, letting $T\to\infty$. Finally, integrating (5.1.9) with respect to μ, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N}(M(T)\varphi)(x)\mu(dx) &= \frac{1}{T}\int_{\mathbb{R}^N}\int_0^T (P_s\varphi)(x)ds\,\mu(dx) = \frac{1}{T}\int_0^T\int_{\mathbb{R}^N}\varphi(x)\mu(dx)ds \\ &= \int_{\mathbb{R}^N}\varphi(x)\mu(dx). \end{aligned}$$

Letting $T\to\infty$ we conclude the proof. $\square$

Remark 5.1.12 The Von Neumann Theorem gives information on the asymptotic behaviour of the semigroup (P_t), as $t\to\infty$. We note that, in general, the limit of $P_t\varphi(x)$ as $t\to\infty$ does not exist, if $\varphi\notin\Sigma$. For example in $\mathbb{R}^2$ consider the Cauchy problem

$$\begin{cases} \xi'(t) = -\eta(t) \\ \eta'(t) = \xi(t) \\ \xi(0)=x_1,\ \eta(0)=x_2 \end{cases}$$

Then $(\xi(t,x),\eta(t,x)) = (x_1\cos t - x_2\sin t, x_1\sin t + x_2\cos t)$, $t\geq 0$, $x\in\mathbb{R}^2$. The semigroup $P_t\varphi(x)=\varphi(\xi(t,x),\eta(t,x))$ is such that $\lim_{t\to\infty}P_t\varphi(x)$ exists only if $x=0$.

If (P_t) is regular, then it can be proved that $\lim_{t\to\infty}(P_t\varphi)(x)=\int_{\mathbb{R}^N}\varphi(y)\mu(dy)$, for all $\varphi\in L^2(\mathbb{R}^N,\mu)$ and $x\in\mathbb{R}^N$. This results, which is due to Doob, means that the underlying stochastic process is stable and $\int_{\mathbb{R}^N}\varphi(y)\mu(dy)$ is the equilibrium.

The next proposition contains the main properties of the subspace Σ. In particular it shows that Σ is a lattice. We remark that if $(A_2, D(A_2))$ is the generator of (P_t) in $L^2(\mathbb{R}^N,\mu)$, then $\Sigma=\ker A_2$.

Proposition 5.1.13 *Let $\varphi,\psi\in\Sigma$. Then the following assertions hold*

(i) $|\varphi|\in\Sigma$,

(ii) $\varphi^+,\varphi^-\in\Sigma$,

(iii) $\varphi\vee\psi, \varphi\wedge\psi\in\Sigma$,

(iv) for all $\lambda \in \mathbb{R}$, the characteristic function of the set $\{x \in \mathbb{R}^N \mid \varphi(x) > \lambda\}$ belongs to Σ.

PROOF. Let us prove (i). By the positivity of (P_t) we infer $|\varphi(x)| = |P_t\varphi(x)| \leq P_t|\varphi|(x)$. Assume, by contradiction, that there exists a Borel set Γ such that $\mu(\Gamma) > 0$ and $|\varphi(x)| < P_t|\varphi|(x)$, for $x \in \Gamma$. Then

$$\int_{\mathbb{R}^N} |\varphi(x)|\mu(dx) < \int_{\mathbb{R}^N} P_t|\varphi|(x)\mu(dx),$$

which contradicts the invariance of μ.

Assertions (ii) and (iii) follow easily from the identities

$$\varphi^+ = \frac{1}{2}(\varphi + |\varphi|), \quad \varphi^- = \frac{1}{2}(\varphi - |\varphi|),$$

$$\varphi \vee \psi = (\varphi - \psi)^+ + \psi, \quad \varphi \wedge \psi = -(\varphi - \psi)^+ + \varphi.$$

In order to prove (iv), it is sufficient to take $\lambda = 0$. Consider $\varphi_n(x) := (n\varphi^+ \wedge 1)(x)$. Then $\lim_{n\to\infty} \varphi_n(x) = \chi_{\{\varphi>0\}}(x)$ and $\varphi_n \in \Sigma$, by (ii) and (iii). By dominated convergence, $\chi_{\{\varphi>0\}}(x) = \lim_{n\to\infty} \varphi_n(x) = \lim_{n\to\infty} P_t\varphi_n(x) = P_t\chi_{\{\varphi>0\}}(x)$. Hence the thesis follows. $\square$

Now, we devote our attention to the case where the limit M_∞ provided by the Von Neumann Theorem is of a particular form.

Definition 5.1.14 *Let μ be an invariant measure for the semigroup (P_t). We say that μ is ergodic if*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T P_t\varphi dt = \overline{\varphi},$$

in $L^2(\mathbb{R}^N, \mu)$, where $\overline{\varphi} = \int_{\mathbb{R}^N} \varphi(x)\mu(dx)$.

Proposition 5.1.15 *μ is ergodic if and only if the dimension of Σ, defined in (5.1.10), is equal to one.*

PROOF. Assume that μ is ergodic. Then, from the Von Neumann Theorem it follows that $M_\infty\varphi = \overline{\varphi}$, for all $\varphi \in L^2(\mathbb{R}^N, \mu)$. Since M_∞ is a projection on Σ, it turns out that Σ is one dimensional.

Conversely, assume that the dimension of Σ is one. Then, there exists a linear continuous functional f on $L^2(\mathbb{R}^N, \mu)$ such that $M_\infty\varphi = f(\varphi)1 = f(\varphi)$, for all $\varphi \in L^2(\mathbb{R}^N, \mu)$. Moreover, the Riesz-Frechèt Theorem yields a function $\varphi_0 \in L^2(\mathbb{R}^N, \mu)$ satisfying $f(\varphi) = \int_{\mathbb{R}^N} \varphi(x)\varphi_0(x)\mu(dx)$. Integrating this identity with respect to μ and recalling the invariance of M_∞ (see Theorem 5.1.11) we find

$$\int_{\mathbb{R}^N} M_\infty\varphi(x)\mu(dx) = \int_{\mathbb{R}^N} \varphi(x)\mu(dx) = \int_{\mathbb{R}^N} \varphi(x)\varphi_0(x)\mu(dx),$$

for all $\varphi \in L^2(\mathbb{R}^N, \mu)$. This leads to $\varphi_0 = 1$ and consequently $M_\infty\varphi = f(\varphi) = \overline{\varphi}$. $\square$

Let μ be an invariant measure for (P_t). A Borel set Γ is said to be *invariant* for the semigroup, if its characteristic function χ_Γ belongs to Σ. Γ is said to be *trivial* if $\mu(\Gamma)$ is equal to 0 or 1.

The next result is a characterization of the ergodicity of an invariant measure in terms of invariant sets.

Proposition 5.1.16 *Let μ be an invariant measure for (P_t). Then μ is ergodic if and only if each invariant set is trivial.*

PROOF. Assume that μ is ergodic and let Γ be an invariant set. Then χ_Γ must be μ-a.e. constant in order to keep Σ one dimensional.

Conversely, suppose that all the invariant sets are trivial and, by contradiction, that μ is not ergodic. Then there exists a nonconstant function $\varphi \in \Sigma$. Therefore, for some $\lambda \in \mathbb{R}$ the set $\{\varphi > \lambda\}$, which is invariant by Proposition 5.1.13, is not trivial. $\square$

An interesting relationship between uniqueness and ergodicity of an invariant measure is contained in the next proposition.

Proposition 5.1.17 *Suppose that there exists a unique invariant measure μ for (P_t). Then it is ergodic.*

PROOF. Assume by contradiction that μ is not ergodic. Then there exists a non trivial invariant set Γ. Define
$$\mu_\Gamma(A) = \frac{\mu(A \cap \Gamma)}{\mu(\Gamma)},$$
for any A Borel set. Since Γ is not trivial, $\mu_\Gamma(\Gamma) \neq \mu(\Gamma)$, hence μ and μ_Γ are distinct. We claim that μ_T is an invariant measure for (P_t). To this aim, it is sufficient to show that
$$\mu_\Gamma(A) = \int_{\mathbb{R}^N} p_t(x, A)\mu_\Gamma(dx),$$
for any Borel set A (see (5.1.4)) or, equivalently, that
$$\mu(A \cap \Gamma) = \int_\Gamma p_t(x, A)\mu(dx).$$
Since Γ is invariant, for all $t \geq 0$ we have $P_t\chi_\Gamma = \chi_\Gamma$ μ-a.e. Then $p_t(x, \Gamma) = \chi_\Gamma(x)$ μ-a.e. and, as a consequence, $p_t(x, A \cap \Gamma) = 0$, μ-a.e. in Γ^c, since $p_t(x, A \cap \Gamma) \leq p_t(x, \Gamma)$. Analogously, $P_t\chi_{\Gamma^c} = \chi_{\Gamma^c}$ μ-a.e., because $P_t\mathbb{1} = \mathbb{1}$. Then $p_t(x, \Gamma^c) = \chi_{\Gamma^c}(x)$ and therefore $p_t(x, A \cap \Gamma^c) = 0$, μ-a.e. in Γ. So we have
$$\begin{aligned}
\int_\Gamma p_t(x, A)\mu(dx) &= \int_\Gamma p_t(x, A \cap \Gamma)\mu(dx) + \int_\Gamma p_t(x, A \cap \Gamma^c)\mu(dx) \\
&= \int_\Gamma p_t(x, A \cap \Gamma)\mu(dx) = \int_{\mathbb{R}^N} p_t(x, A \cap \Gamma)\mu(dx) \\
&= \mu(A \cap \Gamma).
\end{aligned}$$
Thus, we have established that μ_T is an invariant measure for (P_t) and this clearly contradicts the uniqueness of μ. $\square$

Lemma 5.1.18 *Let μ, ν be two ergodic invariant measures of (P_t), with $\mu \neq \nu$. Then μ and ν are singular.*

PROOF. Let Γ be a Borel set such that $\mu(\Gamma) \neq \nu(\Gamma)$. From the Von Neumann Theorem 5.1.11, it follows that $\lim_{T\to\infty} \frac{1}{T}\int_0^T P_s\varphi ds = M_\infty\varphi$ in $L^2(\mathbb{R}^N, \mu)$. In particular, choosing $\varphi = \chi_\Gamma$, we find that there exist a sequence $T_n \to \infty$ and a Borel set M such that $\mu(M) = 1$ and
$$\lim_{n\to\infty} \frac{1}{T_n}\int_0^{T_n} P_s\chi_\Gamma(x)ds = M_\infty\chi_\Gamma(x), \quad \forall x \in M.$$
Since μ is ergodic, $M_\infty\chi_\Gamma = \mu(\Gamma)$, hence
$$\lim_{n\to\infty} \frac{1}{T_n}\int_0^{T_n} P_s\chi_\Gamma(x)ds = \mu(\Gamma), \quad \forall x \in M.$$

Analogously, one can check that there exist a Borel set N with $\nu(N) = 1$ such that

$$\lim_{n\to\infty} \frac{1}{T_n} \int_0^{T_n} P_s \chi_\Gamma(x) ds = \nu(\Gamma), \quad \forall x \in N$$

(without loss of generality we assume that the sequence T_n is the same for ν). Since $\mu(\Gamma) \neq \nu(\Gamma)$, we have that $M \cap N = \emptyset$, hence μ and ν are singular. $\square$

Finally, we are ready to prove Theorem 5.1.8.

Proof of Theorem 5.1.8. Let μ be an invariant measure for the semigroup (P_t). First we show that μ is equivalent to $p_t(x, \cdot)$, for all $t > 0$ and $x \in \mathbb{R}^N$. Let $t_0 > 0$ and $x_0 \in \mathbb{R}^N$ be fixed. By identity (5.1.4) we have

$$\mu(\Gamma) = \int_{\mathbb{R}^N} p_t(x, \Gamma) \mu(dx), \tag{5.1.12}$$

for any Borel set Γ. Let Γ be such that $p_{t_0}(x_0, \Gamma) = 0$. Then, since (P_t) is regular, $p_t(x, \Gamma) = 0$, for all $t > 0$ and $x \in \mathbb{R}^N$. From the integral representation above it follows that $\mu(\Gamma) = 0$. Therefore $\mu << p_{t_0}(x_0, \cdot)$. Conversely, assume that $\mu(\Gamma) = 0$. Then, again from (5.1.12) $p_t(x, \Gamma) = 0$ for some x, hence for every x by the regularity of (P_t). As a consequence, $p_{t_0}(x_0, \cdot) << \mu$.

Let us prove that μ is ergodic. Using Proposition 5.1.16, we show that every invariant set is trivial. Let Γ be a Borel set such that $P_t \chi_\Gamma = \chi_\Gamma$, μ-a.e. Then $p_t(x, \Gamma) = \chi_\Gamma(x)$, μ-a.e. The regularity of (P_t) implies that either $p_t(x, \Gamma) = 0$ μ-a.e. or $p_t(x, \Gamma) = 1$ μ-a.e. From (5.1.12) it follows that $\mu(\Gamma)$ is either 0 or 1, as claimed.

If ν is another invariant measure, then the argument above proves that ν is equivalent to $p_t(x, \cdot)$, for all $t > 0$, $x \in \mathbb{R}^N$ and that ν is ergodic. It turns out that μ and ν are equivalent. If they were different, then Proposition 5.1.18 would imply that μ and ν are singular, which is a contradiction. We conclude that $\mu = \nu$, as stated. $\square$

5.2 Feller semigroups and differential operators

Feller semigroups naturally arise when one deal with second order elliptic operators in spaces of continuous functions. Suppose we are given a second order partial differential operator

$$Au = \sum_{i,j=1}^N q_{ij} D_{ij} u + \sum_{i=1}^N F_i D_i u, \tag{5.2.1}$$

whose coefficients are locally α-Hölder continuous in $\mathbb{R}^N$, $0 < \alpha < 1$, and satisfy

$$q_{ij} = q_{ji}, \quad \sum_{i,j=1}^N q_{ij}(x) \xi_i \xi_j \geq \nu(x) |\xi|^2, \quad \text{for all } x, \xi \in \mathbb{R}^N,$$

with $\inf_K \nu(x) > 0$, for any compact set K of $\mathbb{R}^N$. Under these assumptions it is always possible to associate with A a semigroup $T(t)$ in $C_b(\mathbb{R}^N)$, which yields a bounded classical solution to the parabolic problem

$$\begin{cases} u_t - Au = 0 & \text{in } (0, \infty) \times \mathbb{R}^N \\ u(0, x) = f(x) & x \in \mathbb{R}^N \end{cases} \tag{5.2.2}$$

for every $f \in C_b(\mathbb{R}^N)$. The construction of such a semigroup is based on an approximation procedure which consists of finding a solution to problem (5.2.2) as limit of solutions of parabolic problems in cylinders $(0, \infty) \times B_\rho$, where A is uniformly elliptic. We have already used this construction in Chapters 2, 3 to solve parabolic problems with Neumann or Dirichlet boundary conditions. Here the situation is easier, since we do not have to take any boundary into consideration.

For the sake of completeness, we briefly recall the construction of $T(t)$. Then we give sufficient conditions for the existence of an invariant measure μ for $(T(t))$. We will see also that μ is unique and absolutely continuous with respect to the Lebesgue measure.

5.2.1 Preliminary results

We refer to [38] and the references therein for more details on this argument and the proofs of the results that we are going to show.

Let us fix a ball B_ρ and consider the domain

$$D_\rho(A) = \{u \in C(\overline{B}_\rho) \cap W^{2,p}(B_\rho) \text{ for all } p < \infty \mid u_{|\partial B_\rho} = 0 \text{ and } Au \in C(\overline{B}_\rho)\}. \tag{5.2.3}$$

Then the operator $(A, D_\rho(A))$ generates an analytic semigroup $(T_\rho(t))$ of positive contractions in the space $C(\overline{B}_\rho)$ (see [32, Corollary 3.1.21]) and, for every $f \in C(\overline{B}_\rho)$ the function $u_\rho(t,x) = T_\rho(t)f(x)$ satisfies

$$\begin{cases} D_t u_\rho(t,x) - Au_\rho(t,x) = 0 & \text{in } (0,\infty) \times B_\rho \\ u_\rho(0,x) = f(x) & x \in B_\rho \\ u_\rho(t,x) = 0 & \text{in } (0,\infty) \times \partial B_\rho. \end{cases} \tag{5.2.4}$$

Since the domain $D_\rho(A)$ is not dense in $C(\overline{B}_\rho)$, strong continuity at 0 fails: in fact, $T_\rho(t)f$ converges uniformly to f in $\overline{B}_\rho$, as $t \to 0$, if and only if f vanishes on ∂B_ρ. However, $T_\rho(t)f$ converges to f uniformly in $\overline{B}_{\rho'}$, as $t \to 0$, for every $\rho' < \rho$, hence pointwise in B_ρ. For all $\rho > 0$, there exists a kernel $p_\rho(t,x,y)$ that represents the semigroup $(T_\rho(t))$:

$$T_\rho(t)f(x) = \int_{B_\rho} p_\rho(t,x,y)f(y)dy,$$

for all $f \in C(\overline{B}_\rho)$. Moreover, $p_\rho(t,x,y) > 0$ for $t > 0, x, y \in B_\rho$, $p_\rho(t,x,y) = 0$ for $t > 0, x \in \partial B_\rho, y \in B_\rho$ and for every $y \in B_\rho$, $0 < \varepsilon < \tau$ it belongs to $C^{1+\alpha/2,2+\alpha}((\varepsilon,\tau) \times B_\rho)$ as function of (t,x), and satisfies $D_t p_\rho - Ap_\rho = 0$. If f is positive then $T_\rho(t)f$ is positive and $\|T_\rho(t)f\|_\infty \le \|f\|_\infty$. For all the properties of p_ρ we refer to [24, Chapter 3, Section 7].

An argument based on the classical maximum principle shows that for every $f \in C_b(\mathbb{R}^N)$ the limit $\lim_{\rho\to\infty} T_\rho(t)f$ exists uniformly on compact sets in $\mathbb{R}^N$ and defines a semigroup $(T(t))$ of positive contractions in $C_b(\mathbb{R}^N)$. The main properties of $(T(t))$ are listed in the proposition below.

Proposition 5.2.1 *For every $f \in C_b(\mathbb{R}^N)$, the function $u(t,x) = T(t)f(x)$ belongs to $C^{1+\alpha/2,2+\alpha}_{\mathrm{loc}}((0,\infty) \times \mathbb{R}^N)$ and satisfies the equation*

$$D_t u - Au = 0.$$

Moreover, $T(t)f$ can be represented in the form

$$T(t)f(x) = \int_{\mathbb{R}^N} f(y)p(t,x,y)dy, \tag{5.2.5}$$

where p is a positive function. For almost all $y \in \mathbb{R}^N$, $p(t,x,y)$, as function of (t,x), belongs to $C^{1+\alpha/2,2+\alpha}_{\mathrm{loc}}((0,\infty) \times \mathbb{R}^N)$ and solves $D_t p = Ap$. Finally, $T(t)f$ converges to f uniformly on compact sets of $\mathbb{R}^N$, as $t \to 0$, hence u belongs to $C([0,+\infty[\times\mathbb{R}^N)$ and solves (5.2.2).

We note that the previous proposition establishes, in particular, an integral representation for the semigroup $T(t)$ similar to (5.1.1). Here we get more, since all the measures are absolutely continuous with respect to the Lebesgue measure.

We note also that, since $(T(t))$ is contractive, we have $T(t)\mathbb{1} = \int_{\mathbb{R}^N} p(t,x,y)dy \leq 1$ and there are cases where the strict inequality holds. We will see later a necessary and sufficient condition to have $T(t)\mathbb{1} = \mathbb{1}$ (see Proposition 5.2.7). Finally we observe that, as in the general setting, formula (5.2.5) makes sense also for bounded Borel functions.

As a consequence of the results above, we can prove that $(T(t))$ is irreducible and has the strong Feller property (see Definition 5.1.7).

Proposition 5.2.2 *The semigroup $T(t)$ is irreducible and has the strong Feller property.*

Proof. The irreducibility of $T(t)$ is a consequence of the integral representation (5.2.5) and the positivity of the kernel p. Concerning the strong Feller property, let f be a Borel function and consider a bounded sequence (f_n) in $C_b(\mathbb{R}^N)$ such that $f_n(x)$ converges to $f(x)$, for almost all $x \in \mathbb{R}^N$. From (5.2.5) and the dominated convergence theorem it follows that $T(t)f_n(x)$ converges to $T(t)f(x)$ for all $x \in \mathbb{R}^N$, $t > 0$. Using the interior Schauder estimates (see [30, Theorem IV.10.1]), it turns out that for every fixed $t > 0$, $\rho > 0$ and for all $n \in \mathbb{N}$

$$\|T(t)f_n\|_{C^1(\overline{B}_\rho)} \leq C\|T(t)f_n\|_{C(\overline{B}_{2\rho})} \leq C\|f_n\|_\infty \leq C',$$

with $C' > 0$ independent of n. This implies, by a compactness argument, that there exists a subsequence of $T(t)f_n$ which converges to $T(t)f$ uniformly on compact sets. Therefore $T(t)f \in C_b(\mathbb{R}^N)$. □

Even though $(T(t))$ is not strongly continuous one can define its generator following the approach of [48]. More precisely, let us introduce the operator

$$\begin{aligned}
\widehat{D} &= \Big\{f \in C_b(\mathbb{R}^N) : \sup_{t\in(0,1)} \frac{\|T(t)f - f\|}{t} < \infty \text{ and } \exists g \in C_b(\mathbb{R}^N) \text{ such that} \\
&\qquad \lim_{t\to 0} \frac{(T(t)f)(x) - f(x)}{t} = g(x), \quad \forall\, x \in \mathbb{R}^N\Big\} \\
\widehat{A}f(x) &= \lim_{t\to 0} \frac{(T(t)f)(x) - f(x)}{t}, \quad f \in \widehat{D},\ x \in \mathbb{R}^N
\end{aligned}$$

$(\widehat{A}, \widehat{D})$ is called the *weak generator* of $(T(t))$. It enjoys several properties which are well-known for generators of strongly continuous semigroups. In particular, if $f \in \widehat{D}$, then $T(t)f \in \widehat{D}$ and $\widehat{A}T(t)f = T(t)\widehat{A}f$, for all $t \geq 0$. Moreover, the map $t \to T(t)f(x)$ is continuously differentiable in $[0,\infty[$ for all $x \in \mathbb{R}^N$ and $D_tT(t)f(x) = T(t)\widehat{A}f(x)$. Besides, one can prove that $(0,+\infty) \subset \rho(\widehat{A})$, $\|R(\lambda,\widehat{A})\| \leq 1/\lambda$ and

$$(R(\lambda,\widehat{A})f)(x) = \int_0^{+\infty} e^{-\lambda t}(T(t)f)(x)\,dt, \qquad f \in C_b(\mathbb{R}^N),\ x \in \mathbb{R}^N. \tag{5.2.6}$$

The notion of weak generator is quite general and it allows to study a large class of semigroups on $C_b(E)$ (the so called π-semigroups), for some separable metric space E. In our situation, since the semigroup $(T(t))$ has been constructed starting from a differential operator, it is interesting to point out the relationship existing between $\widehat{A}$ and our operator A. In fact, it can be proved that $\widehat{A}$ is a restriction of A, in the sense specified by the following proposition.

Proposition 5.2.3 *Let $D_{\max}(A)$ be the maximal domain of A in $C_b(\mathbb{R}^N)$:*

$$D_{\max}(A) = \{u \in C_b(\mathbb{R}^N) \cap W^{2,p}_{\rm loc}(\mathbb{R}^N) \text{ for all } p < \infty \mid Au \in C_b(\mathbb{R}^N)\}. \tag{5.2.7}$$

Then $\widehat{D} \subset D_{\max}(A)$ and $\widehat{A}f = Af$, for $f \in \widehat{D}$. The equality $\widehat{D} = D_{\max}(A)$ holds if and only if $\lambda - A$ is injective on $D_{\max}(A)$ for some (hence for all) $\lambda > 0$.

PROOF. Let $\lambda > 0$ be fixed. If $u \in \widehat{D}$, then there exists a unique $f \in C_b(\mathbb{R}^N)$ such that $u = R(\lambda, \widehat{A})f$. We claim that u belongs to $D_{\max}(A)$ and solves the equation $\lambda u - Au = f$. From identity (5.2.6) and the construction of the semigroup $T(t)$, it follows that for every $x \in \mathbb{R}^N$

$$u(x) = \int_0^{+\infty} e^{-\lambda t} \lim_{\rho\to+\infty} (T_\rho(t)f)(x)dt = \lim_{\rho\to+\infty} \int_0^{+\infty} e^{-\lambda t}(T_\rho(t)f)(x)dt,$$

where the last equality follows from the dominated convergence theorem. For each $\rho > 0$ we have

$$\int_0^{+\infty} e^{-\lambda t}(T_\rho(t)f)(x)dt = (R(\lambda, A_\rho)f)(x) =: u_\rho(x), \tag{5.2.8}$$

where A_ρ means the operator A endowed with the domain $D_\rho(A)$ defined in (5.2.3). Therefore the function $u_\rho \in D_\rho(A)$ satisfies

$$\begin{cases} \lambda u_\rho - Au_\rho = f & \text{in } B_\rho, \\ u_\rho = 0 & \text{on } \partial B_\rho. \end{cases}$$

Since $T_\rho(t)$ is contractive, we have

$$\|u_\rho\|_\infty \le \frac{\|f\|_\infty}{\lambda}. \tag{5.2.9}$$

Hence, by difference, we obtain

$$\|Au_\rho\|_\infty \le 2\|f\|_\infty. \tag{5.2.10}$$

For every $R > 0$, the classical interior L^p estimates (see [26, Theorem 9.11]) yield a constant $C > 0$ depending on p, R, N and the operator A such that

$$\|u_\rho\|_{W^{2,p}(B_R)} \le C(\|Au_\rho\|_{L^p(B_{2R})} + \|u_\rho\|_{L^p(B_{2R})}), \tag{5.2.11}$$

for all $\rho > 2R$. From (5.2.9) and (5.2.10) it follows that

$$\|u_\rho\|_{W^{2,p}(B_R)} \le C_1\|f\|_\infty, \tag{5.2.12}$$

with C_1 depending on R, p, N, λ, the operator A but independent of ρ. Choosing $p > N$, (5.2.12) gives a uniform estimate of (u_ρ) in $C^1(\overline{B}_R)$ which allows to apply Ascoli's Theorem and to deduce that a subsequence (u_{ρ_n}) of (u_ρ) converges uniformly to u on compact subsets of $\mathbb{R}^N$. From the equation $\lambda u_{\rho_n} - Au_{\rho_n} = f$ it follows that Au_{ρ_n} converges uniformly on compact sets as well. Therefore, applying (5.2.11) to the difference $u_{\rho_n} - u_{\rho_m}$, we find that u_{ρ_n} converges to u in $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^N)$, hence $u \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^N)$. Taking the limit in the equation satisfied by u_{ρ_n} we deduce that $\lambda u - Au = f$ and, as a consequence, $u \in D_{\max}(A)$. Since $\lambda u - Au = f = \lambda u - \widehat{A}u$, we have $Au = \widehat{A}u$ and the first assertion is proved. As regards the second statement, clearly $\lambda - A$ is bijective from $\widehat{D}$ onto $C_b(\mathbb{R}^N)$. Assume that it is injective also in $D_{\max}(A)$. If $u \in D_{\max}(A)$, there exists $v \in \widehat{D}$ such that $\lambda v - Av = \lambda u - Au$. Therefore $u - v$ belongs to $D_{\max}(A)$ and $\lambda(u - v) - A(u - v) = 0$. From the injectivity of $\lambda - A$ on $D_{\max}(A)$ we deduce that $u = v$ and, consequently, $\widehat{D} = D_{\max}(A)$. □

As a consequence of Proposition 5.2.3, we can write $R(\lambda, A)$ instead of $R(\lambda, \widehat{A})$ (keeping the fact that $R(\lambda, A)$ maps $C_b(\mathbb{R}^N)$ onto $\widehat{D}$ and not onto $D_{\max}(A)$, in general). It is worth stating explicitly a result included in the proof of the above proposition.

Corollary 5.2.4 *For all $\lambda > 0$ and $f \in C_b(\mathbb{R}^N)$, there exists u belonging to $D_{\max}(A)$ such that $\lambda u - Au = f$ and $\|u\|_\infty \le \frac{1}{\lambda}\|f\|_\infty$. Moreover, $u \ge 0$ if $f \ge 0$.*

Remark 5.2.5 Let us consider $f \in C_b(\mathbb{R}^N)$, $f \ge 0$. Then $R(\lambda, A)f$ is a positive solution in $D_{\max}(A)$ of the equation $\lambda u - Au = f$, not unique, in general. In any case, it is the minimal among all the positive solutions of the same equation in $D_{\max}(A)$. Indeed, let $w \in D_{\max}(A)$ be positive and such that $\lambda w - Aw = f$. The function $u_\rho - w \in W^{2,p}(B_\rho) \cap C(\overline{B}_\rho)$, with u_ρ given by (5.2.8), is such that $A(u_\rho - w) \in C(\overline{B}_\rho)$ and satisfies

$$\begin{cases} \lambda(u_\rho - w) - A(u_\rho - w) = 0 & \text{in } B_\rho, \\ u_\rho - w \le 0 & \text{on } \partial B_\rho. \end{cases}$$

We claim that $u_\rho - w \le 0$ in B_ρ. Since $u_\rho - w \in C(\overline{B}_\rho)$, there exists a maximum point $x_0 \in \overline{B}_\rho$. Assume by contradiction that $u_\rho(x_0) - w(x_0) > 0$. Then $x_0 \in B_\rho$. From Corollary A.0.9 we deduce that $A(u_\rho - w)(x_0) \le 0$ and therefore

$$0 = \lambda(u_\rho - w)(x_0) - A(u_\rho - w)(x_0) \ge \lambda(u_\rho - w)(x_0) > 0,$$

which is impossible. Hence $u_\rho(x) - w(x) \le u_\rho(x_0) - w(x_0) \le 0$ for every $x \in B_\rho$. Letting $\rho \to +\infty$ and recalling that $\lim_{\rho \to +\infty} u_\rho = R(\lambda, A)f$, we have $R(\lambda, A)f \le w$, as claimed.

A sufficient condition for the injectivity of $\lambda - A$ on $D_{\max}(A)$ is the existence of a *Liapunov function*, i.e. a function $V \in C^2(\mathbb{R}^N)$, such that $\lim_{|x| \to +\infty} V(x) = +\infty$ and $\lambda V - AV \ge 0$. This assumption leads to growth conditions on the coefficients of A. Indeed, in order to find a Liapunov function, one often considers some simple function V which goes to $+\infty$ as $|x| \to +\infty$, plugs it into $\lambda - A$ and imposes that $\lambda V - AV \ge 0$. By taking for example $V(x) = 1 + |x|^2$, one requires that

$$\sum_{i=1}^N q_{ii}(x) + \sum_{i=1}^N F_i(x)x_i \le \lambda(1 + |x|^2), \quad x \in \mathbb{R}^N.$$

If $\lambda - A$ is injective on $D_{\max}(A)$ then the semigroup $T(t)$ yields the *unique* bounded classical solution to problem (5.2.2).

Proposition 5.2.6 *Suppose that $\lambda - A$ is injective on $D_{\max}(A)$ for some $\lambda > 0$ and let $w \in C^{1,2}((]0, \tau] \times \mathbb{R}^N) \cap C([0, \tau] \times \mathbb{R}^N)$ be a bounded solution of problem (5.2.2). Then $w(t, x) = T(t)f(x)$.*

Proof. By linearity, it is sufficient to prove the statement in the case where w solves problem (5.2.2) with $f = 0$. For $0 < \varepsilon < t \le \tau$ and $x \in \mathbb{R}^N$ we have

$$w(t, x) - w(\varepsilon, x) = \int_\varepsilon^t \frac{d}{ds} w(s, x)ds = \int_\varepsilon^t Aw(s, x)ds = A\int_\varepsilon^t w(s, x)ds. \tag{5.2.13}$$

Since $(A, D_{\max}(A))$ is the weak generator of $T(t)$ (see Proposition 5.2.3), from [48, Proposition 3.4] it follows that it is closed with respect to the π-convergence, defined as

$$f_n \xrightarrow{\pi} f \quad \Longleftrightarrow \quad f_n(x) \to f(x) \quad \text{and} \quad \|f_n\|_\infty \le C.$$

Since $w \in C([0, \tau] \times \mathbb{R}^N)$, we have that $\int_\varepsilon^t w(s, x)ds$ converges to $\int_0^t w(s, x)ds$ as $\varepsilon \to 0$, for every $x \in \mathbb{R}^N$. Moreover $\left\|\int_\varepsilon^t w(s, \cdot)ds\right\|_\infty \le \|w\|_\infty t$, which implies that

$$\int_\varepsilon^t w(s, \cdot)ds \xrightarrow{\pi} \int_0^t w(s, \cdot)ds, \quad \text{as } \varepsilon \to 0.$$

From (5.2.13) we infer that $A\int_\varepsilon^t w(s,x)ds$ converges to $w(t,x)$ when ε goes to zero, for every $x\in\mathbb{R}^N$, and

$$\left\| A\int_\varepsilon^t w(s,\cdot)ds\right\|_\infty = \|w(t,\cdot)-w(\varepsilon,\cdot)\|_\infty \le 2\|w\|_\infty,$$

i.e. $A\int_\varepsilon^t w(s,\cdot)ds \xrightarrow{\pi} w(t,\cdot)$. The closedness of $(A, D_{\max}(A))$ yields

$$\int_0^t w(s,\cdot)ds \in D_{\max}(A) \qquad \text{and} \qquad w(t,x) = A\int_0^t w(s,x)ds, \tag{5.2.14}$$

for $t\le\tau$. Setting $w(\tau+s,x)=T(s)w(\tau,\cdot)(x)$ we obtain a bounded function w which belongs to $C([0,+\infty[\times\mathbb{R}^N)$ and such that (5.2.14) holds for every $t>0$. Indeed, it is clear that the extended function is bounded in $[0,\infty[\times\mathbb{R}^N$. As regards the continuity, by the semigroup law, it is sufficient to show that if $s_n\to 0$ and $x_n\to x$ then $w(\tau+s_n,x_n)\to w(\tau,x)$. To this aim we observe that

$$\begin{aligned}
|w(\tau+s_n,x_n)-w(\tau,x)| &= |T(s_n)w(\tau,\cdot)(x_n)-w(\tau,x)| \\
&\le |T(s_n)w(\tau,\cdot)(x_n)-w(\tau,x_n)| + |w(\tau,x_n)-w(\tau,x)| \\
&\le \sup_{y\in K}|T(s_n)w(\tau,\cdot)(y)-w(\tau,y)| + |w(\tau,x_n)-w(\tau,x)|,
\end{aligned}$$

where K is a compact subset of $\mathbb{R}^N$ such that $x_n\in K$ for all $n\in\mathbb{N}$. Since the semigroup $T(t)$ is strongly continuous with respect to the uniform convergence on compact sets (see Proposition 5.2.1), the first term tends to zero as $n\to\infty$. The second one is infinitesimal, too, by the continuity of w. Now, we claim that (5.2.14) is true for every $t>\tau$. Since

$$\int_0^t w(s,x)ds = \int_0^\tau w(s,x)ds + \int_0^{t-\tau}(T(\sigma)w(\tau,\cdot))(x)d\sigma$$

the claim is proved, because $\int_0^\tau w(s,\cdot)ds\in D_{\max}(A)$ by (5.2.14) and $\int_0^{t-\tau}(T(\sigma)w(\tau,\cdot))(x)d\sigma \in D_{\max}(A)$ by [48, Proposition 3.4].

Using again the closedness of $(A, D_{\max}(A)$ with respect to the π-convergence and Fubini's Theorem we obtain

$$\begin{aligned}
\int_0^{+\infty} e^{-\lambda t}w(t,x)dt &= A\Big(\int_0^{+\infty} e^{-\lambda t}\int_0^t w(s,x)ds\,dt\Big) \\
&= A\Big(\int_0^{+\infty} w(s,x)\int_s^{+\infty} e^{-\lambda t}dt\,ds\Big) \\
&= \frac{1}{\lambda}A\Big(\int_0^{+\infty} e^{-\lambda s}w(s,x)ds\Big).
\end{aligned}$$

It follows that the function $v(x)=\int_0^{+\infty}e^{-\lambda s}w(s,x)ds$ belong to $D_{\max}(A)$ and satisfies $\lambda v - Av = 0$. Since $\lambda - A$ is injective on $D_{\max}(A)$ we infer that $v=0$. This means that the Laplace transform of $w(\cdot,x)$ is identically zero, hence $w=0$. □

Moreover, the following result can be proved.

Proposition 5.2.7 *$\lambda - A$ is injective on $D_{\max}(A)$ if and only if $T(t)\mathbb{1}=\mathbb{1}$, for all $t\ge 0$.*

PROOF. If $\lambda - A$ is injective on $D_{\max}(A)$, then from Proposition 5.2.6 it follows that the semigroup $T(t)$ yields the unique bounded classical solution to problem (5.2.2). Since $\mathbb{1}$ is in fact a bounded classical solution of problem (5.2.2) with initial datum $f=\mathbb{1}$, by uniqueness it turns out that $T(t)\mathbb{1}=\mathbb{1}$.

Conversely, if $T(t)\mathbb{1} = \mathbb{1}$ for all $t \geq 0$, then $R(1, A)\mathbb{1} = \mathbb{1}$ (see (5.2.6)). Let $u \in D_{\max}(A)$ be such that $u - Au = 0$ and $\|u\|_\infty \leq 1$. The function $v = \mathbb{1} - u \in D_{\max}(A)$ is nonnegative and satisfies $v - Av = \mathbb{1}$. On the other hand, by Remark 5.2.5, $R(1, A)\mathbb{1} = \mathbb{1}$ is the minimal positive solution of $w - Aw = \mathbb{1}$, hence $\mathbb{1} \leq \mathbb{1} - u$, i.e. $u \leq 0$. The same argument applied to $-u$ proves that $u \geq 0$ and therefore $u = 0$. □

If $T(t)\mathbb{1} = \mathbb{1}$ then, collecting all the results so far, we have that $(T(t))$ is a Feller semigroup, according to the terminology introduced in the previous section.

5.2.2 Invariant measures

Our aim is to establish now some criteria for the existence of an invariant measure for $T(t)$ in terms of the coefficients of the operator A. Since $T(t)$ is irreducible and has the strong Feller property (see Proposition 5.2.2) we already know that if an invariant measure exists, then it is unique and ergodic (see Theorem 5.1.8). Therefore, we limit our study to the existence part.

We start by a preliminary lemma which is similar to Proposition 5.1.2. We note, however, that here the semigroup is not strongly continuous and A is only its weak generator. For the proof see [38].

Lemma 5.2.8 *Assume that $\lambda - A$ is injective on $D_{\max}(A)$. Then a probability measure μ is invariant for $(T(t))$ if and only if $\int_{\mathbb{R}^N} Af\mu(dx) = 0$, for all $f \in D_{\max}(A)$.*

Proof. Since $(A, D_{\max}(A))$ is the weak generator of $T(t)$, if $u \in D_{\max}(A)$, we have that $T(t)u \in D_{\max}(A)$ and $\frac{d}{dt}T(t)u(x) = (AT(t)u)(x) = (T(t)Au)(x)$. Therefore $\left\|\frac{d}{dt}T(t)u\right\|_\infty \leq \|Au\|_\infty$ and by dominated convergence

$$\frac{d}{dt}\int_{\mathbb{R}^N} T(t)u(x)\mu(dx) = \int_{\mathbb{R}^N} AT(t)u(x)\mu(dx).$$

This shows that μ is an invariant measure for the restriction of $T(t)$ to $D_{\max}(A)$ if and only if $\int_{\mathbb{R}^N} Au\,\mu(dx) = 0$, for every $u \in D_{\max}(A)$. If this is the case and $f \in C_b(\mathbb{R}^N)$, then $f_n = n\int_0^{1/n} T(s)f ds$ belongs to $D_{\max}(A)$ and satisfies $\|f_n\|_\infty \leq \|f\|_\infty$, $f_n(x) \to f(x)$, for every $x \in \mathbb{R}^N$ (see [48, Proposition 3.4]). It follows that $T(t)f_n(x)$ converges to $T(t)f(x)$ (see [38, Proposition 4.6]) and $\|T(t)f_n\|_\infty \leq \|f_n\|_\infty \leq \|f\|_\infty$. Since

$$\int_{\mathbb{R}^N} T(t)f_n(x)\mu(dx) = \int_{\mathbb{R}^N} f_n(x)\mu(dx),$$

by dominated convergence we have

$$\int_{\mathbb{R}^N} T(t)f(x)\mu(dx) = \int_{\mathbb{R}^N} f(x)\mu(dx)$$

and the proof is complete. □

The following result is due to Khas'minskii.

Theorem 5.2.9 (Khas'minskii) *Assume that there exists a function $V \in C^2(\mathbb{R}^N)$ such that $\lim_{|x|\to+\infty} V(x) = +\infty$ and $\lim_{|x|\to+\infty} AV(x) = -\infty$. Then there is an invariant measure μ for $(T(t))$.*

Proof. We observe preliminarily that the existence of a function V satisfying the stated properties implies that $\lambda - A$ is injective on $D_{\max}(A)$, hence $T(t)\mathbb{1} = \mathbb{1}$ (see Proposition 5.2.7) and $(A, D_{\max}(A))$ is the generator of $T(t)$ (see Proposition 5.2.3). Without loss of generality, we can assume that $V \geq 0$ (otherwise we consider $V+c$ instead of V, for a suitable constant c). Recalling Theorem 5.1.6, it is sufficient to prove that the family of measures

$$\frac{1}{T}\int_0^T p(t,x_0,\cdot)dt, \quad T > T_0 \tag{5.2.15}$$

is tight for some $x_0 \in \mathbb{R}^N$ and $T_0 > 0$. Let $M > 0$ be such that $AV(x) \leq M$ for all $x \in \mathbb{R}^N$. Consider $\psi_n \in C^\infty(\mathbb{R})$ such that $\psi_n(t) = t$ for $t \leq n$, ψ_n is constant in $[n+1, +\infty[$ and $\psi_n' \geq 0, \psi_n'' \leq 0$. It is easily seen that $\psi_n \circ V$ belongs to $D_{\max}(A)$. Indeed, $\psi_n \circ V$ is obviously continuous in $\mathbb{R}^N$ and $\sup_{x\in\mathbb{R}^N} |\psi_n(V(x))| \leq \sup_{t\geq 0}\psi_n(t) < +\infty$. It is also clear that $\psi_n \circ V$ and its first and second order derivatives

$$\begin{aligned}
&D_i(\psi_n \circ V)(x) = \psi_n'(V(x))D_iV(x),\\
&D_{ij}(\psi_n \circ V)(x) = \psi_n''(V(x))D_iV(x)D_jV(x) + \psi_n'(V(x))D_{ij}V(x)
\end{aligned}$$

are locally p-summable, for every $p < \infty$. It remains to show that $A(\psi_n \circ V)$ is bounded in $\mathbb{R}^N$. To this aim, we observe that, by the assumption, there exists $R > 0$ such that $V(x) > n+1$ if $|x| > R$. It follows that $\psi_n'(V(x)) = \psi_n''(V(x)) = 0$, if $|x| > R$ and therefore

$$\begin{aligned}
\sup_{x\in\mathbb{R}^N} |A(\psi_n \circ V)(x)| &= \sup_{x\in\mathbb{R}^N}\left|\psi_n'(V(x))AV(x) + \psi_n''(V(x))\sum_{i,j=1}^N q_{ij}(x)D_iV(x)D_jV(x)\right|\\
&= \sup_{|x|\leq R}\left|\psi_n'(V(x))AV(x) + \psi_n''(V(x))\sum_{i,j=1}^N q_{ij}(x)D_iV(x)D_jV(x)\right|\\
&< +\infty.
\end{aligned}$$

Hence we deduce that $u_n(t,\cdot) = T(t)(\psi_n \circ V)(\cdot) \in D_{\max}(A)$ and

$$\begin{aligned}
D_t u_n(t,x) &= T(t)A(\psi_n \circ V)(x) = \int_{\mathbb{R}^N} p(t,x,y)A(\psi_n \circ V)(y)dy\\
&= \int_{\mathbb{R}^N} p(t,x,y)\Big(\psi_n'(V(y))AV(y) + \psi_n''(V(y))\sum_{i,j=1}^N q_{ij}(y)D_iV(y)D_jV(y)\Big)dy
\end{aligned}$$

Integrating this identity and recalling that $\psi_n'' \leq 0$ we have

$$\begin{aligned}
u_n(T,x) - \psi_n(V(x)) &\leq \int_0^T\int_{\mathbb{R}^N} p(t,x,y)\psi_n'(V(y))AV(y)dy\,dt\\
&= \int_0^T\int_E p(t,x,y)\psi_n'(V(y))AV(y)dy\,dt\\
&\quad + \int_0^T\int_{\mathbb{R}^N\setminus E} p(t,x,y)\psi_n'(V(y))AV(y)dy\,dt,
\end{aligned}$$

where $E = \{y \in \mathbb{R}^N \mid 0 \leq AV(y) \leq M\}$. In the first integral we can use dominated convergence since $p(t,x,y)\psi_n'(V(y))AV(y) \leq p(t,x,y)M$. In the second one, where AV is unbounded but negative, we use monotone convergence because $\psi_n' \leq \psi_{n+1}'$. Letting $n \to \infty$ we deduce that

$$\int_{\mathbb{R}^N} p(T,x,y)V(y)dy - V(x) \leq \int_0^T\int_{\mathbb{R}^N} p(t,x,y)AV(y)dy\,dt.$$

Let $\varepsilon, \rho > 0$ be such that $AV(y) \leq -1/\varepsilon$ if $|y| \geq \rho$. It follows that

$$\begin{aligned}
-\frac{V(x)}{T} &\leq \frac{1}{T}\int_0^T\int_{\mathbb{R}^N} p(t,x,y)AV(y)dy\,dt \leq -\frac{1}{\varepsilon T}\int_0^T\int_{\mathbb{R}^N\setminus B_\rho} p(t,x,y)dy\,dt\\
&\quad + \frac{1}{T}\int_0^T\int_{B_\rho} p(t,x,y)AV(y)dy\,dt \leq -\frac{1}{\varepsilon T}\int_0^T\int_{\mathbb{R}^N\setminus B_\rho} p(t,x,y)dy\,dt + M
\end{aligned}$$

hence

$$\frac{1}{T}\int_0^T p(t,x,\mathbb{R}^N\setminus B_\rho)dt \le \varepsilon\left(M+\frac{V(x)}{T}\right),$$

where we have set $p(t,x,\mathbb{R}^N\setminus B_\rho)=\int_{\mathbb{R}^N\setminus B_\rho}p(t,x,y)dy$. Therefore, we have established that the set of the measures (5.2.15) is tight for every fixed $x_0\in\mathbb{R}^N$ and $T_0>0$ and this completes the proof. $\square$

Khas'minkii's Theorem relies upon the existence of suitable supersolutions of the equation $\lambda u - Au = 0$. Next, we give a different criterion, due to Varadhan, to establish the existence of an invariant measure for a special class of operators (see [40, Proposition 2.1]).

Theorem 5.2.10 *Consider the operator*

$$A=\Delta-\langle D\Phi+G,D\rangle,$$

where $\Phi\in C^1(\mathbb{R}^N)$ *and* $G\in C^1(\mathbb{R}^N;\mathbb{R}^N)$. *Assume that* $e^{-\Phi}\in L^1(\mathbb{R}^N)$ *and* $|G|\in L^1(\mathbb{R}^N,\mu)$, *with* $\mu(dx)=ae^{-\Phi(x)}dx$, $a=\|e^{-\Phi}\|_{L^1}^{-1}$. *Suppose also that*

$$\operatorname{div} G=\langle G,D\Phi\rangle, \tag{5.2.16}$$

i.e. $\operatorname{div}(Ge^{-\Phi})=0$. *If* $(T(t))$ *denotes the semigroup associated with* A, *then* $(T(t))$ *is generated by* $(A,D_{\max}(A))$ *and* μ *is its unique invariant measure.*

Proof. Uniqueness follows immediately from the irreducibility and the strong Feller property (see Proposition 5.2.2 and Theorem 5.1.8). For the existence part, we split the proof in two steps.

Step1. The closure $(B,D(B))$ of $(A,C_c^\infty(\mathbb{R}^N))$ generates a strongly continuous semigroup $(S(t))$ in $L^1(\mathbb{R}^N,\mu)$.

Let us prove that $(A,C_c^\infty(\mathbb{R}^N))$ is dissipative in $L^1(\mathbb{R}^N,\mu)$. Let $\lambda>0$ and $u\in C_c^\infty(\mathbb{R}^N)$ be fixed. Multiplying the equation $\lambda u-Au=f$ by sign$u$ and integrating on $\mathbb{R}^N$ with respect to μ we obtain

$$\begin{aligned}\lambda\int_{\mathbb{R}^N}|u|e^{-\Phi}dx-\int_{\mathbb{R}^N}(\Delta u-\langle D\Phi,Du\rangle)\,\mathrm{sign}u\,e^{-\Phi}dx+\int_{\mathbb{R}^N}\langle G,Du\rangle\,\mathrm{sign}u\,e^{-\Phi}dx\\ =\int_{\mathbb{R}^N}f\,\mathrm{sign}u\,e^{-\Phi}dx.\end{aligned}$$

Since $\Delta u-\langle D\Phi,Du\rangle=e^{\Phi}\operatorname{div}(e^{-\Phi}Du)$ and $(Du)\,\mathrm{sign}u=D|u|$ we get

$$\lambda\int_{\mathbb{R}^N}|u|e^{-\Phi}dx-\int_{\mathbb{R}^N}\operatorname{div}(e^{-\Phi}Du)\,\mathrm{sign}u\,dx+\int_{\mathbb{R}^N}\langle G,D|u|\rangle e^{-\Phi}dx=\int_{\mathbb{R}^N}f\,\mathrm{sign}u\,e^{-\Phi}dx.$$

We claim that $\int_{\mathbb{R}^N}\operatorname{div}(e^{-\Phi}Du)\,\mathrm{sign}u\,dx\le 0$. Let $\varphi_n\in C^1(\mathbb{R})$ be such that $|\varphi_n|\le 1$, $\varphi_n'\ge 0$ and $\varphi_n(t)\to\mathrm{sign}t$ for all $t\neq 0$. Then, by dominated convergence, we have

$$\begin{aligned}\int_{\mathbb{R}^N}\operatorname{div}(e^{-\Phi}Du)\,\mathrm{sign}u\,dx &= \lim_{n\to\infty}\int_{\mathbb{R}^N}\operatorname{div}(e^{-\Phi}Du)\varphi_n(u)dx\\ &= -\lim_{n\to\infty}\int_{\mathbb{R}^N}e^{-\Phi}|Du|^2\varphi_n'(u)dx\le 0,\end{aligned}$$

as claimed. Integrating by parts and taking (5.2.16) into account we deduce that

$$\int_{\mathbb{R}^N}\langle G,D|u|\rangle e^{-\Phi}dx=0.$$

It follows that

$$\lambda\int_{\mathbb{R}^N}|u|e^{-\Phi}dx\le\int_{\mathbb{R}^N}|f|e^{-\Phi}dx,$$

which means $\lambda\|u\|_{L^1(\mathbb{R}^N,\mu)} \leq \|\lambda u - Au\|_{L^1(\mathbb{R}^N,\mu)}$.

Next we show that $(I-A)C_c^\infty(\mathbb{R}^N)$ is dense in $L^1(\mathbb{R}^N,\mu)$. Let $g \in L^\infty(\mathbb{R}^N)$ be such that

$$\int_{\mathbb{R}^N} (u - Au) g e^{-\Phi} dx = 0, \qquad \forall\, u \in C_c^\infty(\mathbb{R}^N). \tag{5.2.17}$$

Since, in particular, $ge^{-\Phi} \in L^2_{\rm loc}(\mathbb{R}^N)$, from a classical result of local regularity for distributional solutions of elliptic equations (see [1] for $p=2$ and [2] for general p and also [5]) it follows that $ge^{-\Phi} \in H^1_{\rm loc}(\mathbb{R}^N)$, if λ is sufficiently large and, as a consequence, $g \in H^1_{\rm loc}(\mathbb{R}^N)$. This leads to

$$\int_{\mathbb{R}^N} u\, g\, e^{-\Phi} dx + \int_{\mathbb{R}^N} \langle Du, Dg\rangle e^{-\Phi} dx + \int_{\mathbb{R}^N} \langle G, Du\rangle g\, e^{-\Phi} dx = 0 \tag{5.2.18}$$

for every $u \in H^1(\mathbb{R}^N)$ with compact support. Indeed, if u is such a function, set $u_n = \varrho_n * u$, where ϱ_n is a standard sequence of mollifiers. Then $u_n \in C_c^\infty(\mathbb{R}^N)$ and u_n converges to u in $H^1(\mathbb{R}^N)$, as $n \to \infty$. Moreover, we can find $R > 0$ sufficiently large in such a way that $\operatorname{supp} u_n$ and $\operatorname{supp} u$ are contained in B_R, for every $n \in \mathbb{N}$. Now, each u_n satisfies (5.2.17), hence, integrating by parts, we have

$$\int_{\mathbb{R}^N} u_n\, g\, e^{-\Phi} dx + \int_{\mathbb{R}^N} \langle Du_n, Dg\rangle e^{-\Phi} dx + \int_{\mathbb{R}^N} \langle G, Du_n\rangle g\, e^{-\Phi} dx = 0$$

Letting $n \to \infty$, we obtain (5.2.18). Let η be in $C_c^\infty(\mathbb{R}^N)$ such that $\eta \equiv 1$ in B_1, $0 \leq \eta \leq 1$, $\eta \equiv 0$ in $\mathbb{R}^N \setminus B_2$ and set $\eta_n(x) = \eta(\frac{x}{n})$. Plugging $g\,\eta_n^2$ into (5.2.18) we find

$$\begin{aligned}\int_{\mathbb{R}^N} g^2\eta_n^2 e^{-\Phi} dx \;&+\; \int_{\mathbb{R}^N} \eta_n^2 |Dg|^2 e^{-\Phi} dx + 2\int_{\mathbb{R}^N} \langle D\eta_n, Dg\rangle \eta_n g\, e^{-\Phi} dx \\ &+\; \int_{\mathbb{R}^N} \langle G, Dg\rangle g\, \eta_n^2 e^{-\Phi} dx + 2\int_{\mathbb{R}^N} \langle G, D\eta_n\rangle g^2 \eta_n e^{-\Phi} dx = 0.\end{aligned} \tag{5.2.19}$$

Integrating by parts and recalling (5.2.16) it follows that

$$\int_{\mathbb{R}^N} \langle G, Dg\rangle g\, \eta_n^2 e^{-\Phi} dx = -\int_{\mathbb{R}^N} \langle G, D\eta_n\rangle g^2 \eta_n e^{-\Phi} dx,$$

therefore from (5.2.19) we deduce

$$\begin{aligned}\int_{\mathbb{R}^N} g^2\eta_n^2 e^{-\Phi} dx + \int_{\mathbb{R}^N} \eta_n^2 |Dg|^2 e^{-\Phi} dx \;&=\; -2\int_{\mathbb{R}^N} \langle D\eta_n, Dg\rangle \eta_n g\, e^{-\Phi} dx \\ &\quad - \int_{\mathbb{R}^N} \langle G, D\eta_n\rangle g^2 \eta_n e^{-\Phi} dx \\ &\leq\; \frac{2c}{n} \int_{\mathbb{R}^N} \eta_n |g|\, |Dg| e^{-\Phi} dx + \frac{c}{n}\int_{\mathbb{R}^N} |g|^2\, |G| e^{-\Phi} dx \\ &\leq\; \frac{c}{n}\int_{\mathbb{R}^N} \eta_n^2 |Dg|^2 e^{-\Phi} dx + \frac{c}{n}\, \|g\|_\infty^2 \int_{\mathbb{R}^N} e^{-\Phi} dx \\ &\quad + \frac{c}{n}\, \|g\|_\infty^2 \int_{\mathbb{R}^N} |G| e^{-\Phi} dx.\end{aligned}$$

For n large $1 - \frac{c}{n} > 0$, hence

$$\int_{\mathbb{R}^N} g^2\eta_n^2 e^{-\Phi} dx \leq \frac{c}{n}\|g\|_\infty^2 \int_{\mathbb{R}^N} e^{-\Phi} dx + \frac{c}{n}\|g\|_\infty^2 \int_{\mathbb{R}^N} |G| e^{-\Phi} dx.$$

Letting $n \to \infty$ and using monotone convergence, we find that $g = 0$, which implies that $I - A$ has dense range.

Since $C_c^\infty(\mathbb{R}^N)$ is dense in $L^1(\mathbb{R}^N,\mu)$, from the Lumer-Phillips Theorem (see e.g. [21, Theorem II.3.15]) Step1 follows. We observe that, by construction, $C_c^\infty(\mathbb{R}^N)$ is a core for B. Then μ is an invariant measure for the generated semigroup $(S(t))$, since, integrating by parts we have $\int_{\mathbb{R}^N} Au\,\mu(dx) = 0$, for all $u \in C_c^\infty(\mathbb{R}^N)$ (see Proposition 5.1.2).

Step2. The semigroups $(T(t))$ and $(S(t))$ coincide on $C_b(\mathbb{R}^N)$.

Let first $f \in C_c^\infty(\mathbb{R}^N)$, $f \geq 0$. By construction, $u(t,x) = T(t)f(x)$ is the limit of $u_\rho(t,x)$, as $\rho \to +\infty$, where u_ρ solves (5.2.4). Since f is positive, the classical maximum principle implies that the sequence (u_ρ) increases with ρ. Moreover, if $\mathrm{supp} f \subset B_R$, then $u_R \in C^{1,2}([0,T] \times \overline{B}_R)$. Integrating the equation $D_t u_R = Au_R$ on B_R with respect to μ and using (5.2.16), we find

$$\begin{aligned} D_t \int_{B_R} u_R(t,x)\mu(dx) &= \int_{B_R} Au_R(t,x)\mu(dx) = a\int_{B_R} \mathrm{div}(e^{-\Phi} Du_R)dx \\ &\quad -a\int_{B_R} \langle G, Du_R\rangle e^{-\Phi}dx \\ &= a\int_{\partial B_R} \frac{\partial u_R}{\partial \nu}(t,x)e^{-\Phi}\sigma(dx) + a\int_{B_R} \mathrm{div}G\, u_R\, e^{-\Phi}dx \\ &\quad -a\int_{B_R} \langle G, D\Phi\rangle u_R\, e^{-\Phi}dx - a\int_{\partial B_R} \langle G,\nu\rangle u_R\, e^{-\Phi}dx \\ &= a\int_{\partial B_R} \frac{\partial u_R}{\partial \nu}(t,x)e^{-\Phi}\sigma(dx) \end{aligned}$$

where σ is the surface measure on ∂B_R and ν the outward unit normal vector to B_R. Since $u_R \geq 0$ in B_R and $u_R = 0$ on ∂B_R, it follows that $\frac{\partial u_R}{\partial \nu}(t,x) \leq 0$, hence the map $t \to \int_{B_R} u_R(t,x)\mu(dx)$ is decreasing. This yields

$$\int_{B_R} u_R(t,x)\mu(dx) \leq \int_{B_R} f(x)\mu(dx), \qquad t > 0$$

and, by monotone convergence, $\|T(t)f\|_{L^1(\mathbb{R}^N,\mu)} \leq \|f\|_{L^1(\mathbb{R}^N,\mu)}$. If $f \in C_b(\mathbb{R}^N)$ and $f \geq 0$, let $f_n \in C_c^\infty(\mathbb{R}^N)$ be such that $f_n \geq 0$, $\|f_n\|_\infty \leq \|f\|_\infty$ and $f_n(x) \to f(x)$, for every $x \in \mathbb{R}^N$. Then $T(t)f_n(x) \to T(t)f(x)$ and the same estimate holds by dominated convergence. Finally, since $T(t)$ is positive, we have $|T(t)f| \leq T(t)|f|$, for every $f \in C_b(\mathbb{R}^N)$, hence $\|T(t)f\|_{L^1(\mathbb{R}^N,\mu)} \leq \|f\|_{L^1(\mathbb{R}^N,\mu)}$. It follows that $(T(t))$ can be extended to a strongly continuous semigroup of positive contractions on $L^1(\mathbb{R}^N,\mu)$, denoted by $(\widetilde{T}(t))$, with generator $(\widetilde{A}, D(\widetilde{A}))$.

Let $f \in C_c^\infty(\mathbb{R}^N)$. Then f belongs to $\widehat{D}$, where $\widehat{D}$ is the domain of A as weak generator of $(T(t))$. This means that $\sup_{t>0} \frac{\|T(t)f-f\|_\infty}{t}$ is finite and $\lim_{t\to 0} \frac{T(t)f(x)-f(x)}{t} = Af(x)$, for every $x \in \mathbb{R}^N$. By dominated convergence, the above equality is also true in $L^1(\mathbb{R}^N,\mu)$. Therefore $f \in D(\widetilde{A})$ and $\widetilde{A}f = Af = Bf$. Hence, $C_c^\infty(\mathbb{R}^N)$ is contained in $D(\widetilde{A})$ and $\widetilde{A}$ coincide with B on $C_c^\infty(\mathbb{R}^N)$. If $f \in D(B)$, since $C_c^\infty(\mathbb{R}^N)$ is a core of B, we can find a sequence (f_n) in $C_c^\infty(\mathbb{R}^N)$ such that $f_n \to f$ and $\widetilde{A}f_n = Bf_n \to Bf$ in $L^1(\mathbb{R}^N,\mu)$. Since $(\widetilde{A}, D(\widetilde{A})$ is closed in $L^1(\mathbb{R}^N,\mu)$ it turns out that $f \in D(\widetilde{A})$ and $\widetilde{A}f = Bf$. Thus we have established that $\widetilde{A}$ is an extension of B. Since they are both generators, they must coincide, hence $\widetilde{T}(t) = S(t)$ on $L^1(\mathbb{R}^N,\mu)$. In particular $T(t) = S(t)$ on $C_b(\mathbb{R}^N)$, as claimed. Concerning the last assertion, we observe that $T(t)\mathbf{1} = \mathbf{1}$, since $T(t)\mathbf{1} \leq \mathbf{1}$ and $\int_{\mathbb{R}^N}(T(t)\mathbf{1}-\mathbf{1})e^{-\Phi}dx = 0$. Proposition 5.2.7 concludes the proof. $\square$

Let us consider again A as in (5.2.1). Our next result shows that the invariant measure of $T(t)$, when exists, is absolutely continuous with respect to the Lebesgue measure $|\cdot|$. In this way, we extend the situation of Theorem 5.2.10 to the general case, even though it is not possible any more to know the density explicitly.

Proposition 5.2.11 *Assume that μ is the invariant measure of $T(t)$. Then μ is absolutely continuous with respect to $|\cdot|$ and its density $\varrho(x)$ is strictly positive $|\cdot|$ almost everywhere.*

PROOF. Since $(T(t))$ is regular (see Propositions 5.2.2 and 5.1.9) all the probability measures $p(t,x,\cdot)$ are equivalent. Moreover, μ is equivalent to $p(t,x,\cdot)$ for all $t > 0$ and $x \in \mathbb{R}^N$ (see Theorem 5.1.8). Since $p(t,x,\cdot)$ is absolutely continuous with respect to the Lebesgue measure $|\cdot|$ (see (5.2.5)), it follows that μ is absolutely continuous with respect to $|\cdot|$, too. Let $\varrho \in L^1(\mathbb{R}^N)$ be its density. It is clear that $\varrho \geq 0$. We prove that ϱ is strictly positive $|\cdot|$-a.e. If Γ is a Borel set such that $|\Gamma| > 0$, then $\int_\Gamma \varrho(x)dx = \mu(\Gamma) = P_t\chi_\Gamma = \int_\Gamma p(t,x,y)dy > 0$ since p is positive. Since Γ was arbitrary the thesis follows. □

Remark 5.2.12 As a consequence of the above proposition, we have that if an invariant measure of $(T(t))$ exists, then $T(t)\mathbf{1} = \mathbf{1}$ and therefore $T(t)$ is generated by $(A, D_{\max}(A))$ (see Propositions 5.2.3 and 5.2.7). Indeed, one has $T(t)\mathbf{1} \leq \mathbf{1}$ and $\int_{\mathbb{R}^N}(T(t)\mathbf{1} - \mathbf{1})\varrho(x)dx = 0$, with $\varrho(x) > 0$ $|\cdot|$-a.e. from Proposition 5.2.11.

Moreover, recalling Proposition 5.1.2, we have that $(T(t))$ extends to a strongly continuous semigroup in $L^p(\mathbb{R}^N,\mu)$, for every $1 \leq p < \infty$. Here we have more information, since we can identify the generator $(A_p, D(A_p))$, relating it to the original operator A.

Proposition 5.2.13 *Assume that μ is an invariant measure of $(T(t))$. Then $D_{\max}(A)$ is a core of $(A_p, D(A_p))$ in $L^p(\mathbb{R}^N,\mu)$, hence $(A_p, D(A_p))$ is the closure of $(A, D_{\max}(A))$ in $L^p(\mathbb{R}^N,\mu)$.*

PROOF. We continue to denote by $(T(t))$ the extended semigroup in $L^p(\mathbb{R}^N,\mu)$. In order to prove that $D_{\max}(A)$ is a core of $(A_p, D(A_p))$, it is sufficient to show that

(i) $D_{\max}(A) \subset D(A_p)$;

(ii) $D_{\max}(A)$ is dense in $L^p(\mathbb{R}^N,\mu)$;

(iii) $D_{\max}(A)$ is invariant under the semigroup.

Let $f \in D_{\max}(A)$. Then $\sup_{t>0} \frac{\|T(t)f-f\|_\infty}{t}$ is finite and $\lim_{t\to 0} \frac{T(t)f(x)-f(x)}{t} = Af(x)$, for every $x \in \mathbb{R}^N$. By dominated convergence, we have easily that

$$\left\| \frac{T(t)f - f}{t} - Af \right\|_{L^p(\mathbb{R}^N,\mu)} \to 0, \quad \text{as } t \to 0.$$

Therefore $f \in D(A_p)$ and $A_p f = Af$. Concerning *(ii)*, we show that $C_c^\infty(\mathbb{R}^N)$, which is contained in $D_{\max}(A)$, is dense in $L^p(\mathbb{R}^N,\mu)$. Let first $u \in C_c(\mathbb{R}^N)$. If (ϱ_n) is a standard sequence of mollifiers, then $\varrho_n * u \in C_c^\infty(\mathbb{R}^N)$ converges uniformly to u as $n \to \infty$. Since

$$\int_{\mathbb{R}^N} |\varrho_n * u(x) - u(x)|^p \varrho(x)dx \leq \|\varrho_n * u - u\|_\infty^p,$$

it follows that $\varrho_n * u$ converges to u in $L^p(\mathbb{R}^N,\mu)$, too. This proves that $C_c^\infty(\mathbb{R}^N)$ is dense in $C_c(\mathbb{R}^N)$ with respect to the norm of $L^p(\mathbb{R}^N,\mu)$. Since $C_c(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N,\mu)$ (see [51, Theorem III.3.14]), assertion *(ii)* follows.

Finally, taking into account the fact that $(A, D_{\max}(A))$ generates $(T(t))$ in $C_b(\mathbb{R}^N)$, *(iii)* is clear. At this point, [21, Proposition II.1.7] leads to the conclusion. □

5.3 Characterization of the domain of a class of elliptic operators in $L^p(\mathbb{R}^N,\mu)$

The aim of the present section is to study the following class of operators

$$B = \mathrm{div}(qD) - \langle qD\Phi, D\rangle + \langle G, D\rangle$$

in the space $L^p(\mathbb{R}^N,\mu)$, $1<p<\infty$, where $d\mu = e^{-\Phi}dx$. In particular, our purpose is to provide an explicit description of the domain under which B generates a strongly continuous semigroup in $L^p(\mathbb{R}^N,\mu)$. Our main tools are the results of Chapter 1, where the same problem has been studied for differential operators in $L^p(\mathbb{R}^N)$. In fact, via the transformation $v = e^{-\frac{\Phi}{p}}u$, the operator B on $L^p(\mathbb{R}^N,\mu)$ is similar to an operator A of the form (1.0.1) in the unweighted space $L^p(\mathbb{R}^N)$. Suitable assumptions on the coefficients q, Φ, G allow to apply the generation results of Chapter 1 to the transformed operator so that, via the inverse transformation, we can deduce that B, endowed with the domain

$$\mathcal{D}_\mu = \{u \in W^{2,p}(\mathbb{R}^N,\mu) \mid \langle G, Du\rangle \in L^p(\mathbb{R}^N,\mu)\} \tag{5.3.1}$$

generates a strongly continuous semigroup $(T(t))$ on $L^p(\mathbb{R}^N,\mu)$. We note that, in particular, the measure μ can be the invariant measure of $(T(t))$. This is the case if an additional condition is satisfied (see (A4') below). By $W^{k,p}(\mathbb{R}^N,\mu)$ we mean the weighted Sobolev space

$$W^{k,p}(\mathbb{R}^N,\mu) = \left\{u \in W^{k,p}_{\mathrm{loc}}(\mathbb{R}^N) \mid D^\alpha u \in L^p(\mathbb{R}^N,\mu),\ |\alpha| \le k\right\}.$$

In order to prove that $(B,\mathcal{D}_\mu)$ is a generator, we make the following assumptions on the coefficients:

(A1) $q = (q_{ij})$ is a symmetric matrix, with $q_{ij} \in C^1_b(\mathbb{R}^N)$ and there exists $\nu > 0$ such that $\langle q\xi,\xi\rangle = \sum_{i,j=1}^N q_{ij}(x)\xi_i\xi_j \ge \nu|\xi|^2$, for all $x, \xi \in \mathbb{R}^N$,

(A2) $\Phi \in C^2(\mathbb{R}^N)$, $G \in C^1(\mathbb{R}^N;\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} e^{-\Phi(x)}dx < \infty$,

(A3) for all $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that $|G| + |DG| + |D^2\Phi|^2 \le \varepsilon|D\Phi|^2 + c_\varepsilon$,

(A4) $|\mathrm{div}G - \langle G, D\Phi\rangle| \le \varepsilon|D\Phi|^2 + c_\varepsilon$,

(A4') $\mathrm{div}\, G = \langle G, D\Phi\rangle$.

Since $|\mathrm{div}G| \le \sqrt{N}|DG|$ and (A3) holds, (A4) actually says that $|\langle G, D\Phi\rangle| \le \varepsilon|D\Phi|^2 + c_\varepsilon$. Here and in the sequel, c_ε denotes a nonnegative constant which may go to infinity when ε goes to zero. It may change from line to line, but this is irrelevant to our interests.

We observe that the condition on Φ included in (A3) is satisfied by any polynomial whose homogeneous part of maximal degree is positive definite. However, it fails in $\mathbb{R}^2$ for the function x^2y^2. Moreover, we note that it implies the weaker condition $|D^2\Phi| \le \varepsilon|D\Phi|^2 + c_\varepsilon$, which is assumed in [41] together with a more restrictive assumption on G. If $q_{ij} = \delta_{ij}$ and $\Phi = |x|^2/2$ then we obtain the Ornstein Uhlenbeck operator perturbed with a non symmetric drift G:

$$\Delta - \langle x, D\rangle + \langle G, D\rangle.$$

If G is such that $\langle G(x), x\rangle = 0$ for every $x \in \mathbb{R}^N$, then assumption (A3) is verified if

$$|G(x)| + |DG(x)| \le \varepsilon|x|^2 + c_\varepsilon,$$

i.e. if G and its derivatives grow a little bit less than quadratically. Since $\langle G(x), x\rangle = 0$, this implies automatically (A4). For example, in $\mathbb{R}^2$ one can consider $G(x_1,x_2) = (-x_2, x_1)\times h(x_1,x_2)$, where $h \in C^1(\mathbb{R}^2)$. Since $|G| = |x|\,|h|$, and $|DG|^2 = |x|^2|Dh|^2 + 2h^2 + 2h\langle x, Dh\rangle$, the function h has to satisfy the condition $|h(x)| \le \varepsilon|x| + c_\varepsilon$, for every $\varepsilon > 0$. Then a possible choice is $h(x) = (|x|^2+1)^{\alpha/2}$, with $0 < \alpha < 1$. This situation is excluded in [41].

Replacing (A4) with (A4') we obtain that μ is the invariant measure for the generated semigroup, as we will see in Proposition 5.3.4.

We first need some technical lemmas. These results are completely similar to those of [41] and we give the proof for the sake of completeness. It is useful to observe that one can easily check, as in Lemma 1.3.1, that $C_c^\infty(\mathbb{R}^N)$ is dense in $W^{k,p}(\mathbb{R}^N,\mu)$.

Lemma 5.3.1 *Let $1 < p < \infty$ and assume that $\Phi \in C^2(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} e^{-\Phi(x)}dx < \infty$. If for some $\varepsilon < 1$ there exists $c_\varepsilon > 0$ such that*

$$\Delta\Phi + (p-2)(1+|D\Phi|^2)^{-1}\langle D^2\Phi D\Phi, D\Phi\rangle \le \varepsilon|D\Phi|^2 + c_\varepsilon \tag{5.3.2}$$

then the map $u \to u|D\Phi|$ is bounded from $W^{1,p}(\mathbb{R}^N,\mu)$ to $L^p(\mathbb{R}^N,\mu)$ and the map $u \to |Du|\,|D\Phi|$ is bounded from $W^{2,p}(\mathbb{R}^N,\mu)$ to $L^p(\mathbb{R}^N,\mu)$. Therefore, the operator B is bounded from $\mathcal{D}_\mu$ in $L^p(\mathbb{R}^N,\mu)$.

Proof. Let $1 < p < \infty$ be fixed. Since $C_c^\infty(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N,\mu)$, it is sufficient to prove that

$$\|u|D\Phi|\,\|_{L^p(\mathbb{R}^N,\mu)} \le c(\|u\|_{L^p(\mathbb{R}^N,\mu)} + \|Du\|_{L^p(\mathbb{R}^N,\mu)}),$$

for $u \in C_c^\infty(\mathbb{R}^N)$ and for some constant $c > 0$. Since $t^p \le a(1+t^2)^{\frac{p}{2}-1}t^2 + b$ for all $t \ge 0$ and for some suitable constants $a, b > 0$, we have only to estimate $\int_{\mathbb{R}^N}(1+|D\Phi|^2)^{\frac{p}{2}-1}|D\Phi|^2|u|^p e^{-\Phi}dx$. Integrating by parts and using (5.3.2) we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N}(1+|D\Phi|^2)^{\frac{p}{2}-1}|D\Phi|^2|u|^p e^{-\Phi}dx = -\int_{\mathbb{R}^N}(1+|D\Phi|^2)^{\frac{p}{2}-1}\langle D\Phi, De^{-\Phi}\rangle|u|^p dx = \\
&(p-2)\int_{\mathbb{R}^N}|u|^p(1+|D\Phi|^2)^{\frac{p}{2}-2}\langle D^2\Phi D\Phi, D\Phi\rangle e^{-\Phi}dx + \int_{\mathbb{R}^N}(1+|D\Phi|^2)^{\frac{p}{2}-1}\Delta\Phi|u|^p e^{-\Phi}dx \\
&\quad + p\int_{\mathbb{R}^N}|u|^{p-2}u(1+|D\Phi|^2)^{\frac{p}{2}-1}\langle D\Phi, Du\rangle e^{-\Phi}dx \le \varepsilon\int_{\mathbb{R}^N}(1+|D\Phi|^2)^{\frac{p}{2}-1}|D\Phi|^2|u|^p\,d\mu + \\
&\quad c_\varepsilon\int_{\mathbb{R}^N}(1+|D\Phi|^2)^{\frac{p}{2}-1}|u|^p\,d\mu + p\int_{\mathbb{R}^N}|u|^{p-1}(1+|D\Phi|^2)^{\frac{p}{2}-1}|D\Phi|\,|Du|d\mu.
\end{aligned}$$

Applying the inequality $(1+t^2)^{\frac{p}{2}-1} \le \eta(1+t^2)^{\frac{p}{2}-1}t^2 + c_\eta$, which holds for all $\eta > 0$, we deduce

$$\begin{aligned}
(1-\varepsilon)\int_{\mathbb{R}^N}(1+|D\Phi|^2)^{\frac{p}{2}-1}|D\Phi|^2|u|^p e^{-\Phi}dx &\le c_\varepsilon\,\eta\int_{\mathbb{R}^N}(1+|D\Phi|^2)^{\frac{p}{2}-1}|D\Phi|^2|u|^p d\mu + \\
&\quad c_\varepsilon\,c_\eta\int_{\mathbb{R}^N}|u|^p d\mu + p\int_{\mathbb{R}^N}\left(|u|^{p-1}(1+|D\Phi|^2)^{\frac{p}{2}-1}|D\Phi|\right)|Du|d\mu.
\end{aligned} \tag{5.3.3}$$

Choosing $\eta = \frac{1-\varepsilon}{2c_\varepsilon}$ and using Young's inequality to estimate the last term in (5.3.3), we find that for all $\delta > 0$

$$\begin{aligned}
\frac{1-\varepsilon}{2}\int_{\mathbb{R}^N}(1+|D\Phi|^2)^{\frac{p}{2}-1}|D\Phi|^2|u|^p e^{-\Phi}dx &\le c_\varepsilon\,c_\eta\int_{\mathbb{R}^N}|u|^p d\mu \\
&\quad + \delta\int_{\mathbb{R}^N}|u|^p(1+|D\Phi|^2)^{\left(\frac{p}{2}-1\right)p'}|D\Phi|^{p'}d\mu + c_\delta\int_{\mathbb{R}^N}|Du|^p d\mu,
\end{aligned}$$

where p' is the conjugate exponent of p. Now, the inequality $(1+t^2)^{\left(\frac{p}{2}-1\right)p'}t^{p'} \le k_1(1+t^2)^{\frac{p}{2}-1}t^2 + k_2$, which holds for certain constants $k_1, k_2 > 0$, and a suitable choice of δ conclude the proof. □

Lemma 5.3.2 *Let $1 < p < \infty$ and assume that $\Phi \in C^2(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} e^{-\Phi(x)}dx < \infty$ and such that for all $\varepsilon > 0$ there exists c_ε with the following property*

$$|D^2\Phi| \le \varepsilon|D\Phi|^2 + c_\varepsilon. \tag{5.3.4}$$

Then the map $u \to u\,|D\Phi|^2$ is bounded from $W^{2,p}(\mathbb{R}^N,\mu)$ to $L^p(\mathbb{R}^N,\mu)$.

Proof. Let $u \in C_c^\infty(\mathbb{R}^N)$. Then the vector function $uD\Phi \in W^{1,p}(\mathbb{R}^N)$ and from Lemma 5.3.1 it follows that

$$\begin{aligned}
\|u|D\Phi|^2\|_{L^p(\mathbb{R}^N,\mu)} &\le C\|uD\Phi\|_{W^{1,p}(\mathbb{R}^N,\mu)} \\
&\le C(\|uD\Phi\|_{L^p(\mathbb{R}^N,\mu)} + \|\langle Du, D\Phi\rangle\|_{L^p(\mathbb{R}^N,\mu)} + \|uD^2\Phi\|_{L^p(\mathbb{R}^N,\mu)}).
\end{aligned}$$

Using again Lemma 5.3.1 and applying (5.3.4) we have

$$\|u|D\Phi|^2\|_{L^p(\mathbb{R}^N,\mu)} \le C'(\|u\|_{W^{1,p}(\mathbb{R}^N,\mu)} + \|u\|_{W^{2,p}(\mathbb{R}^N,\mu)} + \varepsilon\|u|D\Phi|^2\|_{L^p(\mathbb{R}^N,\mu)} + c_\varepsilon\|u\|_{L^p(\mathbb{R}^N,\mu)}).$$

Choosing ε sufficiently small we get the statement for $u \in C_c^\infty(\mathbb{R}^N)$. The general case follows by density. □

We observe that under assumptions (A2) and (A3) Lemmas 5.3.1 and 5.3.2 hold.

Now, we are ready to prove the main result of this section. It is useful to introduce the quantities

$$L = \sup_{x\in\mathbb{R}^N}\Big(\sum_{i,j=1}^N |Dq_{ij}(x)|^2\Big)^{\frac{1}{2}} \tag{5.3.5}$$

$$M = \sup_{x\in\mathbb{R}^N}\max_{|\xi|=1}\langle q\xi,\xi\rangle = \sup_{\mathbb{R}^N}\Big(\sum_{i,j=1}^N (q_{ij}(x))^2\Big)^{\frac{1}{2}}$$

Theorem 5.3.3 *Let $1 < p < \infty$ and assume that hypotheses (A1), (A2), (A3) and (A4) are satisfied. Then the operator $(B,\mathcal{D}_\mu)$ generates a positive strongly continuous semigroup $(T(t))$ in $L^p(\mathbb{R}^N,\mu)$.*

Proof. Fix $p \in (1,\infty)$. As pointed out at the beginning of the section, we introduce a transformation in order to deal with an operator in the unweighted space $L^p(\mathbb{R}^N)$. Let us define the isometry

$$J:\quad L^p(\mathbb{R}^N,\mu) \longrightarrow L^p(\mathbb{R}^N)$$
$$u \longmapsto Ju = e^{-\frac{\Phi}{p}}u.$$

A straightforward computation shows that $Bu = J^{-1}\widetilde{B}\,Ju$, for $u \in C_c^\infty(\mathbb{R}^N)$, where

$$\widetilde{B} = \operatorname{div}(qD) + \langle F, D\rangle - V$$

with

$$F = \left(\frac{2}{p}-1\right)qD\Phi + G,$$
$$V = \frac{1}{p}\left[\left(1-\frac{1}{p}\right)\langle qD\Phi, D\Phi\rangle - \operatorname{Tr}(qD^2\Phi) - \langle G, D\Phi\rangle - \sum_{i,j=1}^N D_iq_{ij}D_j\Phi\right].$$

The proof is structured as follows. Setting $U = \frac{1}{p}\left(1-\frac{1}{p}\right)\langle qD\Phi, D\Phi\rangle$, we first prove that

Step1 $A = \operatorname{div}(qD) + \langle F, D\rangle - U$, endowed with the domain

$$\mathcal{D}_p = \{u \in W^{2,p}(\mathbb{R}^N) \mid \langle F, Du\rangle,\ Uu \in L^p(\mathbb{R}^N)\}, \tag{5.3.6}$$

generates a positive strongly continuous semigroup in $L^p(\mathbb{R}^N)$.

Then we deduce that

Step2 $(\widetilde{B},\mathcal{D}_p)$ generates a positive C_0 semigroup in $L^p(\mathbb{R}^N)$.

Finally, we show that

Step3 $(B,\mathcal{D}_\mu)$ generates a positive C_0 semigroup in $L^p(\mathbb{R}^N,\mu)$.

Proof of Step1. We want to show that under assumptions (A1)-(A4) the coefficients of A satisfy the hypotheses of Theorem 1.1.2 with $\sigma = 1$ and $\mu = 0$. More precisely, we claim that there exist a constant $\alpha > 0$, sufficiently small constants $\beta, \theta > 0$, and constants $c_\alpha, c_\beta, c_\theta \geq 0$ such that

(i) $|DU| \leq \alpha U + c_\alpha$,

(ii) $|DF| \leq \beta U + c_\beta$,

(iii) $|F| \leq \theta U + c_\theta$.

As far as (i) is concerned, we have

$$\begin{aligned}
|D_k U| &= \left|\frac{1}{p}\left(1-\frac{1}{p}\right)\sum_{i,j=1}^N D_k q_{ij} D_i\Phi D_j\Phi + \frac{2}{p}\left(1-\frac{1}{p}\right)\sum_{i,j=1}^N q_{ij} D_{ik}\Phi D_j\Phi\right| \\
&\leq \frac{1}{p}\left(1-\frac{1}{p}\right)|D\Phi|^2 \sup_{\mathbb{R}^N}\left(\sum_{i,j=1}^N |D_k q_{ij}|^2\right)^{\frac{1}{2}} \\
&\quad + \frac{2}{p}\left(1-\frac{1}{p}\right)|D\Phi|\left(\sum_{i=1}^N |D_{ik}\Phi|^2\right)^{\frac{1}{2}} \sup_{\mathbb{R}^N}\left(\sum_{i,j=1}^N |q_{ij}|^2\right)^{\frac{1}{2}} \\
&\leq \frac{1}{p}\left(1-\frac{1}{p}\right)L|D\Phi|^2 + \frac{2}{p}\left(1-\frac{1}{p}\right)M|D^2\Phi||D\Phi|,
\end{aligned}$$

where L and M are given in (5.3.5). From (A1) and (A3) and applying the inequality $t \leq \eta\, t^2 + c_\eta$, which holds for every $\eta > 0$, it follows that

$$\begin{aligned}
|D_k U| &\leq \frac{1}{p\,\nu}\left(1-\frac{1}{p}\right)L\langle qD\Phi, D\Phi\rangle + \frac{2}{p}\left(1-\frac{1}{p}\right)M(\varepsilon|D\Phi|^2 + c_\varepsilon|D\Phi|) \\
&\leq \frac{1}{p\,\nu}\left(1-\frac{1}{p}\right)L\langle qD\Phi, D\Phi\rangle + \frac{2}{p\,\nu}\left(1-\frac{1}{p}\right)M\varepsilon\langle qD\Phi, D\Phi\rangle \\
&\quad + \frac{2}{p\,\nu}\left(1-\frac{1}{p}\right)Mc_\varepsilon\eta\langle qD\Phi, D\Phi\rangle + \frac{2}{p}\left(1-\frac{1}{p}\right)Mc_\varepsilon c_\eta \\
&= \alpha U + c_\alpha
\end{aligned}$$

where $\alpha = \frac{L+2M(\varepsilon+c_\varepsilon\eta)}{\nu}$ and $c_\alpha = \frac{2}{p}\left(1-\frac{1}{p}\right)Mc_\varepsilon c_\eta$, for arbitrary $\varepsilon, \eta > 0$. This leads to (i). Now, similar computations yield

$$\begin{aligned}
|DF| &\leq \sqrt{3}\left(\left|\frac{2}{p}-1\right|L|D\Phi| + \left|\frac{2}{p}-1\right|M|D^2\Phi| + |DG|\right) \\
&\leq \sqrt{3}\left(\left|\frac{2}{p}-1\right|\varepsilon(L+M+1)|D\Phi|^2 + c_\varepsilon\right),
\end{aligned}$$

where c_ε depends on ε, p, L, M. Therefore $|DF| \leq \beta U + c_\beta$, with $\beta = O(\varepsilon)$ and $c_\beta > 0$ depending on ε, p, M, L. Finally, condition (iii) follows easily from (A3). Indeed, one has

$$\begin{aligned}
|F| &\leq \sqrt{2}\left(\left|\frac{2}{p}-1\right|M|D\Phi| + |G|\right) \\
&\leq \sqrt{2}\left(\left|\frac{2}{p}-1\right|M\varepsilon|D\Phi|^2 + \left|\frac{2}{p}-1\right|Mc_\varepsilon + \varepsilon|D\Phi|^2 + c_\varepsilon\right) \\
&= \theta U + c_\theta
\end{aligned}$$

with $\theta = O(\varepsilon)$ and c_θ depending on ε, M, p. At this point, assumptions (H1'), (H2'), (H4') and (H5) of Theorem 1.1.2 are satisfied with $\sigma = 1$ and $\mu = 0$. The smallness condition (1.1.7) is guaranteed by a suitable choice of ε and η. Note that the product $\alpha\,\theta$, and not α itself, has to be small. Then Theorem 1.1.2 applies and we find that $(A, \mathcal{D}_p)$ generates a positive, strongly continuous semigroup in $L^p(\mathbb{R}^N)$, with $\mathcal{D}_p$ given by (5.3.6). This concludes the proof of Step1.

Proof of Step2. Let us prove that $\mathrm{div}F + p(V+\lambda_0) \geq 0$, for a suitable $\lambda_0 > 0$. From assumption (A3) we infer

$$\begin{aligned}
\mathrm{div}F + pV \;&= 2\left(\frac{1}{p}-1\right)\sum_{j,k=1}^{N} D_k q_{jk} D_j\Phi + 2\left(\frac{1}{p}-1\right)\mathrm{Tr}(qD^2\Phi) + \left(1-\frac{1}{p}\right)\langle qD\Phi, D\Phi\rangle \\
&\quad + \mathrm{div}G - \langle G, D\Phi\rangle \\
&\geq 2\left(\frac{1}{p}-1\right)\sqrt{N}L|D\Phi| + 2\left(\frac{1}{p}-1\right)M|D^2\Phi| + \left(1-\frac{1}{p}\right)\nu|D\Phi|^2 \\
&\quad - \varepsilon|D\Phi|^2 - c_\varepsilon \\
&\geq \left[\left(2\left(\frac{1}{p}-1\right)(\sqrt{N}L+M)-1\right)\varepsilon + \left(1-\frac{1}{p}\right)\nu\right]|D\Phi|^2 \\
&\quad + 2\left(\frac{1}{p}-1\right)(\sqrt{N}L+M)c_\varepsilon - c_\varepsilon.
\end{aligned}$$

Choosing $\varepsilon > 0$ sufficiently small, we obtain $\mathrm{div}F + pV \geq -p\,\lambda_0$, where $\lambda_0 > 0$ depends on p, ν, L, M. Under this condition, the operator $(\widetilde{B}, \mathcal{D}_p)$ is quasi-dissipative in $L^p(\mathbb{R}^N)$ (see Lemma 1.3.2 and Remark 1.3.4). Moreover, we observe that, setting $W = V - U$, by (A4), (A3) and (A1) respectively, we have

$$\begin{aligned}
|W| &\leq \frac{1}{p}M|D^2\Phi| + \frac{1}{p}|\langle G, D\Phi\rangle| + \frac{\sqrt{N}L}{p}|D\Phi| \\
&\leq \frac{1}{p}(M+1)\varepsilon|D\Phi|^2 + \frac{1}{p}(M+1)c_\varepsilon + \frac{\sqrt{N}}{p}|DG| + \frac{\sqrt{N}L}{p}\varepsilon|D\Phi|^2 + \frac{\sqrt{N}L}{p}c_\varepsilon \\
&\leq \frac{\varepsilon}{p}\left(M+\sqrt{N}+\sqrt{N}L\right)|D\Phi|^2 + \frac{1}{p}\left(M+\sqrt{N}+\sqrt{N}L\right)c_\varepsilon,
\end{aligned}$$

which means

$$|W| \leq \eta U + c_\eta, \tag{5.3.7}$$

for all $\eta > 0$. Then, if $u \in \mathcal{D}_p$ one deduces

$$\|Wu\|_p \leq 2^{1-\frac{1}{p}}(\eta\|Uu\|_p + c_\eta\|u\|_p)$$

and applying estimate (1.3.10) to the operator A we obtain

$$\|Wu\|_p \leq 2^{1-\frac{1}{p}}(\eta c\|Au\|_p + \eta c\|u\|_p + c_\eta\|u\|_p) = \delta\|Au\|_p + c_\delta\|u\|_p \tag{5.3.8}$$

with $\delta > 0$ arbitrarily small. Now, if $\lambda > 0$ is large enough, then $\lambda \in \rho(A)$, since A is the generator of a strongly continuous semigroup. This means that $\lambda - A : \mathcal{D}_p \to L^p(\mathbb{R}^N)$ is invertible, therefore we may write

$$\lambda - \widetilde{B} = \lambda - A + W = [I + WR(\lambda, A)](\lambda - A).$$

It follows that $\lambda - \widetilde{B}$ is invertible on $\mathcal{D}_p$ if and only if $I + WR(\lambda, A)$ is invertible on $L^p(\mathbb{R}^N)$. This is the case if $\|WR(\lambda, A)\| < 1$. Let $f \in L^p(\mathbb{R}^N)$. Applying (5.3.8) with $u = R(\lambda, A)f$ and considering the fact that $(A, \mathcal{D}_p)$ is quasi-dissipative, (see Lemma 1.6.1), we deduce

$$\begin{aligned}
\|WR(\lambda, A)f\|_p &\leq \delta\|AR(\lambda, A)f\|_p + c_\delta\|R(\lambda, A)f\|_p \\
&\leq \delta\lambda\|R(\lambda, A)f\|_p + \delta\|f\|_p + c_\delta\|R(\lambda, A)f\|_p \\
&\leq \left(\frac{\delta\,\lambda}{\lambda - \lambda_p} + \delta + \frac{c_\delta}{\lambda - \lambda_p}\right)\|f\|_p,
\end{aligned}$$

if $\lambda > \lambda_p$, for a suitable λ_p. Choose $\delta < \frac{1}{6}$. Then $\delta\left(1+\frac{\lambda}{\lambda-\lambda_p}\right) < \frac{1}{2}$ for all $\lambda \geq 2\lambda_p$. Let $\lambda \geq 2\lambda_p$ be such that $\lambda > \lambda_p + 2c_\delta$. This implies that $\|WR(\lambda, A)f\|_p \leq a\|f\|_p$, with $a < 1$. Thus, we have established that if λ is large enough, then $\lambda - \widetilde{B}$ is invertible on $\mathcal{D}_p$. This implies also that $(B, \mathcal{D}_p)$ is closed and Step2 follows from the Hille Yosida Theorem [21].

Proof of Step3. As a consequence of Step2, $B = J^{-1}\widetilde{B}J$ with domain $D(B) = \{u \in L^p(\mathbb{R}^N, \mu) \mid Ju \in \mathcal{D}_p\}$ generates a positive C_0-semigroup $(T(t))$ in $L^p(\mathbb{R}^N, \mu)$. We have to show that $D(B) = \mathcal{D}_\mu$.

Let $u \in D(B)$. Then $v = Ju \in \mathcal{D}_p$, so in particular $v \in W^{2,p}(\mathbb{R}^N)$ and $Uv \in L^p(\mathbb{R}^N)$. Since $|D\Phi|^2 \leq \frac{p^2}{(p-1)\nu}U$, we have that $|D\Phi|^2 v \in L^p(\mathbb{R}^N)$. Therefore

$$e^{-\frac{\Phi}{p}} D_j u = \frac{1}{p} v D_j \Phi + D_j v \in L^p(\mathbb{R}^N),$$

since $|D\Phi| \leq |D\Phi|^2 + 1$. Moreover, (A3) and the estimate

$$\|U^{\frac{1}{2}} Dv\|_p \leq K(\|\Delta v\|_p + \|Uv\|_p)$$

(see [41, Proposition 2.3]) yield

$$e^{-\frac{\Phi}{p}} D_{ij} u = \frac{1}{p} v D_{ij}\Phi + D_{ij} v + \frac{1}{p} D_j v D_i \Phi + \frac{1}{p} D_i v D_j \Phi + \frac{1}{p^2} v D_i \Phi D_j \Phi \in L^p(\mathbb{R}^N),$$

i.e. $u \in W^{2,p}(\mathbb{R}^N, \mu)$. Recalling (5.3.7), we have

$$|V| \leq (\eta + 1)U + c_\eta, \tag{5.3.9}$$

hence $Vv \in L^p(\mathbb{R}^N)$. Since $v \in \mathcal{D}_p$, we have that $v \in W^{2,p}(\mathbb{R}^N)$ and $\langle F, Dv\rangle \in L^p(\mathbb{R}^N)$, then $\widetilde{B}v \in L^p(\mathbb{R}^N)$, which implies that $Bu = J^{-1}\widetilde{B}v \in L^p(\mathbb{R}^N, \mu)$. From Lemma 5.3.1 and the fact that $q_{ij} \in C^1_b(\mathbb{R}^N)$ it follows that $\operatorname{div}(qDu)$, $\langle qD\Phi, Du\rangle \in L^p(\mathbb{R}^N, \mu)$. By difference, $\langle G, Du\rangle \in L^p(\mathbb{R}^N, \mu)$ and then $u \in \mathcal{D}_\mu$.

Conversely, let $u \in \mathcal{D}_\mu$ and set $v = Ju$. Then, by Lemma 5.3.1

$$D_j v = e^{-\frac{\Phi}{p}}\left(-\frac{1}{p} u D_j \Phi + D_j u\right) \in L^p(\mathbb{R}^N).$$

Now, Lemma 5.3.2 implies that $|D\Phi|^2 v \in L^p(\mathbb{R}^N)$. Then, since $U \leq \frac{1}{p}\left(1 - \frac{1}{p}\right) M |D\Phi|^2$ and (5.3.9) holds, we obtain that $Uv, Vv \in L^p(\mathbb{R}^N)$. Using again Lemma 5.3.1 and (A3) we get

$$D_{ij} v = e^{-\frac{\Phi}{p}}\left(D_{ij} u - \frac{1}{p} u D_{ij}\Phi + \frac{1}{p^2} u D_i \Phi D_j \Phi - \frac{1}{p} D_j u D_i \Phi - \frac{1}{p} D_i u D_j \Phi\right) \in L^p(\mathbb{R}^N).$$

Therefore $v \in W^{2,p}(\mathbb{R}^N)$. Since $Bu \in L^p(\mathbb{R}^N, \mu)$, we have that $\widetilde{B}v = JBu \in L^p(\mathbb{R}^N)$. By difference, it follows that $\langle F, Dv\rangle \in L^p(\mathbb{R}^N)$. Therefore $u \in D(B)$ and we have proved that $D(B) = \mathcal{D}_\mu$. This concludes the proof. $\square$

In the proposition below, we show that assuming (A4') instead of (A4), the measure μ turns out to be the invariant measure of the semigroup yielded by Theorem 5.3.3.

Proposition 5.3.4 *Assume that (A1), (A2), (A3), (A4') hold. Then μ is, up to a multiplicative constant, the unique invariant measure of the semigroup $(T(t))$ generated by $(B, \mathcal{D}_\mu)$.*

Proof. We claim that $C^2_c(\mathbb{R}^N)$ is a core of B. Recalling the notation introduced in the proof of Theorem 5.3.3, from Lemma 1.3.1 it follows that $C^\infty_c(\mathbb{R}^N)$ is a core of $\widetilde{B}$. This easily implies that $J^{-1}(C_c^\infty(\mathbb{R}^N))$ is a core of B. Indeed, take $u \in \mathcal{D}_\mu$ and consider $v = Ju \in \mathcal{D}_p$. Let (v_n) be a sequence in $C_c^\infty(\mathbb{R}^N)$ such that $v_n \to v$ and $\widetilde{B}v_n \to \widetilde{B}v$ in $L^p(\mathbb{R}^N)$. Set $u_n = J^{-1}v_n$. Then $u_n \in J^{-1}(C_c^\infty(\mathbb{R}^N))$ and $u_n \to u$, $Bu_n = J^{-1}\widetilde{B}v_n \to J^{-1}\widetilde{B}v = Bu$ in $L^p(\mathbb{R}^N, \mu)$. Now, since $J^{-1}(C_c^\infty(\mathbb{R}^N)) \subset C_c^2(\mathbb{R}^N) \subset \mathcal{D}_\mu$ the statement follows. Therefore, in order to show that μ is an invariant measure of $(T(t))$, it is sufficient to prove that $\int_{\mathbb{R}^N} Bu\, d\mu = 0$, for all $u \in C_c^2(\mathbb{R}^N)$ (see Proposition 5.1.2). This follows easily integrating by parts and taking condition (A4') into account. Indeed,

$$
\begin{aligned}
\int_{\mathbb{R}^N} Bu\, d\mu &= \int_{\mathbb{R}^N} \mathrm{div}\,(e^{-\Phi} qDu)dx + \int_{\mathbb{R}^N} \langle G, Du\rangle e^{-\Phi} dx \\
&= -\int_{\mathbb{R}^N} \mathrm{div}\, G\, u\, e^{-\Phi} dx + \int_{\mathbb{R}^N} \langle G, D\Phi\rangle u\, e^{-\Phi} dx = 0.
\end{aligned}
$$

To see that μ is the *unique* invariant measure of $T(t)$, we first note that $T(t)$ is the extension to $L^p(\mathbb{R}^N, \mu)$ of the semigroup generated by $(B, D_{\max}(B))$ in $C_b(\mathbb{R}^N)$, where $D_{\max}(B) = \{u \in C_b(\mathbb{R}^N) \cap W^{2,q}_{\mathrm{loc}}(\mathbb{R}^N)$ for all $q < \infty \mid Au \in C_b(\mathbb{R}^N)\}$ (see Section 5.2). Indeed, since $C_c^2(\mathbb{R}^N)$ is a core for $(B, \mathcal{D}_\mu)$ and since $C_c^2(\mathbb{R}^N)$ is contained in $D_{\max}(B)$, we deduce that $D_{\max}(B)$ is also a core for $(B, \mathcal{D}_\mu)$, hence $(B, \mathcal{D}_\mu)$ is the closure of $(B, D_{\max}(B))$ in $L^p(\mathbb{R}^N, \mu)$. Recalling Proposition 5.2.13, we get that the semigroup generated by $(B, \mathcal{D}_\mu)$ is the extension of that generated in $C_b(\mathbb{R}^N)$, as claimed. At this point, the uniqueness of μ as invariant measure follows, as usual, from the irreducibility and the strong Feller property (see Proposition 5.2.2). $\square$