Chapter 2

Gradient estimates in Neumann parabolic problems in convex regular domains

In the present chapter we study, by means of purely analytic tools, existence, uniqueness and gradient estimates of the solutions to the Neumann problems

$$\begin{cases} u_t(t,x) - \mathcal{A}u(t,x) = 0 & t>0,\ x\in\Omega, \\ \dfrac{\partial u}{\partial \eta}(t,x) = 0 & t>0,\ x\in\partial\Omega, \\ u(0,x) = f(x) & x\in\Omega, \end{cases} \tag{2.0.1}$$

$$\begin{cases} \lambda u(x) - \mathcal{A}u(x) = f(x) & x\in\Omega, \\ \dfrac{\partial u}{\partial \eta}(x) = 0 & x\in\partial\Omega, \end{cases} \tag{2.0.2}$$

where Ω is a regular convex open subset of $\mathbb{R}^N$, η is the unitary outward normal vector to $\partial\Omega$, f is a continuous and bounded function in $\overline{\Omega}$ and $\mathcal{A}$ is the linear second order elliptic operator

$$\mathcal{A} = \sum_{i,j=1}^{N} q_{ij} D_{ij} + \sum_{i=1}^{N} F_i\, D_i - V,$$

whose coefficients are supposed to be regular, possibly unbounded, in $\overline{\Omega}$. The set Ω may be unbounded. Obviously, if $\Omega = \mathbb{R}^N$ we do not require any boundary condition.

Problems (2.0.1) and (2.0.2) are classical in the theory of partial differential equations and they are well understood if the coefficients of $\mathcal{A}$ are bounded. If the coefficients are unbounded and $\Omega = \mathbb{R}^N$, several results of existence, uniqueness and regularity are known, (see [13], [27], [28], [34], [52]) and the overview [38]. Stochastic calculus is a useful tool ([13], [52], [56]); in particular the recent book of Sandra Cerrai [13] contains a deep and exhaustive analysis of what can be proved by stochastic methods.

We consider problem (2.0.1) and we prove that there exists a unique bounded classical solution $u(t,x)$. To do that, we consider the solutions u_n of Neumann problems in a nested sequence Ω_n of bounded domains whose union is Ω, and we prove that u_n converges to a solution of (2.0.1). We remark that one could approximate the solution with solutions of suitable mixed boundary value problems in Ω_n in such a way that for nonnegative initial data the approximating sequence is increasing. This was done by Seizo Itô in his pioneering paper [27]. Although this further property could be of much help in some steps, our techniques to get the gradient bounds do not work with such boundary conditions. Therefore we consider the Neumann boundary condition in each Ω_n. The solution u constructed in such a way is unique, since we assume a Lyapunov type condition which ensures that a maximum principle holds.

Setting $(P_t f)(x) = u(t,x)$, P_t turns out to be a semigroup of linear operators in the space $C_b(\overline{\Omega})$ of the continuous and bounded functions in $\overline{\Omega}$. We remark that in general P_t is not strongly continuous either in $C_b(\overline{\Omega})$ or in its subspace $BUC(\overline{\Omega})$ of the uniformly continuous and bounded functions. This is a typical fact for semigroups associated with elliptic operators with unbounded coefficients. Therefore the generator can not be defined in the classical way. In the literature there are several alternative definitions of generator; here we consider the weak generator introduced by E. Priola (see [48] and also Section 5.2). We prove that it coincides with the realization of $\mathcal{A}$ in $C_b(\overline{\Omega})$ with homogeneous Neumann boundary conditions (see Proposition 2.2.4). In this way, we can prove that the elliptic problem (2.0.2) admits a unique solution, whose second order derivatives exist only in the sense of distributions and are locally p summable for every p.

After we have ensured existence and uniqueness for problems (2.0.1) and (2.0.2), our next step consists in proving gradient estimates. We start by showing that

$$|DP_t f(x)| \le \frac{C_T}{\sqrt{t}}\|f\|_\infty \quad 0<t<T,\ x\in\overline{\Omega},\quad f\in C_b(\overline{\Omega}), \tag{2.0.3}$$

$$|DP_t f(x)| \le C_T(\|f\|_\infty + \|Df\|_\infty) \quad 0\le t\le T,\ x\in\overline{\Omega},\quad f\in C^1_\eta(\overline{\Omega}), \tag{2.0.4}$$

where

$$C^1_\eta(\overline{\Omega}) = \left\{u\in C^1_b(\overline{\Omega}):\ \frac{\partial u}{\partial\eta}(x)=0,\quad x\in\partial\Omega\right\}. \tag{2.0.5}$$

We prove (2.0.3) and (2.0.4) using the Bernstein method, *i. e.* we apply the maximum principle to the equation satisfied by $z_n = u_n^2 + t|Du_n|^2$ (respectively $z_n = u_n^2 + |Du_n|^2$), that gives a bound for z_n independent of n, and then we obtain (2.0.3) (respectively (2.0.4)) letting $n\to\infty$. We observe that the convexity assumption on Ω is crucial at this point, since it leads to the condition $\frac{\partial z_n}{\partial\eta}\le 0$ at the boundary (see Lemma 2.1.3). In the case $\Omega=\mathbb{R}^N$ the previous estimates were proved in [34] with the same method and in [13] with probabilistic methods. As a consequence of (2.0.3) we have an elliptic regularity result, since we can show that the domain of the weak generator of P_t is contained in $C^1_b(\overline{\Omega})$.

Assuming $V\equiv 0$, we prove further gradient estimates. In the case $q_{ij}\equiv\delta_{ij}$ we show that

$$|DP_t f(x)|^p \le e^{k_0 p t} P_t(|Df|^p)(x) \quad t\ge 0,\ x\in\overline{\Omega},\quad f\in C^1_\eta(\overline{\Omega}). \tag{2.0.6}$$

for all $p\ge 1$, where $k_0\in\mathbb{R}$ is determined by the dissipativity condition

$$\sum_{i,j=1}^N D_iF_j(x)\xi_i\xi_j \le k_0|\xi|^2, \quad x\in\Omega,\ \xi\in\mathbb{R}^N. \tag{2.0.7}$$

If the coefficients q_{ij} are not constant we prove the similar estimate

$$|DP_t f(x)|^p \le e^{\sigma_p t} P_t(|Df|^p)(x) \quad t\ge 0,\ x\in\overline{\Omega},\quad f\in C^1_\eta(\overline{\Omega}), \tag{2.0.8}$$

for all $p>1$, where $\sigma_p\in\mathbb{R}$ is a suitable constant. These estimates have interesting consequences. First, if there exists an invariant measure for P_t, that is a probability measure such that

$$\int_\Omega P_t f d\mu = \int_\Omega f d\mu, \quad t\ge 0,\ f\in C_b(\overline{\Omega}),$$

estimates (2.0.6) and (2.0.8) are of much help in the study of the realization of P_t in the spaces $L^p(\Omega,\mu)$, $1 \le p < \infty$ (see Remark 2.4.5 for such consequences and Chapter 5 for the main properties of invariant measures).

Second, we deduce the pointwise estimates

$$
(2.0.9)\qquad
\begin{aligned}
|DP_tf(x)|^p &\le \left(\frac{\sigma_2\nu_0^{-1}}{2(1-e^{-\sigma_2 t})}\right)^{\frac{p}{2}} P_t(|f|^p)(x), \quad t>0,\ p\ge 2,\\
|DP_tf(x)|^p &\le \frac{c_p\sigma_p\nu_0^{-1}}{t^{p/2-1}(1-e^{-\sigma_p t})} P_t(|f|^p)(x), \quad t>0,\ 1<p<2,
\end{aligned}
$$

for $f \in C_b(\overline{\Omega})$, where $c_p > 0$ is a suitable constant. Estimates (2.0.9) give the optimal constant in (2.0.3); moreover integrating over Ω with respect to the invariant measure μ we get the corresponding estimates for DP_tf in $L^p(\Omega,\mu)$, when $f \in L^p(\Omega,\mu)$.

Dissipativity conditions of the type (2.0.7) are of crucial importance to get gradient estimates. Indeed, in section 2.4 we give a counterexample to estimate (2.0.3) for an operator $\mathcal{A} = \Delta + \sum F_iD_i$ where F does not satisfy (2.0.7). Concerning estimate (2.0.6), in the case of variable coefficients q_{ij} the constant σ_p blows up as $p \to 1$, and we do not expect that (2.0.6) holds also for $p = 1$. Estimate (2.0.9) too fails in general for $p = 1$, as we show in the case of the heat semigroup. Finally we show an example related with the Ornstein-Uhlenbeck operator.

2.1 Assumptions and preliminary results

First we state our assumptions that will be kept throughout the chapter. $\Omega \subset \mathbb{R}^N$ is a convex open set with $C^{2+\alpha}$ boundary (see Definition B.0.15). The coefficients of the operator $\mathcal{A}$ are real-valued and belong to $C^{1+\alpha}_{\mathrm{loc}}(\overline{\Omega})$ and satisfy the following conditions:

$$(2.1.1)\qquad q_{ij} = q_{ji}, \quad \sum_{i,j=1}^N q_{ij}(x)\xi_i\xi_j \ge \nu(x)|\xi|^2, \quad x\in\Omega,\ \xi\in\mathbb{R}^N, \quad \inf_{x\in\Omega}\nu(x) = \nu_0 > 0,$$

$$(2.1.2)\qquad |Dq_{ij}(x)| \le M\nu(x), \quad x\in\Omega,\ i,j = 1,...,N,$$

$$(2.1.3)\qquad \sum_{i,j=1}^N D_iF_j(x)\,\xi_i\,\xi_j \le (\beta V(x) + k_0)\,|\xi|^2, \quad x\in\Omega,\ \xi\in\mathbb{R}^N,$$

$$(2.1.4)\qquad V(x)\ge 0, \qquad |DV(x)| \le \gamma(1+V(x)), \quad x\in\Omega,$$

for some constants $M,\gamma \ge 0$, $k_0,\beta\in\mathbb{R}$, $\beta < 1/2$. Moreover, we suppose that there exist a positive function $\varphi \in C^2(\overline{\Omega})$ and $\lambda_0 > 0$ such that

$$(2.1.5)\qquad \lim_{|x|\to+\infty}\varphi(x) = +\infty, \qquad \sup_{\overline{\Omega}}(\mathcal{A}\varphi - \lambda_0\varphi) < +\infty, \qquad \frac{\partial\varphi}{\partial\eta}(x) \ge 0, \quad x\in\partial\Omega.$$

We introduce the following realization of operator $\mathcal{A}$ with homogeneous Neumann boundary condition

$$D(\mathcal{A}) = \left\{ u\in C_b(\overline{\Omega}) \cap \bigcap_{1\le p<\infty} W^{2,p}(\Omega\cap B_R) \text{ for all } R>0:\ \mathcal{A}u\in C_b(\overline{\Omega}),\ \frac{\partial u}{\partial\eta}\Big|_{\partial\Omega} = 0 \right\}.$$

We remark that if $\Omega = \mathbb{R}^N$ our results can be generalized to operators with locally Hölder continuous coefficients satisfying suitable assumptions by a standard convolution approximation, see Remark 2.3.4.

In this section we collect some preliminary results which are the main tools for the study of problems (2.0.1) and (2.0.2). We start by stating maximum principles for such problems, and consequent uniqueness results. For the proofs we refer to Appendix A.

Proposition 2.1.1 *Let $z \in C([0,T]\times\overline{\Omega})\cap C^{0,1}(]0,T]\times\overline{\Omega})\cap C^{1,2}(]0,T]\times\Omega)$ be a bounded function satisfying*

$$\begin{cases} z_t(t,x) - \mathcal{A}z(t,x) \le 0, & 0 < t \le T,\ x \in \Omega, \\ \dfrac{\partial z}{\partial \eta}(t,x) \le 0, & 0 < t \le T,\ x \in \partial\Omega, \\ z(0,x) \le 0 & x \in \Omega. \end{cases}$$

Then $z \le 0$. In particular there exists at most one bounded classical solution of problem (2.0.1).

Proposition 2.1.2 *Let $u \in C_b(\overline{\Omega}) \cap W^{2,p}(\Omega \cap B_R)$ for all $R > 0$ and $p < \infty$, be such that $\mathcal{A}u \in C_b(\overline{\Omega})$ and*

$$\begin{cases} \lambda u(x) - \mathcal{A}u(x) \le 0, & x \in \Omega, \\ \dfrac{\partial u}{\partial \eta}(x) \le 0, & x \in \partial\Omega, \end{cases} \tag{2.1.6}$$

for some $\lambda \ge \lambda_0$. Then $u \le 0$. In particular, there exists at most one solution in $D(\mathcal{A})$ of problem (2.0.2).

The following lemma is of crucial importance for our estimates; it holds for convex domains and this is the reason why we have assumed that Ω is convex.

Lemma 2.1.3 *Let Λ be a convex open set with C^1 boundary, not necessarily bounded. Let $u \in C^2(\overline{\Lambda})$ such that $\partial u/\partial\eta(x) = 0$ for all $x \in \partial\Lambda$. Then the function $v := |Du|^2$ satisfies*

$$\frac{\partial v}{\partial \eta}(x) \le 0, \quad x \in \partial\Lambda.$$

Proof. Let us introduce the notation $\dfrac{\partial \eta}{\partial \tau} = \left(\dfrac{\partial \eta_1}{\partial \tau}, \cdots, \dfrac{\partial \eta_N}{\partial \tau}\right)$, where the derivatives are understood in local coordinates. Since Ω is convex, we have $\tau \cdot \dfrac{\partial \eta}{\partial \tau}(x) \ge 0$ for all $x \in \partial\Omega$ and all vector τ tangent to $\partial\Omega$ at x (see [25, section V.B]). By assumption, $Du(x)\cdot\eta(x) = 0$ for all $x \in \partial\Omega$ and then differentiating we get

$$\frac{\partial}{\partial \tau}(Du(x)\cdot\eta(x)) = D^2u(x)\tau\cdot\eta(x) + Du(x)\cdot\frac{\partial \eta}{\partial \tau}(x) = 0, \qquad x \in \partial\Lambda,$$

for every vector τ tangent to $\partial\Omega$. For $\tau = Du(x)$ we have

$$\frac{\partial v}{\partial \eta}(x) = 2D^2u(x)\tau\cdot\eta(x) = -2\tau\cdot\frac{\partial \eta}{\partial \tau}(x) \le 0, \qquad x \in \partial\Omega.$$

□

Now we recall some known results about Neumann problems in bounded domains. Let Λ be a bounded open set in $\mathbb{R}^N$ with $C^{2+\alpha}$ boundary. Consider the realization of the operator $\mathcal{A}$ in $C(\overline{\Lambda})$ with homogeneous Neumann boundary condition

$$D_\eta(A) = \left\{ u \in W^{2,p}(\Lambda) \text{ for all } p < +\infty \ :\ \mathcal{A}u \in C(\overline{\Lambda}),\ \frac{\partial u}{\partial \eta}(x) = 0,\ x \in \partial\Lambda \right\}, \tag{2.1.7}$$

and $Au = \mathcal{A}u$ for all $u \in D_\eta(A)$.

It is well known that $(A, D_\eta(A))$ generates a strongly continuous analytic positive semigroup $(S(t))$ of contractions in the space $C(\overline{\Lambda})$ (see e.g. [32, Section 3.1.5]). It follows that for all $f \in C(\overline{\Lambda})$ the function $u(t,x) = (S(t)f)(x)$ has the following properties

(i) $u \in C([0,+\infty[;C(\overline{\Lambda})) \cap C^1(]0,+\infty[;C(\overline{\Lambda}))$,

(ii) $u(t,\cdot) \in D_\eta(A)$, for all $t>0$,

(iii) u is the unique solution of the Neumann problem

$$\begin{cases} D_t u(t,x) - Au(t,x) = 0 & t>0,\ x\in\Lambda, \\ \dfrac{\partial u}{\partial \eta}(t,x) = 0 & t>0,\ x \in \partial\Lambda, \\ u(0,x) = f(x) & x \in \overline{\Lambda}. \end{cases} \tag{2.1.8}$$

satisfying (i) and (ii).

Actually the function u enjoys further regularity, as it is shown below.

Lemma 2.1.4 *The following properties hold*

(a) $u \in C^{1+\alpha/2,2+\alpha}([\varepsilon,T]\times\overline{\Lambda})$ *for all* $0<\varepsilon<T<+\infty$ *and*

$$\|u\|_{C^{1+\alpha/2,2+\alpha}([\varepsilon,T]\times\overline{\Lambda})} \leq C\,\|u\|_{C([0,T]\times\overline{\Lambda})} \tag{2.1.9}$$

for a suitable constant $C = C(\varepsilon,T,\Lambda)>0$.

(b) $D_i u \in C^{1+\alpha/2,2+\alpha}([\varepsilon,T]\times\overline{\Lambda}')$, *for all* $i=1,...,N$, $0<\varepsilon<T<+\infty$ *and* Λ' *open set with* $\overline{\Lambda}' \subset \Lambda$. *In particular* $u \in C^{1,3}(]0,+\infty[\times\Lambda)$.

Proof. (a) Assume first that $f \in C^{2+\alpha}(\overline{\Lambda})$ and $\partial f/\partial\eta = 0$ on $\partial\Lambda$. Then there exists a function $v \in C^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\Lambda})$, for all $T>0$, which solves (2.1.8) (see [30, Theorem IV.5.3]). By uniqueness $v(t,x) = u(t,x)$.

Now take $f \in C(\overline{\Lambda})$ and consider a sequence $(f_n) \subseteq C^{2+\alpha}(\overline{\Lambda})$ with $\partial f_n/\partial\eta = 0$ on $\partial\Lambda$, which converges to f in $C(\overline{\Lambda})$. Let $v_n \in C^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\Lambda})$, for all $T>0$, be the solution of problem (2.1.8) with initial datum f_n. Fix $0<\varepsilon'<\varepsilon<T$, then the following Schauder estimate holds

$$\|v_n\|_{C^{1+\alpha/2,2+\alpha}([\varepsilon,T]\times\overline{\Lambda})} \leq C\|v_n\|_{C([\varepsilon',T]\times\overline{\Lambda})}, \qquad n\in\mathbb{N} \tag{2.1.10}$$

where $C = C(\varepsilon,\varepsilon',T,\Lambda)>0$ (see Theorem C.1.1).

On the other hand, the maximum principle implies that if $z \in C([0,T]\times\overline{\Lambda}) \cap C^1(]0,T]\times\overline{\Lambda}) \cap C^{1,2}(]0,T]\times\Lambda)$ solves problem (2.1.8) then

$$\|z\|_{C([0,T]\times\overline{\Lambda})} \leq \|f\|_{C(\overline{\Lambda})}.$$

Applying this estimate and (2.1.10) to the difference $v_n - v_m$ we get

$$\begin{aligned} \|v_n - v_m\|_{C([0,T]\times\overline{\Lambda})} &\leq \|f_n - f_m\|_{C(\overline{\Lambda})}, & n,m\in\mathbb{N}, \\ \|v_n - v_m\|_{C^{1+\alpha/2,2+\alpha}([\varepsilon,T]\times\overline{\Lambda})} &\leq C\,\|f_n - f_m\|_{C(\overline{\Lambda})}, & n,m\in\mathbb{N}. \end{aligned}$$

It follows that (v_n) is a Cauchy sequence in $C^{1+\alpha/2,2+\alpha}([\varepsilon,T]\times\overline{\Lambda})$ and in $C([0,T]\times\overline{\Lambda})$, consequently it converges to a function $\overline{v} \in C^{1+\alpha/2,2+\alpha}([\varepsilon,T]\times\overline{\Lambda}) \cap C([0,T]\times\overline{\Lambda})$. Iterating the same argument we find a function $v \in C^{1+\alpha/2,2+\alpha}_{\rm loc}(]0,+\infty[\times\overline{\Lambda}) \cap C([0,+\infty[\times\overline{\Lambda})$ which solves problem (2.1.8) with datum f. Again, by uniqueness, $v(t,x) = u(t,x)$. Estimate (2.1.9) is clear from (2.1.10) $n\to\infty$.

(b) The statement follows from [29, Theorem 8.12.1] since the coefficients of A belong to $C^{1+\alpha}(\overline{\Lambda})$. □

Next we prove a gradient estimate for $S(t)f$, using Bernstein's method (see [34, Theorem 2.4]). It is worth observing that, since Λ is bounded, this result is well-known. Actually, our interest is not in the estimate itself but rather in the fact that the constant C_T in (2.1.11) does not depend on the domain Λ, when it is convex. This will be an important step in the study of problem (2.0.1).

Proposition 2.1.5 *Let Λ be a bounded convex open set with $C^{2+\alpha}$ boundary. For all fixed $T>0$ there exists a constant $C_T>0$ independent of Λ such that*

$$|DS(t)f(x)| \le \frac{C_T}{\sqrt{t}}\,\|f\|_\infty \qquad 0<t\le T,\ x\in\overline{\Lambda} \tag{2.1.11}$$

for every $f\in C(\overline{\Lambda})$.

Proof. We may suppose that $V\ge 1$; the general case follows considering the operator $A' = A - I$. Assume first that $f\in D_\eta(A)$; set $u(t,x)=(S(t)f)(x)$ and define the function

$$v(t,x)=u^2(t,x)+a\,t\,|Du(t,x)|^2\ , \qquad t\ge 0,\ x\in\Lambda,$$

where $a>0$ is a parameter that will be chosen later. Then we have $v\in C^{1,2}(]0,T]\times\Lambda)\cap C^{0,1}(]0,T]\times\overline{\Lambda})$; moreover, since $f\in D_\eta(A)$, we have $u\in C([0,T];D_\eta(A))$; in particular $Du\in C([0,T]\times\overline{\Lambda})$ and then $v\in C([0,T]\times\overline{\Lambda})$.

We claim that for a suitable value of $a>0$ independent of Λ, we have

$$v_t(t,x)-\mathcal{A}v(t,x)\le 0, \qquad 0<t<T,\ x\in\Lambda, \tag{2.1.12}$$

$$\frac{\partial v}{\partial\eta}(t,x)\le 0 \qquad 0<t<T,\ x\in\partial\Lambda; \tag{2.1.13}$$

then the maximum principle implies

$$v(t,x)\le \sup_{x\in\overline{\Lambda}} v(0,x)=\|f\|_\infty^2 \qquad 0\le t\le T,\ x\in\overline{\Lambda},$$

which yields (2.1.11) with $C_T=a^{-1/2}$.

The boundary condition (2.1.13) follows from Lemma 2.1.3. For (2.1.12), a straightforward computation shows that v satisfies the equation

$$v_t(t,x)-\mathcal{A}v(t,x)=a|Du(t,x)|^2-2\sum_{i,j=1}^N q_{ij}(x)\,D_iu(t,x)\,D_ju(t,x)+g_1(t,x)+g_2(t,x),$$

where

$$\begin{aligned} g_1(t,x) &= 2\,a\,t\sum_{i,j=1}^N D_iF_j(x)\,D_iu(t,x)\,D_ju(t,x)-a\,t\,V(x)|Du(t,x)|^2\\ &\quad -2\,a\,t\,u(t,x)Du(t,x)\cdot DV(x)-V(x)u^2(t,x),\\ g_2(t,x) &= 2\,a\,t\left(\sum_{i,j,k=1}^N D_kq_{ij}(x)D_ku(t,x)D_{ij}u(t,x)-\sum_{i,j,k=1}^N q_{ij}(x)D_{ik}u(t,x)D_{jk}u(t,x)\right). \end{aligned}$$

Let us estimate the function g_1. Using (2.1.3), (2.1.4) and recalling that $V\ge 1$ we get for all $\varepsilon>0$

$$\begin{aligned} g_1 &\le 2at(\beta V+k_0)|Du|^2-atV|Du|^2+2a\gamma C_\varepsilon t(1+V)|u|^2+2a\gamma\varepsilon t(1+V)|Du|^2-Vu^2\\ &\le at(2\beta-1+2\gamma\varepsilon)V|Du|^2+(4a\gamma C_\varepsilon t-1)Vu^2+2at(k_0+\gamma\varepsilon)|Du|^2, \end{aligned}$$

where $C_\varepsilon > 0$ is a constant. Since $\beta < 1/2$ we can choose $\varepsilon = \varepsilon(\beta, \gamma)$ such that $(2\beta - 1 + 2\gamma\varepsilon) < 0$ and we get

$$g_1 \leq (4\, a\, \gamma\, C_\varepsilon t - 1) V u^2 + 2\, a\, t (k_0 + \gamma\varepsilon) |Du|^2. \tag{2.1.14}$$

Concerning g_2, from (2.1.2) we have

$$\begin{aligned}
\sum_{i,j,k=1}^N D_k q_{ij} D_k u D_{ij} u &\leq M\nu(x) \sum_{k=1}^N |D_k u| \sum_{i,j=1}^N |D_{ij} u| \\
&\leq M N^{3/2} \nu(x) |Du| \left(\sum_{i,j=1}^N (D_{ij} u)^2 \right)^{1/2} \\
&\leq \nu(x) \sum_{i,j=1}^N (D_{ij} u)^2 + \frac{1}{4} M^2 N^3 \nu(x) |Du|^2,
\end{aligned}$$

and therefore

$$\begin{aligned}
g_2(t,x) &\leq 2\, a\, t \left(\nu(x) \sum_{i,j=1}^N (D_{ij} u)^2 + \frac{1}{4} M^2 N^3 \nu(x) |Du|^2 - \nu(x) \sum_{i,j=1}^N (D_{ij} u)^2 \right) \\
&= \frac{1}{2}\, a\, t\, M^2 N^3\, \nu(x) |Du|^2. \qquad (2.1.15)
\end{aligned}$$

Estimates (2.1.14) and (2.1.15) imply that

$$\begin{aligned}
v_t(t,x) - \mathcal{A} v(t,x) &\leq \left\{ a + 2\, at(k_0 + \gamma\varepsilon)) + \left(\frac{1}{2} a\, t\, M^2 N^3 - 2 \right) \nu(x) \right\} |Du(t,x)|^2 \\
&\quad + (4\, a\, \gamma\, C_\varepsilon t - 1) V(x) u^2(t,x) \\
&\leq \left\{ a + 2\, aT(k_0^+ + \gamma\varepsilon)) + \left(\frac{1}{2} a\, T\, M^2 N^3 - 2 \right) \nu(x) \right\} |Du(t,x)|^2 \\
&\quad + (4\, a\, \gamma\, C_\varepsilon T - 1) V(x) u^2(t,x),
\end{aligned}$$

for all $t \in]0,T]$ and $x \in \Lambda$. It is clear now that there exists a sufficiently small value $a > 0$ which depends on $\nu_0, M, k_0, \beta, \gamma, N, T$ but not on Λ such that (2.1.12) holds.

If $f \in C(\overline{\Lambda})$ the statement follows easily using the semigroup law, since $S(t)$ is analytic:

$$|DS(t)f(x)| = |DS(t/2)S(t/2)f(x)| \leq \frac{\sqrt{2}\, C_T}{\sqrt{t}} \|S(t/2)f\|_\infty \leq \frac{\sqrt{2}\, C_T}{\sqrt{t}} \|f\|_\infty.$$

□

2.2 Construction of the associated semigroup

In this section we prove that there exist bounded solutions to problems (2.0.1) and (2.0.2), we show that there exists a semigroup $(P_t)_{t \geq 0}$ in $C_b(\overline{\Omega})$ which yields the solution of (2.0.1) and we study the main properties of P_t.

We consider a nested sequence $\{\Omega_n\}_{n \in \mathbb{N}}$ of convex bounded open sets with $C^{2+\alpha}$ boundary such that

$$\bigcup_{n \in \mathbb{N}} \Omega_n = \Omega, \qquad \partial\Omega \subset \bigcup_{n \in \mathbb{N}} \partial\Omega_n.$$

We denote the domain of the realization of $\mathcal{A}$ in Ω_n with

$$D_n(\mathcal{A}) = \left\{ u \in W^{2,p}(\Omega_n) \text{ for all } p < \infty : \ \mathcal{A}u \in C(\overline{\Omega}_n), \ \frac{\partial u}{\partial \eta}(x) = 0, \ x \in \partial\Omega_n \right\}. \tag{2.2.1}$$

and we denote the associated semigroup with $(T_n(t))_{t\geq 0}$. Here is the existence theorem for problem (2.0.1).

Theorem 2.2.1 *For every $f \in C_b(\overline{\Omega})$ there exists a unique bounded solution $u(t,x)$ of problem (2.0.1) belonging to $C([0,+\infty[\times\overline{\Omega}) \cap C^{1+\alpha/2,2+\alpha}_{\text{loc}}(]0,+\infty[\times\overline{\Omega})$. Moreover*

$$u(t,x) = \lim_{n\to\infty} (T_n(t))f(x), \qquad t \geq 0, \ x \in \overline{\Omega}. \tag{2.2.2}$$

Setting $P_t f = u(t,\cdot)$, then $(P_t)_{t\geq 0}$ is a positive contraction semigroup in $C_b(\overline{\Omega})$. Moreover

$$\|DP_t f\|_\infty \leq \frac{C_T}{\sqrt{t}} \|f\|_\infty \qquad 0 < t \leq T, \tag{2.2.3}$$

where C_T is the same as in (2.1.11).

Proof. Set $u_n(t,x) = (T_n(t)f)(x)$. Let $\Omega' \subset \Omega$ be a bounded open set and $0 < \varepsilon < T$. From [30, Theorem IV.10.1] it follows that if $\Omega'' \subset \Omega$ is a bounded open set such that $\Omega' \subset \Omega''$ and $\text{dist}\,(\Omega', \Omega \setminus \Omega'') > 0$, then there exists a constant $C = C(\varepsilon, T, \Omega', \Omega'') > 0$ such that

$$\|u_n\|_{C^{1+\alpha/2,2+\alpha}([\varepsilon,T]\times\overline{\Omega}')} \leq C\|u_n\|_{C([0,T]\times\overline{\Omega}'')}. \tag{2.2.4}$$

Hence

$$\|u_n\|_{C^{1+\alpha/2,2+\alpha}([\varepsilon,T]\times\overline{\Omega}')} \leq C\|f\|_\infty,$$

for all $n \in \mathbb{N}$ such that $\Omega'' \subset \Omega_n$, and therefore the sequence $(u_n)_{n\in\mathbb{N}}$ is relatively compact in $C^{1,2}([\varepsilon,T]\times\overline{\Omega}')$. Considering an increasing sequence of domains $[\varepsilon_n, T_n] \times \overline{\Omega}_n'$ whose union is $]0,+\infty[\times\overline{\Omega}$ and using a diagonal procedure we can conclude that there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ (possibly dependent on f) such that

$$\exists \lim_{k\to\infty} u_{n_k}(t,x) = u(t,x), \qquad t > 0, \ x \in \overline{\Omega},$$

where $u \in C^{1+\alpha/2,2+\alpha}_{\text{loc}}(]0,+\infty[\times\overline{\Omega})$. Moreover $(u_{n_k})_{k\in\mathbb{N}}$ converges to u in $C^{1,2}([\varepsilon,T]\times\overline{\Omega}')$ for all $0 < \varepsilon < T$ and for all bounded open set $\Omega' \subset \Omega$.

We prove that u is a bounded classical solution of problem (2.0.1). The function u is a solution of the equation $u_t - \mathcal{A}u = 0$ in $]0,+\infty[\times\Omega$. This follows letting $k \to \infty$ in the equation satisfied by u_{n_k}. Moreover since

$$|u(t,x)| \leq \|f\|_\infty, \qquad t > 0, \ x \in \overline{\Omega},$$

then u is bounded in $]0,+\infty[\times\overline{\Omega}$. The boundary condition

$$\frac{\partial u}{\partial \eta}(t,x) = 0, \quad t > 0, \ x \in \partial\Omega.$$

follows immediately since u_{n_k} converges to u in $C^{1,2}([\varepsilon,T]\times\overline{\Omega}')$ for all $0 < \varepsilon < T$ and $\Omega' \subset \Omega$ bounded open set. Finally we prove that u is continuous at $(0,x_0)$ with value $f(x_0)$ for all $x_0 \in \overline{\Omega}$. Consider two neighborhoods $U_1 \subset U_0$ of x_0. Set $\Omega_0 = U_0 \cap \Omega$ and $\Omega_1 = U_1 \cap \Omega$ and suppose that Ω_0 is convex and has $C^{2+\alpha}$ boundary. Let $\theta \in C^2(\overline{\Omega}_0)$ be such that $\theta = 0$ in a neighborhood of $\Omega \cap \partial U_0$, $\theta = 1$ in $\overline{\Omega}_1$ and $\partial\theta/\partial\eta = 0$ in $U_0 \cap \partial\Omega$. Define

$$v_n(t,x) = \theta(x)u_n(t,x), \qquad t > 0, \ x \in \Omega_0.$$

Then v_n satisfies the boundary condition

$$\frac{\partial v_n}{\partial \eta}(t,x) = \theta(x)\frac{\partial u_n}{\partial \eta}(t,x) + u_n(t,x)\frac{\partial \theta}{\partial \eta}(x) = 0, \tag{2.2.5}$$

for all $t > 0$ and $x \in \partial\Omega_0$ and for all n such that $\Omega_0 \subset \Omega_n$. Moreover v_n satisfies the equation

$$D_t v_n(t,x) - \mathcal{A} v_n(t,x) = \psi_n(t,x), \qquad t > 0,\ x \in \Omega_0,$$

where

$$\psi_n(t,x) = -u_n(t,x)(\mathcal{A} + V(x))\theta(x) - 2\sum_{i,j=1}^{N} q_{ij}(x) D_i u_n(t,x) D_j \theta(x).$$

Since $T_n(t)$ satisfies the gradient estimate (2.1.11), it follows that there exists a constant $C > 0$ such that

$$\|\psi_n(t)\|_\infty \le \frac{C}{\sqrt{t}} \qquad 0 < t \le T, \tag{2.2.6}$$

for all $n \in \mathbb{N}$. Let $T(t)$ be the strongly continuous analytic semigroup generated by the realization of $\mathcal{A}$ in $C(\overline{\Omega}_0)$ with Neumann boundary conditions. From [32, Proposition 4.1.2] it follows that $v_n(t)$ can be written as

$$v_n(t) = T(t)(\theta f) + \int_0^t T(t-s)\psi_n(s)ds.$$

Since $v_n = u_n$ in $\overline{\Omega}_1$, if $(t,x) \in]0,T[\times\overline{\Omega}_1$ we have

$$|u_{n_k}(t,x) - f(x_0)| \le |T(t)(\theta f)(x) - f(x_0)| + \int_0^t \|T(t-s)\psi_{n_k}(s)\|_\infty ds.$$

Using (2.2.6) and letting $k \to \infty$ we get

$$|u(t,x) - f(x_0)| \le |T(t)(\theta f)(x) - f(x_0)| + \int_0^t \frac{C}{\sqrt{s}} ds$$

which shows that u is continuous at $(0,x_0)$. Since $x_0 \in \overline{\Omega}$ is arbitrary, we conclude that u is continuous in $[0,T]\times\overline{\Omega}$. Thus we have proved that u is a bounded classical solution of problem (2.0.1).

We claim that the whole sequence $(u_n)_{n\in\mathbb{N}}$ converges to u in $C^{1,2}([\varepsilon,T]\times\overline{\Omega}')$ for all $0 < \varepsilon < T$, $\Omega' \subset \Omega$ bounded open set. Indeed consider any subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$. The previous argument can be applied to $(u_{n_k})_{k\in\mathbb{N}}$ and it follows that there is a subsequence $(u_{n_{k_j}})_{j\in\mathbb{N}}$ and a function v such that v is a classical bounded solution of problem (2.0.1) and $(u_{n_{k_j}})_{j\in\mathbb{N}}$ converges to v. But from Proposition 2.1.1 it follows that $u = v$. This show that the whole sequence converges to u.

Writing $(P_t f)(x) = u(t,x)$, we get the positivity of P_t directly from the positivity of $T_n(t)$. The semigroup law for the linear operators P_t follows in a standard way from uniqueness.

Finally, according to Proposition 2.1.5, for all $T > 0$ there exists a constant $C_T > 0$ such that

$$|DT_n(t)f(x)| \le \frac{C_T}{\sqrt{t}}\|f\|_\infty, \quad 0 < t \le T,\ x \in \overline{\Omega}_n,$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ we get (2.2.3). □

The next proposition shows some continuity properties of P_t that will be useful in the sequel.

Proposition 2.2.2 *If $(f_n)_{n\in\mathbb{N}} \subset C_b(\overline{\Omega})$ is a bounded sequence which converges pointwise in Ω to a function $f \in C_b(\overline{\Omega})$, then $(P_t f_n)(x)$ converges to $(P_t f)(x)$ in $C^{1,2}([\varepsilon, T] \times \overline{\Omega}')$ for all $0 < \varepsilon < T$ and all bounded sets $\Omega' \subset \Omega$. If (f_n) converges to f uniformly on compact subsets of $\overline{\Omega}$, then $(P_t f_n)(x)$ converges to $(P_t f)(x)$ uniformly in $[0, T] \times \overline{\Omega}'$ for all $T > 0$ and all bounded sets $\Omega' \subset \Omega$.*

Finally P_t can be represented in the form

$$(P_t f)(x) = \int_\Omega f(y) p(t, x; dy), \quad t > 0, \; x \in \overline{\Omega}, \tag{2.2.7}$$

where $p(t, x; dy)$ is a positive finite Borel measure on Ω.

Proof. We may assume that $f = 0$. Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $C_b(\overline{\Omega})$ that converges pointwise to zero in Ω, and set $u_n(t, x) = P_t f_n(x)$. Using the local Schauder estimate (2.2.4) and the maximum principle it follows that the sequence (u_n) is bounded in $C^{1+\alpha/2,2+\alpha}([\varepsilon, T] \times \overline{\Omega}')$ for all $0 < \varepsilon < T$ and all bounded $\Omega' \subset \Omega$. Therefore there exist a subsequence u_{n_k}, and a function $u \in C^{1,2}(]0, +\infty[\times\overline{\Omega})$ such that u_{n_k} converges to u in $C^{1,2}([\varepsilon, T] \times \overline{\Omega}')$ for all $0 < \varepsilon < T$ and for all bounded $\Omega' \subset \Omega$. The function u is a bounded solution of the equation

$$u_t - \mathcal{A}u = 0 \quad \text{in } (0, +\infty) \times \Omega,$$

and it satisfies the boundary condition

$$\frac{\partial u}{\partial \eta} = 0 \quad \text{in } (0, +\infty) \times \partial\Omega.$$

Now we show that u is continuous up to $t = 0$ and that $u(0, x) = 0$ in order to conclude that $u \equiv 0$, by Proposition 2.1.1. Let Ω_0, Ω_1 and θ be as in the proof of Theorem 2.2.1 and set $v_n(t, x) = \theta(x) u_n(t, x)$. Then we can write

$$v_n(t) = T(t)(\theta f_n) + \int_0^t T(t - s)\psi_n(s) ds,$$

where $T(t)$ is the semigroup generated by the realization of $\mathcal{A}$ in $C(\overline{\Omega}_0)$ with Neumann boundary condition and

$$\psi_n(t, x) = -u_n(t, x)(\mathcal{A} + V(x))\theta(x) - 2\sum_{i,j=1}^N q_{ij}(x) D_i u_n(t, x) D_j \theta(x).$$

Using the gradient estimate (2.2.3) and the boundedness of $(f_{n_k})_{k\in\mathbb{N}}$ it follows that

$$|v_{n_k}(t, x)| \le |(T(t)(\theta f_{n_k}))(x)| + C\sqrt{t}, \quad x \in \overline{\Omega}_0, \; 0 \le t \le T, \; k \in \mathbb{N}, \tag{2.2.8}$$

where $C > 0$ is a constant independent of $k \in \mathbb{N}$. For all $1 < p < +\infty$ the semigroup $(T(t))$ extends to an analytic semigroup in $L^p(\Omega_0)$ (see [32, Section 3.1.1]), and for $p > N$ the domain of the generator of $T(t)$ in $L^p(\Omega_0)$ is embedded in $C(\overline{\Omega}_0)$; since θf_{n_k} converges to zero in $L^p(\Omega_0)$ it follows that $T(t)(\theta f_{n_k})$ converges to zero uniformly in $\overline{\Omega}_0$. Thus letting $k \to \infty$ in (2.2.8) we get

$$|u(t, x)| \le C\sqrt{t}, \quad 0 < t < T, \; x \in \overline{\Omega}_1,$$

which implies that u is continuous at $(0, x_0)$ for all $x_0 \in \overline{\Omega}_1$. Since $\Omega_1 \subset \Omega$ is arbitrary, we obtain that u is continuous at $t = 0$ with $u(0, x) = 0$.

Therefore $u \equiv 0$ and the subsequence u_{n_k} converges to zero in $C^{1,2}([\varepsilon, T] \times \overline{\Omega}')$ for all $0 < \varepsilon < T$ and bounded $\Omega' \subset \Omega$. As in the proof of Theorem 2.2.1 one can prove that the whole sequence $(u_n)_{n\in\mathbb{N}}$ converges to zero in $C^{1,2}([\varepsilon, T] \times \overline{\Omega}')$ for all $0 < \varepsilon < T$ and bounded $\Omega' \subset \Omega$, as stated.

Suppose now that $(f_n)_{n\in\mathbb{N}}$ converges to zero uniformly on compact subsets of $\overline{\Omega}$. By (2.2.8) we have

$$|u_n(t,x)| \le \|T(t)(\theta f_n)\|_\infty + C\sqrt{t} \le \|\theta f_n\|_\infty + C\sqrt{t}, \qquad x \in \overline{\Omega}_1,\ 0 \le t \le T,$$

where $C > 0$ does not depend on $n \in \mathbb{N}$. Therefore for all $\varepsilon > 0$ we have

$$\|u_n\|_{C([0,T]\times\overline{\Omega}_1)} \le \|\theta f_n\|_\infty + C\sqrt{\varepsilon} + \|u_n\|_{C([\varepsilon,T]\times\overline{\Omega}_1)}.$$

Taking into account the first step of the proof this yields

$$\limsup_{n\to\infty} \|u_n\|_{C([0,T]\times\overline{\Omega}_1)} \le C\sqrt{\varepsilon},$$

that is u_n converges to zero uniformly in $[0,T]\times\overline{\Omega}_1$. Since Ω_1 is arbitrary, the conclusion follows.

We can prove now (2.2.7). By the Riesz representation theorem, for every $x \in \overline{\Omega}$ there exists a positive finite Borel measure $p(t,x;dy)$ in Ω such that

$$(P_t f)(x) = \int_\Omega f(y)p(t,x;dy), \qquad f \in C_0(\Omega). \tag{2.2.9}$$

If $f \in C_b(\overline{\Omega})$, we consider a bounded sequence $(f_n)_{n\in\mathbb{N}} \subset C_0(\Omega)$ which converges to f uniformly on compact sets of Ω. Writing (2.2.9) for f_n and letting $n \to +\infty$ we obtain the statement for $f \in C_b(\overline{\Omega})$, by dominated convergence. □

Using the semigroup law we extend estimate (2.2.3) to the whole half-line $[0,+\infty[$.

Corollary 2.2.3 *For all $\omega > 0$ there exists $C_\omega > 0$ such that*

$$\|DP_t f\|_\infty \le C_\omega \frac{e^{\omega t}}{\sqrt{t}} \|f\|_\infty, \qquad t > 0,\ f \in C_b(\overline{\Omega}). \tag{2.2.10}$$

Proof. Fix $\omega > 0$ and let $T = T(\omega) > 0$ such that $e^{\omega t}t^{-1/2} \ge 1$, for all $t > T(\omega)$. By (2.2.3) for all $t \in]0,T]$ we have

$$\|DP_t f\|_\infty \le \frac{C_T}{\sqrt{t}} \|f\|_\infty \le C_T \frac{e^{\omega t}}{\sqrt{t}} \|f\|_\infty\ , \qquad 0 < t \le T,$$

while for all $t > T$

$$\|DP_t f\|_\infty = \|DP_T\, P_{t-T} f\|_\infty \le \frac{C_T}{\sqrt{T}} \|P_{t-T} f\|_\infty \le \frac{C_T}{\sqrt{T}} \|f\|_\infty \le \frac{C_T}{\sqrt{T}} \frac{e^{\omega t}}{\sqrt{t}} \|f\|_\infty, \quad t > T.$$

So the statement follows with $C_\omega = \max\left\{C_T, \frac{C_T}{\sqrt{T}}\right\}$. □

We remark that the semigroup $(P_t)_{t\ge0}$ is not strongly continuous in $C_b(\overline{\Omega})$ in general: this is shown by the example $\Omega = \mathbb{R}^N$ and $\mathcal{A} = \Delta$. As in the case $\Omega = \mathbb{R}^N$ (see Section 5.2), we can introduce the weak generator $(\widehat{A}, D(\widehat{A}))$ defined by

$$\begin{aligned} D(\widehat{A}) &= \Big\{ f \in C_b(\overline{\Omega}) : \sup_{t\in(0,1)} \frac{\|P_t f - f\|}{t} < \infty \ \text{ and } \ \exists g \in C_b(\overline{\Omega}) \ \text{ such that} \\ &\qquad \lim_{t\to0} \frac{(P_t f)(x) - f(x)}{t} = g(x), \quad \forall\, x \in \overline{\Omega} \Big\} \\ \widehat{A} f(x) &= \lim_{t\to0} \frac{(P_t f)(x) - f(x)}{t}, \quad f \in D(\widehat{A}),\ x \in \overline{\Omega}. \end{aligned}$$

The following results are proved in [48]: if $f \in D(\widehat{A})$, then $P_t f \in D(\widehat{A})$ and $\widehat{A} P_t f = P_t \widehat{A} f$, for all $t \geq 0$. Moreover one has $(0, +\infty) \subset \rho(\widehat{A})$, $\|R(\lambda, \widehat{A})\| \leq 1/\lambda$ and

$$(R(\lambda, \widehat{A}) f)(x) = \int_0^{+\infty} e^{-\lambda t} (P_t f)(x)\, dt, \qquad x \in \overline{\Omega}, \tag{2.2.11}$$

and $R(\lambda, \widehat{A})$ is surjective from $C_b(\overline{\Omega})$ onto $D(\widehat{A})$ for all $\lambda > 0$.

Our aim now is to show that in fact $\widehat{A}$ coincides with the operator $\mathcal{A}$. This result is well known in the case where $\Omega = \mathbb{R}^N$. More precisely, one can prove that $\widehat{A} \subset \mathcal{A}$. If it is assumed that a Liapunov function exists, then one can check that also the other inclusion holds. We refer to Section 5.2, where the main properties concerning Feller semigroups in $\mathbb{R}^N$ are collected. If Ω is not the whole space, then the same result holds, but in proving it we have to pay attention to the boundary. Indeed, the main point in the proof below consists in applying suitable interior L^p estimates which involve also a part of $\partial\Omega$ (see (2.2.13)).

Proposition 2.2.4 *For all $f \in C_b(\overline{\Omega})$ and $\lambda > 0$, the function $u = R(\lambda, \widehat{A}) f$ belongs to $D(\mathcal{A})$ and solves problem (2.0.2). Moreover $D(\widehat{A}) = D(\mathcal{A})$ and $\widehat{A} v = \mathcal{A} v$ for all $v \in D(\mathcal{A})$.*

PROOF. Let $f \in C_b(\overline{\Omega})$ and let $u = R(\lambda, \widehat{A}) f$. For all $n \in \mathbb{N}$, let $u_n = R_n(\lambda, \mathcal{A}) f \in D_n(\mathcal{A})$, where $R_n(\lambda, \mathcal{A})$ is the resolvent of the operator $(\mathcal{A}, D_n(\mathcal{A}))$, that is

$$u_n(x) = \int_0^{+\infty} e^{-\lambda t} (T_n(t) f)(x) dt, \quad x \in \overline{\Omega}_n.$$

Taking into account the contractivity of $T_n(t)$, we have

$$\|u_n\|_\infty \leq \frac{1}{\lambda} \|f\|_\infty, \qquad \|\mathcal{A} u_n\|_\infty \leq 2\, \|f\|_\infty \tag{2.2.12}$$

for all $n \in \mathbb{N}$, and then from Theorem 2.2.1 and by dominated convergence it follows that

$$\lim_{n \to \infty} u_n = u,$$

pointwise in $\overline{\Omega}$ and in $L^p(\Omega_k)$, for all $k \in \mathbb{N}$. Furthermore, by Theorem C.2.1 we have

$$\|u_n - u_m\|_{W^{2,p}(\Omega_k)} \leq c(p, k) \Big(\|u_n - u_m\|_{L^p(\Omega_{k+1})} \Big), \qquad n, m > k, \tag{2.2.13}$$

for all $p \in (1, +\infty)$, where $c(p, k) > 0$ is a constant. Consequently u_n converges to u in $W^{2,p}(\Omega_k)$, for all $k \in \mathbb{N}$. Hence $u \in W^{2,p}(\Omega \cap B_R)$, for all $R < \infty$. Moreover by Sobolev embedding u_n converges to u in $C^1(\overline{\Omega}_k)$ for all $k \in \mathbb{N}$, and then we deduce that $\partial u / \partial \eta = 0$ in $\partial\Omega$. Finally, letting $n \to \infty$ in the equation $\lambda u_n - \mathcal{A} u_n = f$, it follows that $\lambda u - \mathcal{A} u = f$ in Ω. Therefore u belongs to $D(\mathcal{A})$ and it is a solution of problem (2.0.2).

In particular, since $R(\lambda, \widehat{A})$ is surjective from $C_b(\overline{\Omega})$ onto $D(\widehat{A})$, it follows that $D(\widehat{A}) \subset D(\mathcal{A})$. Conversely, let $u \in D(\mathcal{A})$ and define $f = \lambda u - \mathcal{A} u \in C_b(\overline{\Omega})$, where $\lambda \geq \lambda_0$ (see (2.1.5)). Then the function $v = R(\lambda, \widehat{A}) f$ is a bounded solution of problem (2.0.2). By Proposition 2.1.2 we have $u = v$, and in particular $u \in D(\widehat{A})$. □

A consequence of the gradient estimate (2.2.10) is that $D(\mathcal{A})$ is continuously embedded in $C_b^1(\overline{\Omega})$.

Proposition 2.2.5 *$D(\mathcal{A}) \subseteq C_b^1(\overline{\Omega})$. Moreover for all $\omega > 0$ there exists a constant $M_\omega > 0$ such that:*

$$\|Du\|_\infty \leq M_\omega \, \|u\|_\infty^{\frac{1}{2}} \, \|(\mathcal{A} - \omega)\, u\|_\infty^{\frac{1}{2}} \tag{2.2.14}$$

for all $u \in D(\mathcal{A})$.

PROOF. Let $u \in D(\mathcal{A})$, $\omega > 0$ and $\lambda > 0$. Then the function $f = (\lambda + \omega)\, u - \mathcal{A}u$ belongs to $C_b(\overline{\Omega})$ and

$$u(x) = (R(\lambda + \omega, \widehat{A})f)(x) = \int_0^{+\infty} e^{-(\lambda+\omega)\,t}(P_t f)(x)\, dt\ , \qquad x \in \overline{\Omega}.$$

By using estimate (2.2.10), we may differentiate under the integral sign obtaining

$$Du(x) = \int_0^{+\infty} e^{-(\lambda+\omega)\,t}(DP_t f)(x)\, dt, \qquad x \in \Omega$$

and

$$|Du(x)| \leq C_\omega \int_0^{+\infty} \frac{e^{-\lambda\, t}}{\sqrt{t}}\, dt\, \|f\|_\infty = \frac{M_\omega}{\sqrt{\lambda}}\|f\|_\infty\ , \qquad x \in \Omega,$$

where $M_\omega > 0$ is a constant. Therefore

$$\|Du\|_\infty \leq M_\omega\left(\sqrt{\lambda}\|u\|_\infty + \frac{\|(\mathcal{A} - \omega)\, u\|_\infty}{\sqrt{\lambda}}\right),$$

and, taking the minimum over λ, (2.2.14) follows. □

With the same technique as in Proposition 2.1.5 we get the following gradient estimate.

Proposition 2.2.6 *For every $T > 0$ there exists $C_T > 0$ such that*

$$\|DP_t f\|_\infty \leq C_T\left(\|f\|_\infty + \|Df\|_\infty\right)\ , \qquad 0 \leq t \leq T, \tag{2.2.15}$$

for every $f \in C^1_\eta(\overline{\Omega})$ (see (2.0.5)).

PROOF. We may suppose that $V \geq 1$; the general case follows considering the operator $\mathcal{A}' = \mathcal{A} - I$. We give the proof by steps; first we prove that there exists a constant $C_T > 0$ such that

$$|DT_n(t)f(x)| \leq C_T(\|f\|_\infty + \|Df\|_\infty)\ , \qquad 0 \leq t \leq T,\ x \in \overline{\Omega}_n, \tag{2.2.16}$$

for every $n \in \mathbb{N}$ and $f \in C^1_\eta(\overline{\Omega}_n)$. Since $D_n(\mathcal{A})$ (see (2.2.1)) is dense in $C^1_\eta(\overline{\Omega}_n)$, it is enough to prove (2.2.16) for $f \in D_n(\mathcal{A})$.

Let $f \in D_n(\mathcal{A})$ and define

$$w(t, x) = u^2(t, x) + a\,|Du(t, x)|^2, \qquad t > 0,\ x \in \Omega_n\ ,$$

where $u(t, x) = (T_n(t)f)(x)$ and $a > 0$ is a constant. Then $w \in C([0, T] \times \overline{\Omega}_n) \cap C^{0,1}(]0, T] \times \overline{\Omega}_n) \cap C^{1,2}(]0, T] \times \Omega_n)$ and from Lemma 2.1.3 it follows that

$$\frac{\partial w}{\partial \eta}(t, x) \leq 0, \qquad t > 0,\ x \in \partial\Omega_n.$$

Moreover w satisfies the equation

$$w_t(t, x) - \mathcal{A}w(t, x) = -2\sum_{i,j=1}^N q_{ij}(x)\, D_i u(t, x)\, D_j u(t, x) + h_1(t, x) + h_2(t, x),$$

where

$$\begin{aligned} h_1(t, x) &= 2\,a\sum_{i,j=1}^N D_i F_j(x)\, D_i u(t, x)\, D_j u(t, x) - aV(x)|Du(t, x)|^2 \\ &\quad -2\,au(t, x)Du(t, x)\cdot DV(x) - V(x)u^2(t, x), \\ h_2(t, x) &= 2\,a\left(\sum_{i,j,k=1}^N D_k q_{ij}(x)D_k u(t, x)D_{ij}u(t, x) - \sum_{i,j,k=1}^N q_{ij}(x)D_{ik}u(t, x)D_{jk}u(t, x)\right). \end{aligned}$$

The same estimates of the proof of Proposition 2.1.5 show that there exists a value of $a > 0$ independent of n such that

$$w_t(t,x) - \mathcal{A}w(t,x) \le 0, \qquad 0 \le t \le T,\ x \in \Omega_n\ .$$

Therefore the classical maximum principle yields

$$w(t,x) \le \sup_{x\in\overline{\Omega}_n} w(0,x) \le (\|f\|_\infty^2 + a\,\|Df\|_\infty^2), \qquad 0 \le t \le T,\ x \in \overline{\Omega}_n,$$

which implies (2.2.16) with $C_T = a^{-1/2} \vee 1$.

Let now $f \in C^1_\eta(\overline{\Omega})$. For all $k \in \mathbb{N}$, let $\theta_k \in C^1_b(\overline{\Omega})$ be a function with bounded support such that

$$0 \le \theta_k \le 1, \quad \|D\theta_k\|_\infty \le L,$$
$$\theta_k = 1 \ \text{ in } \Omega_k, \quad \frac{\partial\theta_k}{\partial\eta} = 0 \ \text{ in } \partial\Omega,$$

where $L > 0$ is a constant independent of $k \in \mathbb{N}$, and set $f_k = \theta_k f$. Then for all $n \in \mathbb{N}$ such that $\mathrm{supp}\,(\theta_k) \subset \Omega_n$ we have

$$\frac{\partial f_k}{\partial\eta}(x) = \frac{\partial\theta_k}{\partial\eta}(x)f(x) + \theta_k(x)\frac{\partial f}{\partial\eta}(x) = 0, \qquad x \in \partial\Omega_n,$$

that is $f_k \in C^1_\eta(\overline{\Omega}_n)$. Then $T_n(t)f_k$ satisfies estimate (2.2.16), and letting $n \to +\infty$ we get

$$|DP_t f_k(x)| \le C_T(\|f_k\|_\infty + \|Df_k\|_\infty) \le C_T((1+L)\|f\|_\infty + \|Df\|_\infty)\,, \qquad 0 \le t \le T,\ x \in \overline{\Omega}.$$

Taking into account Proposition 2.2.2 and letting $k \to \infty$ the statement follows. □

As a consequence we get the following result which will be used in the sequel.

Proposition 2.2.7 *If $f \in C^1_\eta(\overline{\Omega})$ then the function $DP_t f$ is continuous in $[0,+\infty) \times \overline{\Omega}$.*

Proof. Let $f \in C^1_\eta(\overline{\Omega})$. Taking account of Theorem 2.2.1 we have only to prove that $DP_t f$ is continuous at $t = 0$. Let $x_0 \in \overline{\Omega}$ be fixed and Ω_0, Ω_1, θ and $T(t)$ as in the proof of Theorem 2.2.1. We set

$$v(t,x) = \theta(x)(P_t f)(x), \qquad t \ge 0,\ x \in \overline{\Omega}_0,$$

and we prove that Dv is continuous at $(0, x_0)$; since $v(t,x) = (P_t f)(x)$ for all $x \in \Omega_1$ then the conclusion follows. We can write

$$v(t) = T(t)(\theta f) + \int_0^t T(t-s)\psi(s)ds,$$

where

$$\psi(t,x) = -P_t f(x)(\mathcal{A} + V(x))\theta(x) - 2\sum_{i,j=1}^N q_{ij}(x)D_iP_t f(x)D_j\theta(x).$$

From Proposition 2.2.6 it follows that

$$\|\psi(t)\|_\infty \le C_T(\|f\|_\infty + \|Df\|_\infty), \qquad 0 \le t \le T,$$

for some $C_T > 0$, where T is fixed, and then by (2.1.11) we have

$$\|DT(t-s)\psi(s)\|_\infty \le \frac{C}{\sqrt{t-s}}(\|f\|_\infty + \|Df\|_\infty), \qquad 0 < s < t \le T.$$

for some $C > 0$. Therefore

$$|Dv(t,x) - Df(x_0)| \le |DT(t)(\theta f)(x) - Df(x_0)| + 2C\sqrt{t}(\|f\|_\infty + \|Df\|_\infty),$$

for all $0 < t \le T$, $x \in \overline{\Omega}_0$. Taking account of

$$\lim_{(t,x)\to(0,x_0)} |DT(t)(\theta f)(x) - Df(x_0)| = 0, \tag{2.2.17}$$

we conclude that Dv is continuous at $(0, x_0)$. Relation (2.2.17) is immediate if $\theta f \in D_\eta(\mathcal{A})$, where $D_\eta(\mathcal{A})$ is the domain of the generator of $T(t)$, as in (2.1.7). Indeed in this case $T(t)(\theta f)$ belongs to $C([0,\infty); D_\eta(\mathcal{A}))$ and $D_\eta(\mathcal{A}) \subset C^1_\eta(\overline{\Omega}_0)$. In general we have $\theta f \in C^1_\eta(\overline{\Omega}_0)$ (see (2.2.5)), and (2.2.17) follows by approximation, since $D_\eta(\mathcal{A})$ is dense in $C^1_\eta(\overline{\Omega}_0)$. □

Remark 2.2.8 In the case $\Omega = \mathbb{R}^N$ the compactness of P_t in $C_b(\mathbb{R}^N)$ has been studied in [39]. The results extend to the case $\Omega \neq \mathbb{R}^N$, with the same proofs adapted to the Neumann problem. Assume that $V \equiv 0$, *i. e.* consider the conservative case where $P_t\mathbb{1} = \mathbb{1}$. First, P_t is compact in $C_b(\overline{\Omega})$ for all $t > 0$ if and only if for all $t, \varepsilon > 0$ there exists a bounded set $\Omega' \subset \Omega$ such that $p(t, x, \Omega') \ge 1 - \varepsilon$ for all $x \in \overline{\Omega}$. Secondly, if there exists a positive function $\psi \in C^2$ such that

$$\lim_{|x|\to+\infty} \psi(x) = +\infty, \qquad \frac{\partial\psi}{\partial\eta}(x) = 0, \ \ x \in \partial\Omega, \qquad \mathcal{A}\psi(x) \le -g(\psi(x)), \ \ x \in \Omega,$$

where $g : [0, +\infty[\to \mathbb{R}$ is a convex function such that $\lim_{x\to+\infty} g(x) = +\infty$ and $1/g$ is integrable at $+\infty$, then P_t is compact in $C_b(\overline{\Omega})$ for all $t > 0$.

2.3 Pointwise gradient estimates

In the whole section we assume that $V \equiv 0$ which implies that $P_t\mathbb{1} = \mathbb{1}$ for all $t > 0$, by uniqueness. Actually this is a necessary condition for the estimates that we are going to prove. Indeed, taking $f = \mathbb{1}$ in (2.3.1) it follows that $P_t\mathbb{1} = \mathbb{1}$.

Proposition 2.3.1 *Suppose $q_{ij}(x) \equiv \delta_{ij}$ for all $i, j = 1, ..., N$. Then for every $p \ge 1$ and $f \in C^1_\eta(\overline{\Omega})$ we have*

$$|DP_tf(x)|^p \le e^{pk_0t}P_t(|Df|^p)(x), \quad t \ge 0, \ x \in \overline{\Omega}. \tag{2.3.1}$$

Proof: It is sufficient to prove the case $p = 1$. For $p > 1$, we observe that since $P_t\mathbb{1} = \mathbb{1}$ the measures $p(t, x; dy)$ given by Proposition 2.2.2 are probability measures, and then Jensen's inequality yields

$$|DP_tf(x)|^p \le \left(e^{k_0t}P_t(|Df|)(x)\right)^p \le e^{k_0pt}P_t(|Df|^p)(x).$$

Let $f \in C^1_\eta(\overline{\Omega})$ and let $\varepsilon > 0$ be fixed. Set $u(t, x) = P_tf(x)$ and define the function

$$w(t,x) = \left(|Du(t,x)|^2 + \varepsilon\right)^{\frac{1}{2}}, \quad t > 0, \ x \in \Omega.$$

From Proposition 2.2.6 and Proposition 2.2.7 it follows that w is bounded and continuous in $[0, +\infty[\times\overline{\Omega}$. Since $u \in C^{1+\alpha/2,2+\alpha}_{\mathrm{loc}}(]0, +\infty[\times\overline{\Omega})$ (see Theorem 2.2.1), we have that $w \in C^{0,1}(]0, +\infty[\times\overline{\Omega})$. Finally, from [29, Theorem 8.12.1] we deduce that $w \in C^{1,2}(]0, +\infty[\times\Omega)$. From Lemma 2.1.3 it follows that

$$\frac{\partial w}{\partial\eta}(t,x) = \frac{1}{2}\left(|Du(t,x)|^2 + \varepsilon\right)^{-\frac{1}{2}} \frac{\partial}{\partial\eta}|Du|^2(t,x) \le 0, \quad t > 0, \ x \in \partial\Omega.$$

A straightforward computation shows that w satisfies the equation

$$w_t(t,x) - \mathcal{A}w(t,x) = g_1(t,x) + g_2(t,x)$$

where

$$\begin{aligned} g_1 &= \left(|Du|^2+\varepsilon\right)^{-\frac{1}{2}} \sum_{i,j=1}^N (D_iF_j)(D_iu)(D_ju) \\ g_2 &= \left(|Du|^2+\varepsilon\right)^{-\frac{3}{2}} \sum_{i=1}^N \left(\sum_{j=1}^N (D_ju)(D_{ij}u)\right)^2 - \left(|Du|^2+\varepsilon\right)^{-\frac{1}{2}} \sum_{i,j=1}^N (D_{ij}u)^2 \,. \end{aligned}$$

We estimate now the functions g_1 and g_2. Since

$$\begin{aligned} \left(|Du|^2+\varepsilon\right)^{-\frac{3}{2}} \sum_{i=1}^N \left(\sum_{j=1}^N D_juD_{ij}u\right)^2 &\le \left(|Du|^2+\varepsilon\right)^{-\frac{3}{2}} |Du|^2 \sum_{i,j=1}^N (D_{ij}u)^2 \\ &\le \left(|Du|^2+\varepsilon\right)^{-\frac{1}{2}} \sum_{i,j=1}^N (D_{ij}u)^2. \end{aligned}$$

it follows that $g_2 \le 0$. On the other hand using (2.1.3) we obtain

$$g_1(t,x) \le k_0 \left(|Du(t,x)|^2+\varepsilon\right)^{-\frac{1}{2}} |Du(t,x)|^2 = k_0 w - k_0\varepsilon \left(|Du(t,x)|^2+\varepsilon\right)^{-\frac{1}{2}} \,.$$

If $k_0 \ge 0$ we have immediately

$$g_1(t,x) \le k_0 w\,,$$

whereas if $k_0 < 0$, we have

$$g_1(t,x) \le k_0 w - k_0\sqrt{\varepsilon}\,.$$

In any case we obtain

$$w_t - \mathcal{A}w \le k_0\,(w-\delta_\varepsilon)$$

where

$$\delta_\varepsilon = \begin{cases} 0 & k_0 \ge 0, \\ \sqrt{\varepsilon} & k_0 < 0. \end{cases}$$

Therefore the function $v = w - \delta_\varepsilon$ satisfies

$$\begin{cases} v_t(t,x) - \mathcal{A}v(t,x) \le k_0\, v(t,x) & t>0,\ x\in\Omega, \\ \dfrac{\partial v}{\partial \eta}(t,x) \le 0 & t>0,\ x\in\partial\Omega, \\ v(0,x) = \left(|Df(x)|^2+\varepsilon\right)^{\frac{1}{2}} - \delta_\varepsilon & x\in\overline{\Omega}. \end{cases}$$

On the other hand, the function

$$z(t,x) = e^{k_0t}P_t\left((|Df|^2+\varepsilon)^{\frac{1}{2}}\right)(x), \qquad t>0,\ x\in\Omega,$$

solves the problem

$$\begin{cases} z_t(t,x) - \mathcal{A}z(t,x) = k_0 z(t,x) & t>0,\ x\in\Omega, \\ \dfrac{\partial z}{\partial \eta}(t,x) = 0 & t>0,\ x\in\partial\Omega, \\ z(0,x) = (|Df(x)|^2+\varepsilon)^{\frac{1}{2}} & x\in\overline{\Omega}. \end{cases}$$

Therefore Proposition 2.1.1 applied to $v-z$ and to the operator $\mathcal{A}+k_0I$ yields $v \le z$, that is

$$\left(|Du(t,x)|^2+\varepsilon\right)^{\frac{1}{2}} - \delta_\varepsilon \le e^{k_0t}P_t\left((|Df|^2+\varepsilon)^{\frac{1}{2}}\right)(x) \quad t\ge 0,\ x\in\overline{\Omega}.$$

Letting $\varepsilon \to 0$ estimate (2.3.1) with $p=1$ follows. □

We now consider the case of variable second order coefficients. Under the assumption

$$\sum_{i,j=1}^{N} (Dq_{ij}(x)\cdot\xi)^2 \le q_0\nu(x)|\xi|^2, \quad x\in\Omega,\ \xi\in\mathbb{R}^N, \tag{2.3.2}$$

which is slightly stronger than (2.1.2), we generalize the previous result when $p>1$.

Proposition 2.3.2 *Suppose that (2.3.2) holds. Then*

$$|DP_tf(x)|^p \le e^{\sigma_p t}P_t(|Df|^p)(x), \quad t\ge 0,\ x\in\overline{\Omega}, \tag{2.3.3}$$

for all $p>1$ and $f\in C^1_\eta(\overline{\Omega})$, where $\sigma_p = pk_0 + \frac{p}{4}q_0$ if $p\ge 2$ and $\sigma_p = pk_0 + \frac{p}{4(p-1)}q_0$ if $1<p<2$.

Proof. Let $f\in C^1_\eta(\overline{\Omega})$ be fixed. We first prove the statement for $p=2$. Consider the function

$$w(t,x) = |Du(t,x)|^2, \quad t>0,\ x\in\Omega,$$

where $u(t,x)=(P_tf)(x)$; then $w\in C([0,+\infty[\times\overline{\Omega})\cap C^{0,1}(]0,+\infty[\times\overline{\Omega})\cap C^{1,2}(]0,+\infty[\times\Omega)$, and from Lemma 2.1.3 we have

$$\frac{\partial w}{\partial\eta}(t,x)\le 0, \quad t>0,\ x\in\partial\Omega.$$

Moreover it is readily seen that

$$w_t(t,x) - \mathcal{A}w(t,x) = f_0(t,x),$$

where

$$f_0 = 2\left(\sum_{i,j,k} D_kq_{ij}D_kuD_{ij}u + \sum_{j,k} D_kF_jD_kuD_ju - \sum_{i,j,k} q_{ij}D_{ik}uD_{jk}u\right).$$

From (2.3.2) it follows that

$$\begin{aligned}
\sum_{i,j,k=1}^{N} D_kq_{ij}(x)D_kuD_{ij}u &\le \left(\sum_{i,j=1}^{N}(D_{ij}u)^2\right)^{1/2}\left(\sum_{i,j=1}^{N}(Dq_{ij}\cdot Du)^2\right)^{1/2}\\
&\le \left(\sum_{i,j=1}^{N}(D_{ij}u)^2\right)^{1/2}\left(q_0\nu(x)|Du|^2\right)^{1/2}\\
&\le \nu(x)\sum_{i,j=1}^{N}(D_{ij}u)^2 + \frac14 q_0|Du|^2, \qquad (2.3.4)
\end{aligned}$$

and then using (2.1.3) we get

$$\begin{aligned}
f_0(t,x) &\le 2\left(\nu(x)\sum_{i,j=1}^{N}(D_{ij}u)^2 + \frac14 q_0|Du|^2 + k_0|Du|^2 - \nu(x)\sum_{i,j=1}^{N}(D_{ij}u)^2\right)\\
&= \left(2k_0+\frac{q_0}{2}\right)|Du|^2 = \sigma_2|Du|^2
\end{aligned}$$

On the other hand the function

$$z(t,x) = e^{\sigma_2 t}P_t(|Df|^2)(x), \qquad t>0,\ x\in\Omega,$$

is the solution of the problem

$$\begin{cases} z_t(t,x) - \mathcal{A}z(t,x) = \sigma_2 z(t,x) & t>0,\ x\in\Omega, \\ \dfrac{\partial z}{\partial\eta}(t,x) = 0 & t>0,\ x\in\partial\Omega, \\ z(0,x) = |Df(x)|^2 & x\in\overline{\Omega}. \end{cases}$$

Using Proposition 2.1.1 we can conclude that $w \le z$, that is (2.3.3) with $p=2$.

Now the case $p>2$ follows easily applying Jensen's inequality:

$$|DP_tf(x)|^p \le \left(e^{\sigma_2 t}P_t(|Df|^2)(x)\right)^{\frac{p}{2}} \le e^{\sigma_p t}P_t(|Df|^p)(x), \quad t>0,\ x\in\Omega.$$

Assume $1<p<2$. Fix $\varepsilon>0$ and define the function

$$w(t,x) = \left(|Du(t,x)|^2+\varepsilon\right)^{\frac{p}{2}},$$

where $u(t,x) = (P_tf)(x)$. Then $w\in C([0,+\infty[\times\overline{\Omega})\cap C^{0,1}(]0,+\infty[\times\overline{\Omega})\cap C^{1,2}(]0,+\infty[\times\Omega)$, and from Lemma 2.1.3 we have

$$\frac{\partial w}{\partial\eta}(t,x) = \frac{p}{2}\left(|Du(t,x)|^2+\varepsilon\right)^{\frac{p}{2}-1}\frac{\partial}{\partial\eta}|Du(t,x)|^2 \le 0, \quad t>0,\ x\in\partial\Omega.$$

Moreover it turns out that

$$w_t(t,x) - \mathcal{A}w(t,x) = f_1(t,x)+f_2(t,x),$$

where

$$f_1 = p\left(|Du|^2+\varepsilon\right)^{\frac{p-2}{2}} f_0$$

$$f_2 = p\,(2-p)\left(|Du|^2+\varepsilon\right)^{\frac{p-4}{2}}\sum_{i,j,k,h} q_{ij}D_kuD_{jk}uD_huD_{ih}u$$

Taking into account (2.3.4) for all $\delta>0$ we have

$$f_1 \le p\left(|Du|^2+\varepsilon\right)^{\frac{p-2}{2}}\left(\delta\nu(x)\sum_{i,j=1}^N (D_{ij}u)^2 + \frac{1}{4\delta}q_0|Du|^2 + k_0|Du|^2 - \sum_{i,j,k=1}^N q_{ij}D_{jk}uD_{ik}u\right).$$

As far as f_2 is concerned, we set $A_{kh} = \sum_{i,j=1}^N q_{ij}D_{jk}uD_{ih}u$ and we observe that, since the matrix $A=(A_{kh})$ is symmetric and nonnegative definite, we have $\sum_{k,h=1}^N A_{kh}D_huD_ku \le \mathrm{Tr}(A)|Du|^2$, where $\mathrm{Tr}(A)$ denotes the trace of A. Therefore

$$\begin{aligned} f_2 &= p(2-p)\left(|Du|^2+\varepsilon\right)^{\frac{p-4}{2}}\sum_{k,h=1}^N A_{kh}D_kuD_hu \\ &\le p(2-p)\left(|Du|^2+\varepsilon\right)^{\frac{p-2}{2}}\sum_{i,j,k=1}^N q_{ij}D_{jk}uD_{ik}u. \end{aligned}$$

Choosing $\delta = p-1$ we get

$$\begin{aligned} f_1+f_2 &\le p\left(|Du|^2+\varepsilon\right)^{\frac{p-2}{2}}\Bigg((p-1)\nu(x)\sum_{i,j=1}^N (D_{ij}u)^2 + \Big(\frac{q_0}{4(p-1)}+k_0\Big)|Du|^2 \\ &\quad +(1-p)\sum_{i,j,k=1}^N q_{ij}D_{jk}uD_{ik}u\Bigg) \\ &\le \left(pk_0+\frac{p}{4(p-1)}q_0\right)\left(|Du|^2+\varepsilon\right)^{\frac{p-2}{2}}|Du|^2 = \sigma_p w - \varepsilon\sigma_p\left(|Du|^2+\varepsilon\right)^{\frac{p-2}{2}}, \end{aligned}$$

which implies

$$w_t - \mathcal{A}w \le \sigma_p(w - \delta_\varepsilon),$$

where

$$\delta_\varepsilon = \begin{cases} 0 & \text{if } \ \sigma_p \ge 0, \\ \varepsilon^{\frac{p}{2}} & \text{if } \ \sigma_p < 0. \end{cases}$$

Now the conclusion of the proof is the same as in Proposition 2.3.1: applying Proposition 2.1.1 to compare with $z(t,x) = e^{\sigma_p t} P_t((|Df|^2 + \varepsilon)^{\frac{p}{2}})$ we deduce

$$\left(|Du(t,x)|^2 + \varepsilon\right)^{\frac{p}{2}} - \delta_\varepsilon \le e^{\sigma_p t} P_t\left((|Df|^2 + \varepsilon)^{\frac{p}{2}}\right)(x), \qquad t \ge 0, \ x \in \overline{\Omega},$$

and then (2.3.3) follows letting $\varepsilon \to 0$. □

In the following proposition we deduce from (2.3.3) another type of pointwise gradient estimate. The basic idea of the proof is taken from [7] where it is considered the case $p = 2$.

Proposition 2.3.3 *Assume that (2.3.2) holds. Then for all $f \in C_b(\overline{\Omega})$ we have*

$$(2.3.5) \qquad |DP_t f(x)|^p \le \left(\frac{\sigma_2 \nu_0^{-1}}{2(1 - e^{-\sigma_2 t})}\right)^{\frac{p}{2}} P_t(|f|^p)(x), \qquad t > 0, \ x \in \overline{\Omega},$$

for all $p \ge 2$, and

$$(2.3.6) \qquad |DP_t f(x)|^p \le \frac{c_p \nu_0^{-1} \sigma_p}{t^{p/2-1}(1 - e^{-\sigma_p t})} P_t(|f|^p)(x), \qquad t > 0, \ x \in \overline{\Omega},$$

for all $1 < p < 2$, where $c_p = 2^p/(p(p-1))^{p/2}$ and σ_p is given by Proposition 2.3.2. When $\sigma_p = 0$ in (2.3.5) and (2.3.6) we replace $\sigma_p/(1 - e^{-\sigma_p t})$ by $1/t$.

Proof. We prove that $T_n(t)f$ satisfies estimates (2.3.5) and (2.3.6) for $x \in \overline{\Omega}_n$, for all $n \in \mathbb{N}$; then the conclusion follows letting $n \to \infty$. Fix $n \in \mathbb{N}$ and set $T_t = T_n(t)$, for simplicity. Note that T_t satisfies estimate (2.3.3) for all the functions in $C^1_\eta(\overline{\Omega}_n)$.

First we consider the case $p = 2$. Let $f \in C_b(\overline{\Omega})$, fix $t > 0$ and set

$$\Phi(s) = T_s\left((T_{t-s}f)^2\right), \quad 0 \le s \le t - \varepsilon \ ,$$

where $\varepsilon > 0$. From the analiticity of T_t it follows that $g = T_{t-s}f \in D_n(\mathcal{A})$, for all $0 \le s \le t - \varepsilon$ (we recall that $D_n(\mathcal{A})$ is the domain of the generator of T_t, defined in (2.2.1)). Moreover from a direct calculation it is readily seen that $g^2 \in D_n(\mathcal{A})$ and

$$\Phi'(s) = \mathcal{A}T_s(g^2) - 2\,T_s(g\mathcal{A}g) = T_s(\mathcal{A}(g^2) - 2g\mathcal{A}g) = 2T_s(\langle qDg, Dg\rangle).$$

Thus

$$\Phi(t-\varepsilon) - \Phi(0) = T_{t-\varepsilon}((T_\varepsilon f)^2) - (T_t f)^2 = 2\int_0^{t-\varepsilon} T_s(\langle qDT_{t-s}f, DT_{t-s}f\rangle)\, ds.$$

Now, applying Proposition 2.3.2 to $T_{t-s}f$ we obtain

$$T_s(\langle qDT_{t-s}f, DT_{t-s}f\rangle) \ge \nu_0 T_s(|DT_{t-s}f|^2) \ge \nu_0 e^{-\sigma_2 s}|DT_t f|^2,$$

so that

$$T_{t-\varepsilon}((T_\varepsilon f)^2) - (T_t f)^2 \ge 2\,\nu_0 |DT_t f|^2 \int_0^{t-\varepsilon} e^{-\sigma_2 s}\, ds = \frac{2\nu_0(1 - e^{-\sigma_2(t-\varepsilon)})}{\sigma_2}|DT_t f|^2 \ ,$$

and then

$$|DT_tf|^2 \le \frac{\sigma_2\,\nu_0^{-1}}{2(1-e^{-\sigma_2(t-\varepsilon)})}\Big(T_{t-\varepsilon}((T_\varepsilon f)^2)-(T_tf)^2\Big) \le \frac{\sigma_2\,\nu_0^{-1}}{2(1-e^{-\sigma_2(t-\varepsilon)})}\,T_{t-\varepsilon}((T_\varepsilon f)^2).$$

Letting $\varepsilon \to 0$ we obtain our claim.

If $p > 2$, using Jensen's inequality we get

$$|DT_tf|^p \le \left(\frac{\sigma_2\,\nu_0^{-1}}{2(1-e^{-\sigma_2 t})}T_t(f^2)\right)^{\frac{p}{2}} \le \left(\frac{\sigma_2\,\nu_0^{-1}}{2(1-e^{-\sigma_2 t})}\right)^{\frac{p}{2}} T_t(|f|^p).$$

Now assume $1 < p < 2$. Let first $f \in C_b(\overline{\Omega})$ with $f \ge \delta$ for some $\delta > 0$. Fix $t, \varepsilon > 0$ and define the function

$$\Psi(s) = T_s\left((T_{t-s}f)^p\right) \qquad 0 \le s \le t-\varepsilon.$$

Then $g = T_{t-s}f \ge \delta > 0$ and a straightforward computation shows that

$$\mathcal{A}(g^p) = pg^{p-1}\mathcal{A}g + p(p-1)g^{p-2}\langle qDg, Dg\rangle, \qquad \frac{\partial g^p}{\partial \eta} = pg^{p-1}\frac{\partial g}{\partial \eta}$$

which imply that $g^p \in D_n(\mathcal{A})$, since $g \in D_n(\mathcal{A})$. Moreover

$$\Psi'(s) = T_s\Big(\mathcal{A}(g^p) - pg^{p-1}\mathcal{A}g\Big) = p(p-1)\,T_s\Big((T_{t-s}f)^{p-2}\langle qDT_{t-s}f, DT_{t-s}f\rangle\Big),$$

and hence

$$T_{t-\varepsilon}((T_\varepsilon f)^p) - (T_tf)^p = p(p-1)\int_0^{t-\varepsilon} T_s\Big((T_{t-s}f)^{p-2}\langle qDT_{t-s}f, DT_{t-s}f\rangle\Big)\,ds. \tag{2.3.7}$$

Applying Proposition 2.3.2 and Hölder's inequality we get for all $\beta \in \mathbb{R}$

$$\begin{aligned}
|DT_tf|^p &= |DT_sT_{t-s}f|^p \le e^{\sigma_p s}T_s(|DT_{t-s}f|^p)\\
&= e^{\sigma_p s}T_s\Big(|DT_{t-s}f|^p\,(T_{t-s}f)^{-\beta}\,(T_{t-s}f)^{\beta}\Big)\\
&\le e^{\sigma_p s}\left\{T_s\left(|DT_{t-s}f|^2\,(T_{t-s}f)^{-\frac{2\beta}{p}}\right)\right\}^{p/2}\left\{T_s\,(T_{t-s}f)^{\frac{2\beta}{2-p}}\right\}^{1-p/2}\\
&\le e^{\sigma_p s}\nu_0^{-1}\left\{T_s\left(\langle qDT_{t-s}f, DT_{t-s}f\rangle\,(T_{t-s}f)^{-\frac{2\beta}{p}}\right)\right\}^{p/2}\left\{T_s\,(T_{t-s}f)^{\frac{2\beta}{2-p}}\right\}^{1-p/2}.
\end{aligned}$$

Choosing $\beta = p(2-p)/2$ and using Jensen's and Young's inequalities we get for all $\delta > 0$

$$\begin{aligned}
|DT_tf|^p &\le \nu_0^{-1}e^{\sigma_p s}\left\{T_s\left(\langle qDT_{t-s}f, DT_{t-s}f\rangle\,(T_{t-s}f)^{p-2}\right)\right\}^{p/2}\left\{T_s\,(T_{t-s}f)^p\right\}^{1-p/2}\\
&\le \nu_0^{-1}e^{\sigma_p s}\left\{T_s\left(\langle qDT_{t-s}f, DT_{t-s}f\rangle\,(T_{t-s}f)^{p-2}\right)\right\}^{p/2}\left\{T_t(f^p)\right\}^{1-p/2}\\
&\le \nu_0^{-1}e^{\sigma_p s}\left\{\frac{p}{2}\delta^{\frac{2}{p}}T_s\left(\langle qDT_{t-s}f, DT_{t-s}f\rangle\,(T_{t-s}f)^{p-2}\right) + \Big(1-\frac{p}{2}\Big)\,\delta^{\frac{2}{p-2}}T_t(f^p)\right\},
\end{aligned}$$

so that

$$\nu_0e^{-\sigma_p s}|DT_tf|^p \le \frac{p}{2}\delta^{\frac{2}{p}}T_s\left(\langle qDT_{t-s}f, DT_{t-s}f\rangle\,(T_{t-s}f)^{p-2}\right) + \Big(1-\frac{p}{2}\Big)\,\delta^{\frac{2}{p-2}}T_t(f^p).$$

Integrating from 0 to $t-\varepsilon$ and using (2.3.7) we get

$$\begin{aligned}
\frac{\nu_0(1-e^{-\sigma_p(t-\varepsilon)})}{\sigma_p}|DT_tf|^p &\le \frac{p}{2}\delta^{\frac{2}{p}}\int_0^{t-\varepsilon} T_s\left(\langle qDT_{t-s}f, DT_{t-s}f\rangle\,(T_{t-s}f)^{p-2}\right)ds\\
&\quad + \Big(1-\frac{p}{2}\Big)\,\delta^{\frac{2}{p-2}}\,(t-\varepsilon)\,T_t(f^p)\\
&= \frac{p}{2}\delta^{\frac{2}{p}}\,\frac{T_{t-\varepsilon}((T_\varepsilon f)^p) - (T_tf)^p}{p(p-1)} + \Big(1-\frac{p}{2}\Big)\,\delta^{\frac{2}{p-2}}\,(t-\varepsilon)\,T_t(f^p)
\end{aligned}$$

and then letting $\varepsilon \to 0$

$$|DT_t f|^p \le \frac{\nu_0^{-1}\sigma_p}{1-e^{-\sigma_p t}} T_t(f^p) \left(\frac{p}{2}\delta^{\frac{2}{p}} \frac{1}{p(p-1)} + \left(1-\frac{p}{2}\right)\delta^{\frac{2}{p-2}}\, t \right).$$

Taking the optimal choice $\delta = \{p(p-1)t\}^{\frac{p(2-p)}{4}}$ we finally obtain

$$|DT_t f|^p \le \frac{\nu_0^{-1}\sigma_p}{[p(p-1)]^{p/2} t^{p/2-1}(1-e^{-\sigma_p t})} T_t(f^p). \tag{2.3.8}$$

If $f \in C_b(\overline{\Omega})$ and $f \ge 0$ then (2.3.8) follows by approximating f with $f + \frac{1}{n}$ and using Proposition 2.2.2. If $f \in C_b(\overline{\Omega})$ then

$$\begin{aligned} |DT_t f|^p &= |DT_t(f^+ - f^-)|^p \le 2^{p-1}(|DT_t(f^+)|^p + |DT_t(f^-)|^p) \\ &\le \frac{2^{p-1}\nu_0^{-1}\sigma_p}{[p(p-1)]^{p/2} t^{p/2-1}(1-e^{-\sigma_p t})} \left(T_t((f^+)^p) + T_t((f^-)^p)\right) \\ &\le \frac{2^{p}\nu_0^{-1}\sigma_p}{[p(p-1)]^{p/2} t^{p/2-1}(1-e^{-\sigma_p t})} T_t(|f|^p), \end{aligned}$$

which concludes the proof. □

Remark 2.3.4 If $\Omega = \mathbb{R}^N$, we can consider the case of operators with locally Hölder continuous but not differentiable coefficients. In the case of differentiable coefficients, (2.1.2) and (2.1.3) are consequences of

$$|q_{ij}(x) - q_{ij}(y)| \le M\nu(x)|x-y|, \qquad x, y \in \Omega, \tag{2.3.9}$$

$$(F(x) - F(y))\cdot(x-y) \le (\beta V(x) + k_0)|x-y|^2, \qquad x, y \in \Omega. \tag{2.3.10}$$

Assume that the coefficients q_{ij} and F_i belong to $C^{\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ and satisfy (2.3.9) and (2.3.10), and assume that $V \in C^{1+\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ and it satisfies (2.1.4). If one considers a standard family of mollifiers $(\zeta_\varepsilon)_{\varepsilon>0}$ and define $q^\varepsilon_{ij} = q_{ij} * \zeta_\varepsilon$ and $F^\varepsilon_i = F_i * \zeta_\varepsilon$, then the functions q^ε_{ij} and F^ε_i are regular and satisfy (2.3.9) and (2.3.10) with the same constants q_0, β, k_0 for all $\varepsilon > 0$. Therefore q^ε_{ij} and F^ε_i satisfy (2.1.2) and (2.1.3); if $\mathcal{A}^\varepsilon$ denotes the operator with coefficients q^ε_{ij}, F^ε_i and V, and if P^ε_t denotes the associated semigroup, then P^ε_t satisfies all the gradient estimates that we have proved, with the same constants for all $\varepsilon > 0$. As $\varepsilon \to 0$ we get the gradient estimates for the semigroup P_t associated with the operator with coefficients q_{ij}, F_i and V. Indeed from the interior estimates [30, Theorem IV.10.1] it follows that $P^\varepsilon_t f \to P_t f$ in $C^{1,2}_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^N)$.

2.4 Consequences and counterexamples

The aim of this section is to show on one hand some consequences of the gradient estimates proved so far and on the other two counterexamples to some of them.

We start by giving a new formulation of the uniform gradient estimate (2.2.3): now we precise how the constant C_T depends on the operator $\mathcal{A}$. This allows us to deduce a Liouville type theorem.

Corollary 2.4.1 *Suppose that $V \equiv 0$ and (2.3.2) holds. Then for every $f \in C_b(\overline{\Omega})$*

$$\|DP_t f\|_\infty \le \left(\frac{\nu_0^{-1}\sigma_2}{2(1-e^{-\sigma_2 t})} \right)^{\frac{1}{2}} \|f\|_\infty\ , \qquad t > 0\ ,$$

if $\sigma_2 \neq 0$ *and*

$$\|DP_t f\|_\infty \le \left(\frac{1}{2\,\nu_0\,t}\right)^{\frac{1}{2}} \|f\|_\infty\ , \qquad t > 0\ ,$$

if $\sigma_2 = 0$.

The proof is an easy consequence of Proposition (2.3.3) with $p = 2$.

Proposition 2.4.2 *Suppose that* $V \equiv 0$*, (2.3.2) holds and* $\sigma_2 = 2k_0 + \frac{1}{2}q_0 \le 0$*. If* $f \in D(\mathcal{A})$ *is such that* $\mathcal{A}f = 0$ *then* f *is constant.*

Proof. Let $f \in D(\mathcal{A})$ and $\mathcal{A}f = 0$. Then $P_t f = f$, for all $t \ge 0$. Applying Corollary 2.4.1 and letting $t \to +\infty$ it turns out that $Df \equiv 0$ and consequently f is constant. □

Now we assume that $(P_t)_{t\ge0}$ extends to a contractive semigroup in $L^1_\mu(\Omega) = L^1(\Omega,\mu)$, for some measure μ. Then, by interpolation, P_t extends to a contractive semigroup in $L^p_\mu(\Omega)$ for all $1 \le p < \infty$.

In this situation, the pointwise gradient estimates of Section 2.3 imply global gradient estimates with respect to the L^p-norm. Moreover, if $(A_p, D(A_p))$ denotes the generator of P_t in $L^p_\mu(\Omega)$, we deduce that $D(A_p)$ embeds continuously in $W^{1,p}_\mu(\Omega)$.

Proposition 2.4.3 *Suppose that* $V \equiv 0$ *and (2.3.2) holds. For all* $f \in L^p_\mu(\Omega)$*, we have* $P_t f \in W^{1,p}_\mu(\Omega)$ *and*

$$\|DP_t f\|_p \le \left(\frac{\nu_0^{-1}\sigma_2}{2(1-e^{-\sigma_2 t})}\right)^{\frac{1}{2}} \|f\|_p\ , \qquad t>0,\ p\ge 2 \tag{2.4.1}$$

$$\|DP_t f\|_p \le t^{\frac{1}{p}-\frac{1}{2}} \left(\frac{c_p\,\nu_0^{-1}\sigma_p}{1-e^{-\sigma_p t}}\right)^{\frac{1}{p}} \|f\|_p\ , \qquad t>0,\ 1<p<2. \tag{2.4.2}$$

In the case where $\sigma_p = 0$*,* $\sigma_p/(1-e^{-\sigma_p t})$ *is replaced by* $1/t$*.*

Proof. Fix $p \ge 2$. If $f \in C_b(\overline{\Omega}) \cap L^p_\mu(\Omega)$ integrating (2.3.5) it follows that $P_t f \in W^{1,p}_\mu(\Omega)$ and it satisfies (2.4.1). If $f \in L^p_\mu(\Omega)$, take a sequence $(f_n) \subset C_b(\overline{\Omega}) \cap L^p_\mu(\Omega)$ that converges to f in $L^p_\mu(\Omega)$. Writing (2.4.1) for $f_n - f_m$ it follows that $P_t f_n$ is a Cauchy sequence in $W^{1,p}_\mu(\Omega)$. Therefore $P_t f \in W^{1,p}_\mu(\Omega)$ and it satisfies (2.4.1). The case $1 < p < 2$ follows similarly from (2.3.6). □

Corollary 2.4.4 *Suppose that* $V \equiv 0$*. For all* $p > 1$ *and* $\omega > 0$ *there exists* $C = C(p,\omega) > 0$ *such that*

$$\|DP_t f\|_p \le C\,\frac{e^{\omega t}}{\sqrt{t}}\,\|f\|_p\ , \qquad t>0, \tag{2.4.3}$$

for every $f \in L^p_\mu$*. Consequently,* $D(A_p) \subset W^{1,p}_\mu(\Omega)$ *and for all* $\omega > 0$ *there exists* $M_\omega > 0$ *such that*

$$\|Du\|_p \le M_\omega \|u\|_p^{\frac{1}{2}}\ \|(A_p-\omega)u\|_p^{\frac{1}{2}} \tag{2.4.4}$$

for all $u \in D(A_p)$*.*

PROOF. Fix $T > 0$. From Proposition 2.4.3 it follows that $\|DP_tf\|_p \le C_T t^{-1/2}\|f\|_p$ for every $t \in]0, T[$ and $f \in L^p_\mu(\Omega)$ for some constant $C_T > 0$. Therefore arguing as in Corollary 2.2.3 we get (2.4.3).

For the second statement, fix $\omega, \lambda > 0$. Let $f \in C_b(\overline{\Omega}) \cap L^p_\mu(\Omega)$ and set $u = R(\lambda + \omega, \mathcal{A})f$. Then

$$Du(x) = \int_0^{+\infty} e^{-(\lambda+\omega)t}(DP_tf)(x)dt, \qquad x \in \overline{\Omega}.$$

As in Proposition 2.2.5, with estimate (2.2.10) replaced by (2.4.3), we deduce that

$$\|Du\|_p \le M_\omega \|u\|_p^{\frac{1}{2}} \ \|(A_p - \omega)u\|_p^{\frac{1}{2}}.$$

Since $C_b(\overline{\Omega}) \cap L^p_\mu(\Omega)$ is dense in $L^p_\mu(\Omega)$, $R(\lambda, \mathcal{A})(C_b(\overline{\Omega}) \cap L^p_\mu(\Omega))$ is a core for $(A_p, D(A_p))$. Thus, the general case $u \in D(A_p)$ easily follows from the previous step by approximation. □

Remark 2.4.5 We note that, in particular, one may take as μ the invariant measure of P_t (when it exists), which is, by definition, a Borel probability measure such that

$$\int_\Omega P_t f d\mu = \int_\Omega f d\mu,$$

for all $t \ge 0$ and $f \in C_b(\overline{\Omega})$ (we refer to Chapter 5 for more details concerning invariant measures). In this case estimate (2.0.6) and (2.0.8) have interesting consequences. (2.0.6) with $p = 1$ and $k_0 < 0$ yields the hypercontractivity of (P_t) in $L^2(\Omega, \mu)$, which means that for every $f \in L^2(\Omega, \mu)$ one has

$$\|P_tf\|_{L^{q(t)}(\Omega,\mu)} \le \|f\|_{L^2(\Omega,\mu)}, \tag{2.4.5}$$

where $q(t) = 1 + e^{\lambda t}$ for a suitable $\lambda > 0$. One can check that (2.4.5) is equivalent to the logarithmic Sobolev inequality

$$\int_\Omega |f|^2 \log |f| d\mu \le \|f\|^2_{L^2(\Omega,\mu)} \log \|f\|_{L^2(\Omega,\mu)} + \frac{2}{\lambda}\int_\Omega |Df|^2 d\mu,$$

for every $f \in W^{1,2}(\Omega, \mu)$.

(2.0.8) with $p = 2$ and $\sigma_2 < 0$ yields the Poincaré inequality in $W^{1,2}(\Omega, \mu)$

$$\int_\Omega |f - \overline{f}\,|^2 d\mu \le C \int_\Omega |Df|^2 d\mu, \tag{2.4.6}$$

where $\overline{f} = \int_\Omega f d\mu$. As a consequence, one obtains the spectral gap for the generator A_2 of (P_t) in $L^2(\Omega, \mu)$, which means that

$$\sigma(A_2) \setminus \{0\} \subseteq \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \le -1/C\}$$

where C is determined by (2.4.6).

We do not enter in the details of such consequences, but we limit ourselves to mention them. We refer to [20, Section 10.5].

Example 2.4.6 This example shows that Proposition 2.3.3 fails in general for $p = 1$. Consider the heat semigroup in $\mathbb{R}$

$$P_tf(x) = \frac{1}{(4\pi t)^{1/2}} \int_\mathbb{R} e^{-\frac{(x-y)^2}{4t}} f(y)dy, \qquad t > 0,\ x \in \mathbb{R}$$

generated by the operator $\mathcal{A}u(x) = u''(x)$. The derivative is given by

$$DP_t f(x) = \frac{1}{2t(4\pi t)^{1/2}} \int_{\mathbb{R}} (y-x) e^{-\frac{(x-y)^2}{4t}} f(y)dy, \qquad t > 0,\ x \in \mathbb{R}.$$

Fix $R > 0$. Let $f \in C_b(\mathbb{R})$ be such that $0 \le f \le 1$, $f(x) = 0$ for $x < R - R^{-1}$ and $f(x) = 1$ for $x > R$. Then

$$P_t f(0) \le \frac{1}{(4\pi t)^{1/2}} \int_{R-R^{-1}}^{\infty} e^{-\frac{|y|^2}{4t}} dy, \qquad DP_t f(0) \ge \frac{1}{2t(4\pi t)^{1/2}} \int_R^{\infty} y e^{-\frac{|y|^2}{4t}} dy.$$

Therefore

$$DP_t f(0) \ge c_R P_t f(0), \qquad c_R = \frac{1}{2t} \int_R^{\infty} y e^{-\frac{|y|^2}{4t}} dy \left(\int_{R-R^{-1}}^{\infty} e^{-\frac{|y|^2}{4t}} dy \right)^{-1}.$$

Using the De L'Hôspital rule, it is readily seen that $c_R \to +\infty$ as $R \to +\infty$. This means that no pointwise estimate similar to (2.3.5) can hold for $p = 1$.

With the next counterexample we show that gradient estimate (2.2.3) is not true in general without assuming the dissipativity condition (2.1.3). In particular we show an example in which $D(\mathcal{A})$ is not contained in $C^1_\eta(\overline{\Omega})$.

Example 2.4.7 Consider in $\Omega = \mathbb{R}$ the operator

$$\mathcal{A}u(x) = u''(x) + B'(x)u'(x) = e^{-B(x)} \left(e^{B(x)} u'(x) \right)', \qquad x \in \mathbb{R},$$

where $B \in C^2(\mathbb{R})$ is such that $Q(x) = e^{B(x)} \int_0^x e^{-B(t)} dt \in L^1(\mathbb{R})$. Then, in particular $e^B \in L^1(\mathbb{R})$. Let $D(\mathcal{A}) = \{u \in C^2(\mathbb{R}) \cap C_b(\mathbb{R}) \,:\, \mathcal{A}u \in C_b(\mathbb{R})\}$. It follows from [55, page 242] (see also [40, Proposition 2.1]) that $(\mathcal{A}, D(\mathcal{A}))$ is the generator of a semigroup in $C_b(\mathbb{R})$ having $e^{B(x)}dx$ as its invariant measure.

If $f \in C_b(\mathbb{R})$, then the function

$$u(x) = C_1 + \int_0^x e^{-B(t)} \left(C_2 + \int_0^t f(s) e^{B(s)} ds \right) dt, \tag{2.4.7}$$

for arbitrary $C_1, C_2 \in \mathbb{R}$, is the general solution of the equation $\mathcal{A}u = f$. Assuming that

$$\int_{-\infty}^{+\infty} f(t) e^{B(t)} dt = 0, \tag{2.4.8}$$

and setting

$$C_2 = -\int_0^{+\infty} f(t) e^{B(t)} dt = \int_{-\infty}^0 f(t) e^{B(t)} dt,$$

for $x > 0$ (2.4.7) gives

$$\begin{aligned} u(x) &= C_1 - \int_0^x e^{-B(t)} \int_t^{+\infty} f(s) e^{B(s)} ds\, dt \\ &= C_1 - \int_0^{+\infty} e^{B(s)} f(s) \int_0^{s \wedge x} e^{-B(t)} dt\, ds. \end{aligned}$$

It follows that

$$|u(x)| \le |C_1| + \|f\|_\infty \int_0^{+\infty} Q(s) ds, \qquad x > 0,$$

which implies that u is bounded at $+\infty$. Similarly, since $Q \in L^1(]-\infty, 0[)$, u is bounded at $-\infty$. Since $\mathcal{A}u = f$, we conclude that $u \in D(\mathcal{A})$. The derivative of u is given by

$$u'(x) = -e^{-B(x)} \int_x^{+\infty} f(s) e^{B(s)} ds, \qquad x \in \mathbb{R}.$$

We claim that we can choose the functions B and f so that $Q \in L^1(\mathbb{R})$, (2.4.8) holds but u' is not bounded. To this aim, take

$$B(x) = -x^4 + \log h(x),$$

where $h \in C^2(\mathbb{R})$ satisfies

$$\begin{cases} h(x) = \varepsilon_n & \text{if } \ x = n - \dfrac{\delta_n}{2}, \ n \in \mathbb{N}, \\ \varepsilon_n \le h(x) \le 1 & \text{if } \ n - \delta_n < x < n, \ n \in \mathbb{N}, \\ h(x) = 1 & \text{otherwise,} \end{cases}$$

with

$$\varepsilon_n = \frac{1}{n} e^{\left(n - \frac{1}{2}\right)^4 - \left(n + \frac{1}{2}\right)^4}, \qquad \delta_n = \frac{e^{-n^4}}{n^2} \varepsilon_n \,.$$

As a consequence of this choice

$$Q(x) = e^{-x^4} \int_0^x e^{t^4} dt, \quad x < 0, \qquad Q(x) = h(x) e^{-x^4} \int_0^x \frac{e^{t^4}}{h(t)} dt, \quad x > 0.$$

Using the De L'Hôspital rule one sees that $\lim_{x \to -\infty} x^3 Q(x) = 1/4$ and hence that $Q \in L^1(]-\infty, 0[)$. If $x > 0$ then

$$\begin{aligned} Q(x) &\le e^{-x^4} \int_0^x \frac{e^{t^4}}{h(t)} dt \le e^{-x^4} \int_0^x e^{t^4} dt + e^{-x^4} \sum_{n=1}^{[x]+1} \int_{n-\delta_n}^n \frac{e^{n^4}}{\varepsilon_n} dt \\ &\le e^{-x^4} \int_0^x e^{t^4} dt + e^{-x^4} \sum_{n=1}^{\infty} \frac{\delta_n e^{n^4}}{\varepsilon_n} = e^{-x^4} \int_0^x e^{t^4} dt + e^{-x^4} \sum_{n=1}^{\infty} \frac{1}{n^2}, \end{aligned}$$

which shows that $Q \in L^1(]0, +\infty)$. Let $f \in C_b(\mathbb{R})$ be such that $f(x) = 1$ for all $x > 0$ and (2.4.8) holds. Then

$$u'(x) = -\frac{e^{x^4}}{h(x)} \int_x^{\infty} h(t) e^{-t^4} dt, \qquad x > 0$$

and in particular, at $x_n = n - \delta_n/2$

$$\begin{aligned} |u'(x_n)| &= \frac{e^{{x_n}^4}}{\varepsilon_n} \int_{x_n}^{+\infty} h(t) e^{-t^4} dt \ge \frac{e^{\left(n - \frac{1}{2}\right)^4}}{\varepsilon_n} \int_n^{n+\frac{1}{2}} e^{-t^4} dt \\ &\ge \frac{e^{\left(n - \frac{1}{2}\right)^4}}{2\varepsilon_n} e^{-\left(n + \frac{1}{2}\right)^4} = \frac{n}{2}, \end{aligned}$$

which implies that $u'(x)$ is unbounded at $+\infty$.

Therefore we have shown that the function u belongs to $D(\mathcal{A})$ but not to $C_b^1(\mathbb{R})$. This means that the gradient estimate (2.2.3) cannot be true. We note that in this situation the dissipativity assumption (2.1.3) fails since B'' is unbounded from above.

Example 2.4.8 We see now an example of a Neumann problem in a domain Ω with Lipschitz continuous boundary. In spite of the lower regularity of $\partial\Omega$, the associated semigroup satisfies the gradient estimate (2.3.1). Consider the Ornstein-Uhlenbeck operator

$$\mathcal{A}u(x) = \frac{1}{2}\Delta u(x) - x \cdot Du(x), \qquad x \in \mathbb{R}^N.$$

If we set

$$\begin{aligned} N(m,\sigma^2)(y) &= \frac{1}{\left(\sqrt{2\pi}\,\sigma\right)^N} e^{-\frac{|y-m|^2}{2\sigma^2}}, \qquad \sigma > 0,\ m, y \in \mathbb{R}^N, \\ \Gamma(t,x,y) &= N(e^{-t}x, 1-e^{-2t})(y), \quad t > 0,\ x, y \in \mathbb{R}^N, \end{aligned}$$

then the Ornstein-Uhlenbeck semigroup in $C_b(\mathbb{R}^N)$ is given by the formula

$$(U_t f)(x) = \int_{\mathbb{R}^N} f(y)\Gamma(t,x,y)dy, \quad t > 0,\ x \in \mathbb{R}^N.$$

We fix $k \in \mathbb{N}$, $0 \le k < N$ and we consider the domain $\Omega = \{x \in \mathbb{R}^N : x_{k+1}, ..., x_N > 0\}$. We define now the Ornstein-Uhlenbeck operator in Ω with Neumann boundary conditions. For $k+1 \le j \le N$ consider the reflections

$$\theta_j : \mathbb{R}^N \to \mathbb{R}^N, \qquad \theta_j x = (x_1, ..., x_{j-1}, -x_j, x_{j+1}, ..., x_N), \qquad x \in \mathbb{R}^N,$$

and the family

$$\Lambda = \{\theta = \theta_{i_1} \circ \cdots \circ \theta_{i_n},\ k+1 \le i_j \le N,\ i_j < i_h \text{ if } j < h,\ 1 \le n \le N-k\}.$$

Moreover if $f \in C_b(\overline{\Omega})$ we define the extension $Ef \in C_b(\mathbb{R}^N)$ by

$$(Ef)(x) = f(x_1, ..., x_k, |x_{k+1}|, ..., |x_N|), \quad x \in \mathbb{R}^N.$$

The Ornstein-Uhlenbeck semigroup in Ω is given by the formula

$$(P_t f)(x) = (U_t Ef)(x) = \int_{\mathbb{R}^N} (Ef)(y)\Gamma(t,x,y)dy, \qquad t > 0,\ x \in \Omega.$$

With the changes of variable $y' = \theta y$ and using the identity $\Gamma(t,x,\theta y) = \Gamma(t,\theta x,y)$ for all $\theta \in \Lambda$, we get

$$\begin{aligned} (P_t f)(x) &= \int_\Omega f(y)\Big\{\Gamma(t,x,y) + \sum_{\theta\in\Lambda}\Gamma(t,x,\theta y)\Big\}dy \\ &= \int_\Omega f(y)\Big\{\Gamma(t,x,y) + \sum_{\theta\in\Lambda}\Gamma(t,\theta x,y)\Big\}dy \qquad (2.4.9) \end{aligned}$$

The Neumann boundary condition can be verified in the following way. Let $x \in \partial\Omega$ be such that $x_j = 0$ for some $j \in \{k+1, ..., N\}$ and $x_i \neq 0$ for all $i \in \{k+1, ..., N\}$, $i \neq j$. Then the outward unit normal vector is $\eta(x) = -e_j$. For all $\theta \in \Lambda$ the normal derivative of the function $\Gamma(t,\theta x,y)$ is

$$\frac{\partial}{\partial x_j}\Gamma(t,\theta x,y) = \frac{(\pm y_j - e^{-t}x_j)e^{-t}}{(1-e^{-2t})}\Gamma(t,\theta x,y), \quad t > 0,\ x, y \in \Omega,$$

where in the right hand side we have the sign $+$ if θ does not contain the reflection θ_j and the sign $-$ otherwise. Let now $\theta \in \Lambda$ such that it does not contain the reflection θ_j and let $\theta' = \theta_j \circ \theta \in \Lambda$; then if $x_j = 0$ we have $\theta x = \theta' x$ and

$$\frac{\partial}{\partial x_j}\Gamma(t,\theta x,y) + \frac{\partial}{\partial x_j}\Gamma(t,\theta' x,y) = \frac{y_j}{(1-e^{-2t})}\Gamma(t,\theta x,y) - \frac{y_j}{(1-e^{-2t})}\Gamma(t,\theta' x,y) = 0,$$

for all $t > 0$ and $y \in \Omega$. Thus the Neumann boundary condition for $P_t f$ follows coupling in the sum in formula (2.4.9) all the maps $\theta \in \Lambda$ that does not contain the reflection θ_j with the respective maps $\theta' = \theta_j \circ \theta$. In this way all the terms of the sum are considered and the normal derivative turns out to be zero.

Since $DU_t Ef(x) = e^{-t} U_t(DEf)(x)$ for all $x \in \mathbb{R}^N$, we have

$$|DP_t f(x)| \leq e^{-t} U_t(|DEf|)(x) = e^{-t} P_t(|Df|)(x), \qquad t \geq 0,\ x \in \overline{\Omega},$$

that is P_t satisfies the gradient estimate (2.3.1) for $p = 1$ and hence for all $p \geq 1$.