Chapter 7

Binary operations on bivariate d.f.'s

Let H be a binary operation on $[0,1]$ and let Δ^2 be the set of bivariate d.f.'s. A binary operation η on Δ^2 is said to be *induced pointwise* by H if, for all A and B in Δ^2 and for all $(x,y) \in \overline{\mathbb{R}}^2$,

$$\eta(A,B)(x,y) = H(A(x,y), B(x,y)). \tag{7.1}$$

The function $\eta(A,B) : [0,1]^2 \to [0,1]$ given by (7.1) is called *composition* of A and B via H.

The major result of this chapter is the characterization of the induced pointwise operations on the set Δ^2 (section 7.2). A similar operation has been studied, in the univariate case, by C. Alsina *et al.* ([4]) in order to solve some problems arising in the theory of probabilistic metric spaces. However, in the bivariate case, the characterization is quite different and involves the new notion of "P–increasing function", a generalization of the 2–increasing functions, here introduced and studied (section 7.1). Section 7.3 is devoted mainly to questions related to the Fréchet classes and the convergence of d.f.'s. We conclude with some remarks of this problem on the class of copulas (section 7.4). These results can be also found in [45, 48, 38].

7.1 P–increasing functions

The focus of this section is on the new concept of P–increasing function, which will be needed for the characterization of induced pointwise operations on bivariate d.f.'s.

Definition 7.1.1. A function $H : [0,1]^2 \to [0,1]$ is said to be *P–increasing* (i.e. *probabilistically increasing*) if, and only if,

$$H(s_1,t_1) + H(s_4,t_4) \geq \max\left[H(s_2,t_2) + H(s_3,t_3), H(s_3,t_2) + H(s_2,t_3)\right], \tag{7.2}$$

for all $s_i, t_i \in [0,1]$ ($i \in \{1,2,3,4\}$) such that

$$s_1 \leq s_2 \wedge s_3 \leq s_2 \vee s_3 \leq s_4, \qquad t_1 \leq t_2 \wedge t_3 \leq t_2 \vee t_3 \leq t_4, \tag{7.3}$$

$$s_1 + s_4 \geq s_2 + s_3, \qquad t_1 + t_4 \geq t_2 + t_3. \tag{7.4}$$

Here we present a geometric interpretation of the P–increasing property.

Given s_i, t_i ($i \in \{1,2,3,4\}$) as in Definition 7.1.1, let

$$u_1 := s_2 \wedge s_3, \qquad u_4 := s_2 \vee s_3, \qquad v_1 := t_2 \wedge t_3, \qquad v_4 := t_2 \vee t_3.$$

Set

$$\begin{array}{llll} \mathbf{p} = (s_1,t_1), & \mathbf{q} = (s_4,t_1), & \mathbf{r} = (s_4,t_4), & \mathbf{s} = (s_1,t_4) \\ \mathbf{p'} = (u_1,v_1), & \mathbf{q'} = (u_4,v_1), & \mathbf{r'} = (u_4,v_4), & \mathbf{s'} = (u_1,v_4) \end{array}$$

Consider the rectangle R_1 with vertices **p**, **q**, **r** and **s**, and the rectangle R_2 with vertices **p'**, **q'**, **r'** and **s'**. Hence $R_2 \subseteq R_1$ and conditions (7.3) and (7.4) imply that the centre of R_2 lies below and to the left of the centre of R_1 (unless $R_1 = R_2$).

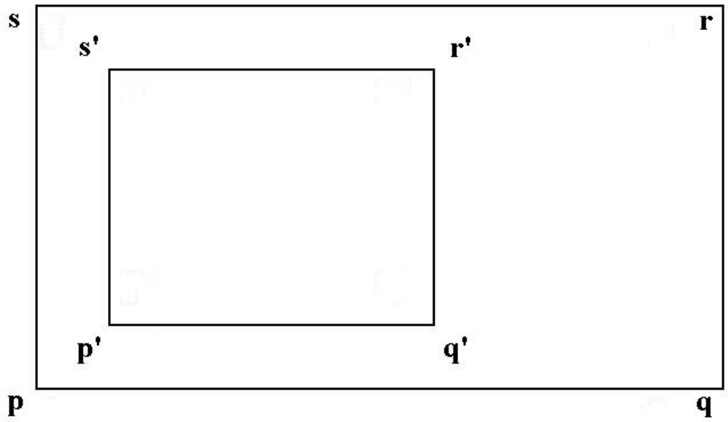

Figure 7.1: Geometric interpretation of the P–increasing property

Now, there are four choices for (u_1, v_1) – namely (s_2,t_2), (s_2,t_3), (s_3,t_2) and (s_3,t_3) – each leading to corresponding choices for the other vertices of R_2. For example, if

$(u_1, v_1) = (s_2, t_2)$ then $(u_4, v_4) = (s_3, t_3)$, and so on. In each case, (7.2) yields the two inequalities

$$H(\mathbf{p}) + H(\mathbf{r}) \geq H(\mathbf{p'}) + H(\mathbf{r'}),$$
$$H(\mathbf{p}) + H(\mathbf{r}) \geq H(\mathbf{q'}) + H(\mathbf{s'}).$$

In particular, when $R_1 = R_2$, the above inequalities establish that the P–increasing property implies the 2–increasing property.

Remark 7.1.1. Notice that conditions (7.3) and (7.4) on the points s_i and t_i ($i = 1, 2, 3, 4$) ensure that $(s_2, s_3) \prec_w (s_1, s_4)$ and $(t_2, t_3) \prec_w (t_1, t_4)$.

Remark 7.1.2. In the sequel, in order to prove that a function H is P–increasing, we restrict ourselves to showing that, for all s_i, t_i as in Definition 7.1.1,

$$H(s_1, t_1) + H(s_4, t_4) \geq H(s_2, t_2) + H(s_3, t_3), \tag{7.5}$$

instead of inequality (7.2) that can be easily obtained by means of a relabelling of the points. In fact, this was the primary definition of P–increasing function (see [45]). The equivalent definition given above was suggested by A. Sklar in a personal communication and it is adopted here because of its straightforward geometrical interpretation.

The P–increasing property is connected with the property of being directionally convex ([147, 111, 99]). We recall that a function $H : [0,1]^2 \to [0,1]$ is called *directionally convex* if, for all s_i, t_i ($i \in \{1, 2, 3, 4\}$) in $[0,1]$ such that (7.3) holds together with the condition, stronger than (7.4),

$$s_1 + s_4 = s_2 + s_3, \qquad t_1 + t_4 = t_2 + t_3, \tag{7.6}$$

we have

$$H(s_1, t_1) + H(s_4, t_4) \geq H(s_2, t_2) + H(s_3, t_3).$$

Theorem 7.1.1. *For a function $H : [0,1]^2 \to [0,1]$ the following statements are equivalent:*

(a) *H is P–increasing;*

(b) *H is directionally convex and increasing in each place.*

Proof. (a) $\Longrightarrow$ (b): Given a P–increasing function H, it suffices to show that H is increasing in each place. Consider $b \in [0,1]$ and, for all $i \in \{1, 2, 3, 4\}$, take s_i and t_i as in Definition 7.1.1, but satisfying the further conditions $s_1 = s_2$ and $t_i = b$. Hence

$$H(s_4, b) - H(s_3, b) - H(s_2, b) + H(s_2, b) \geq 0,$$

from which $H(s_4, b) \geq H(s_3, b)$, viz. $t \mapsto H(t, b)$ is increasing. The isotony of H in the other variable is established in an analogous manner.

(b) $\Longrightarrow$ (a): Let the s_1's and the t_i's ($i \in \{1, 2, 3, 4\}$) be as in Definition 7.1.1 and choose v_4 and w_4 in $[0, 1]$ such that $v_4 \in [s_2 \vee s_3, s_4]$, $w_4 \in [t_2 \vee t_3, t_4]$ and

$$s_1 + v_4 = s_2 + s_3, \qquad t_1 + w_4 = t_2 + t_3.$$

Hence

$$H(s_2, t_2) + H(s_3, t_3) \leq H(s_1, t_1) + H(v_4, w_4) \leq H(s_1, t_1) + H(s_4, t_4),$$

which is the desired conclusion. ☐

In particular, by using a characterization of the directionally convex functions ([111, Theorem 2.5]), we can obtain the following

Theorem 7.1.2. *A function* $H : [0, 1]^2 \to [0, 1]$ *is* P*–increasing if, and only if, the following statements hold:*

(a) H *is* 2*–increasing;*

(b) H *is increasing in each place;*

(c) H *is convex in each place.*

Note that the convex combinations of two P–increasing functions are P–increasing.

Corollary 7.1.1. *Let* $H : [0, 1]^2 \to [0, 1]$ *be* P*–increasing. The following statements hold:*

(a) H *is jointly continuous on* $[0, 1[^2$*;*

(b) $H \leq \Pi$.

Proof. (a): By classical properties of convex functions, it follows that every P–increasing function $H : [0, 1]^2 \to [0, 1]$ is continuous in each variable on $[0, 1[$ and then, in view of Proposition 2.1.2, it is jointly continuous on $[0, 1[^2$.

(b) If there exists (x_0, y_0) in $]0, 1[$ such that $H(x_0, y_0) > x_0 y_0$, then the horizontal section of H at y_0 is not be convex and, thus, H is not be P–increasing. ☐

Corollary 7.1.2. *Let* $H : [0, 1]^2 \to \mathbb{R}$ *be twice differentiable. Then* H *is* P*–increasing if, and only if, all the derivatives of the first and the second order of* H *are greater than (or equal to)* 0 *on* $[0, 1]^2$.

Example 7.1.1. The copulas Π and W are P–increasing, and so is their convex sum $C_\alpha = \alpha\Pi + (1 - \alpha)W$. But, the copula M is not P–increasing; in fact, if we consider s_i and t_i in $[0, 1]$ ($i \in \{1, 2, 3, 4\}$) such that

$$s_1 = 2/10 \leq s_2 = 3/10 = s_3 \leq s_4 = 5/10,$$
$$t_1 = 0 \leq t_2 = 3/10 = t_3 \leq t_4 = 1,$$

then

$$M(2/10,0) - M(3/10,3/10) - M(3/10,3/10) + M(5/10,1) = -1/10 < 0.$$

Notice that P–increasing copulas are associated with a random pair (X,Y) that is both SD$(X|Y)$ and SD$(Y|X)$ (see Proposition 1.7.3). For example, we can consider the family of copulas given, for every $\alpha \in \,]-1,0]$, by

$$C_\alpha(x,y) = xy + \alpha xy(1-x)(1-y),$$

which is a subclass of the FGM class (see Example 1.6.3).

Important examples of P–increasing functions are given by the following result.

Proposition 7.1.1. *Let f and g be increasing and convex functions from $[0,1]$ into $[0,1]$. Let $H : [0,1]^2 \to [0,1]$ be P–increasing. Then, the function $H_{f,g}$ defined by*

$$H_{f,g}(x,y) := H(f(x),g(y))$$

is P–increasing.

Proof. From Proposition 3.2.1, it follows that the function $H_{f,g}$ is a 2–increasing agop. Moreover, every horizontal (resp., vertical) section of H is convex, because it is composition of the convex and increasing horizontal (resp., vertical) section of A with f (resp. g). Now, the desired assertion follows from Theorem 7.1.2. □

Example 7.1.2. For every $\alpha, \beta \geq 1$, $\Lambda_{\alpha,\beta}(x,y) := \lambda x^\alpha + (1-\lambda)y^\beta$ ($\lambda \in [0,1]$) and $\Pi_{\alpha,\beta}(x,y) := x^\alpha \cdot y^\beta$ are P–increasing. In particular, the weighted arithmetic mean is P–increasing, but it is not the case of the weighted geometric mean. Consider, for instance, s_i and t_i in $[0,1]$ ($i \in \{1,2,3,4\}$) given by

$$s_1 = 0 < s_2 = \frac{4}{10} = s_3 < s_4 = \frac{8}{10}, \qquad t_1 = \frac{4}{10} < t_2 = \frac{7}{10} = t_3 < t_4 = 1,$$

then

$$\sqrt{s_1\,t_1} + \sqrt{s_4\,t_4} - \sqrt{s_2\,t_2} - \sqrt{s_3\,t_3} = \frac{\sqrt{80}}{10} - \frac{\sqrt{112}}{10} < 0.$$

7.2 Induced pointwise operations on d.f.'s

Here we characterize the induced pointwise operations on Δ^2.

Lemma 7.2.1. *If H is a 2–increasing agop, then, for all s, s', t, t' in $[0,1]$, it satisfies the condition*

$$|H(s',t') - H(s,t)| \leq |H(s',1) - H(s,1)| + |H(1,t') - H(1,t)|\,.$$

Family	Parameters
$\Lambda_{\alpha,\beta}(x,y) := \lambda x^\alpha + (1-\lambda) y^\beta$	$\alpha, \beta \geq 1$
$\Pi_{\alpha,\beta}(x,y) := x^\alpha \cdot y^\beta$	$\alpha, \beta \geq 1$
$F_\alpha(x,y) := \alpha xy + (1-\alpha)\max\{x+y-1, 0\}$	$\alpha \in [0,1]$
$G_\alpha(x,y) := xy + \alpha xy(1-x)(1-y)$	$\alpha \in [-1,0]$
$S_\alpha(x,y) := xy + \alpha \frac{\sin \pi x}{x} \frac{\sin \pi y}{y}$	$\alpha \in [-1,0]$
$M_\alpha(x,y) := xy + \alpha \min\{x, 1-x\} \min\{y, 1-y\}$	$\alpha \in [-1,0]$

Table 7.1: Family of P–increasing functions

Proof. Let s and s' be in $[0,1]$ with $s \leq s'$. Then, for every $t \in [0,1]$,

$$H(s',1) - H(s,1) \geq H(s',t) - H(s,t).$$

Similarly, for all $s \in [0,1]$ and for t and t' in $[0,1]$, with $t \leq t'$,

$$H(1,t') - H(1,t) \geq H(s,t') - H(s,t).$$

Therefore, for all s, s', t, t' in $[0,1]$, we have

$$\begin{aligned} |H(s',t') - H(s,t)| &\leq |H(s',t') - H(s,t')| + |H(s,t') - H(s,t)| \\ &\leq |H(s',1) - H(s,1)| + |H(1,t') - H(1,t)| . \end{aligned}$$ □

Theorem 7.2.1. *For a function* $H : [0,1]^2 \to [0,1]$ *the following statements are equivalent:*

(a) *H induces pointwise a binary operation η on Δ^2;*

(b) *H fulfils the conditions*

  (b.1) *$H(0,0) = 0$ and $H(1,1) = 1$,*

  (b.2) *H is P–increasing,*

  (b.3) *H is left–continuous in each place.*

Proof. (a) $\Longrightarrow$ (b): Let H induce pointwise the binary operation η on Δ^2, viz. for all A and B in Δ^2 and $(x,y) \in \overline{\mathbb{R}}^2$, the function

$$\eta(A,B)(x,y) := H(A(x,y), B(x,y))$$

is in Δ^2. For all 2–d.f.'s A and B we have

$$H(0,0) = H\left(A(x,-\infty), B(x,-\infty)\right) = \eta(A,B)(x,-\infty) = 0$$

and

$$H(1,1) = H\left(A(+\infty,+\infty), B(+\infty,+\infty)\right) = \eta(A,B)(+\infty,+\infty) = 1.$$

Let s_i and t_i be in $[0,1]$ ($i \in \{1,2,3,4\}$) such that (7.3) and (7.4) hold. Hence, there exist two d.f.'s A and B in Δ^2 and four points x_1, x_2, y_1, y_2 in $\overline{\mathbb{R}}$, with $x_1 \le x_2$ and $y_1 \le y_2$, such that

$$\begin{aligned} s_1 &= A(x_1,y_1), \quad s_2 = A(x_1,y_2), \quad s_3 = A(x_2,y_1), \quad s_4 = A(x_2,y_2), \\ t_1 &= B(x_1,y_1), \quad t_2 = B(x_1,y_2), \quad t_3 = B(x_2,y_1), \quad t_4 = B(x_2,y_2). \end{aligned}$$

Since $\eta(A,B)$ is 2–increasing,

$$\eta(A,B)(x_1,y_1) + \eta(A,B)(x_2,y_2) - \eta(A,B)(x_1,y_2) - \eta(A,B)(x_2,y_1) \ge 0,$$

which, with the above positions, is equivalent to

$$H(s_1,t_1) + H(s_4,t_4) \ge H(s_2,t_2) + H(s_3,t_3).$$

But we may exchange s_2 and s_3 and find a bivariate d.f. A' such that

$$s_1 = A'(x_1,y_1), \quad s_3 = A'(x_1,y_2), \quad s_2 = A'(x_2,y_1), \quad s_4 = A'(x_2,y_2).$$

Hence, with B unchanged, we have

$$H(s_1,t_1) + H(s_4,t_4) \ge H(s_3,t_2) + H(s_2,t_3),$$

from which it follows (7.2).

In order to prove (b.3), let s be any point in $[0,1]$ and let $\{s_n\}$ be any sequence in $[0,1]$ that increases to s, $s_n \uparrow s$. Let A and B be in Δ^2 such that (i) the margin $F(x) := A(x,+\infty)$ of A is continuous and strictly increasing and (ii) the margin $G(x) := B(x,+\infty)$ of B is constant on $\mathbb{R}$ and equal to t, $G(x) = t$ for all $x \in \mathbb{R}$. Thus the sequence $\{x_n\}$, where $x_n := F^{-1}(s_n)$ for all $n \in \mathbb{N}$, converges to $x := F^{-1}(s)$, $x_n \uparrow x$. Now, for all $t \in [0,1]$

$$\begin{aligned} &H(s_n,t) = H\left(F(x_n), G(x_n)\right) = H\left(A(x_n,+\infty), B(x_n,+\infty)\right) \\ &= \eta(A,B)(x_n,+\infty) \xrightarrow{n\to+\infty} \eta(A,B)(x,+\infty) \\ &= H\left(A(x,+\infty), B(x,+\infty)\right) = H\left(F(x), G(x)\right) = H(s,t). \end{aligned}$$

In an analogous manner, the function $t \mapsto \eta(A,B)(s,t)$ is proved to be left–continuous for all $s \in [0,1]$.

(b) $\Longrightarrow$ (a): Let H satisfy conditions (b.1) through (b.3) and define an operation η on Δ^2 via

$$\eta(A,B)(x,y) := H(A(x,y), B(x,y)) \qquad \text{for all } A, B \in \Delta^2.$$

It is a straightforward matter to verify that $\eta(A,B)$ thus defined satisfies the boundary conditions $\eta(A,B)(+\infty,+\infty)=1$, and $\eta(A,B)(t,-\infty)=0=\eta(A,B)(-\infty,t)$ for all $t\in\mathbb{R}$. Moreover, given x,x',y,y' in $\mathbb{R}$ with $x\le x'$ and $y\le y'$, we have

$$\begin{aligned}\eta(A,B)(x',y')-\eta(A,B)(x',y)-\eta(A,B)(x,y')+\eta(A,B)(x,y)\\ =H(A(x',y'),B(x',y'))-H(A(x',y),B(x',y))\\ -H(A(x,y'),B(x,y'))+H(A(x,y),B(x,y)).\end{aligned}$$

Now, take

$$\begin{aligned}&s_1=A(x,y),\quad s_2=A(x',y),\quad s_3=A(x,y'),\quad s_4=A(x',y')\\ &t_1=B(x,y),\quad t_2=B(x',y),\quad t_3=B(x,y'),\quad t_4=B(x',y');\end{aligned}$$

then s_i and t_i ($i\in\{1,2,3,4\}$) satisfy (7.3) and (7.4) and, because H is P–increasing, it follows that $\eta(A,B)$ is 2-increasing. Thus it remains to verify that $\eta(A,B)$ is left–continuous in each variable. Let x be in $\mathbb{R}$, let y be any point in $\overline{\mathbb{R}}$, and let $\{x_n\}$ be a sequence of reals such that $x_n\uparrow x$. Hence

$$\begin{aligned}&|\eta(A,B)(x_n,y)-\eta(A,B)(x,y)|\\ &\qquad=|H\left(A(x_n,y),B(x_n,y)\right)-H\left(A(x,y),B(x,y)\right)|\xrightarrow[n\to+\infty]{}0,\end{aligned}$$

since $s\mapsto A(s,y)$ and $s\mapsto B(s,y)$ are left–continuous and Proposition 2.1.2 holds. In an analogous manner, $t\mapsto\eta(A,B)(x,t)$ is proved to be left–continuous for all $x\in\overline{\mathbb{R}}$. This completes the proof. ☐

The class of all functions that induce pointwise a binary operation on Δ^2 shall be denoted by $\mathcal{P}$. In particular, notice that if H is in $\mathcal{P}$, then H is a binary aggregation operator.

Theorem 7.2.1 is similar to the characterization of induced pointwise operations on Δ, which is reproduced here (see [4]).

Theorem 7.2.2. *For a function $H:[0,1]^2\to[0,1]$ the following statements are equivalent:*

(a') *H induces pointwise a binary operation η on Δ, viz. for every F and G in Δ, $\eta(F,G)(t):=H(F(t),G(t))$ is a d.f.;*

(b') *H fulfils the conditions*

(b.1') *$H(0,0)=0$ and $H(1,1)=1$,*

(b.2') *H is increasing in each variable,*

(b.3') *H is left–continuous in each place.*

Because every P–increasing function satisfies (b.2') (see section 7.1), every function in $\mathcal{P}$ induces pointwise also a binary operation on Δ.

7.3 Some connected questions

Let A and B be bivariate d.f.'s defined for all $x, y \in \overline{\mathbb{R}}$ by

$$A(x,y) = C\left(F_1(x), G_1(y)\right) \qquad \text{and} \qquad B(x,y) = D\left(F_2(x), G_2(y)\right),$$

where F_i, G_i $(i = 1, 2)$ are their respective margins and C and D are their respective copulas (we adopt, if necessary, the method of bilinear interpolation in order to single out one copula, see [140]). In other words, A and B are, respectively, in the Fréchet classes $\Gamma(F_1, G_1)$ and $\Gamma(F_2, G_2)$. If H is in $\mathcal{P}$, we can obtain some information on the margins of the pointwise induced d.f. $\eta(A, B)$ defined as in (7.1).

Proposition 7.3.1. *Under the above assumptions, $\eta(A, B)$ belongs to the Fréchet class determined by the (unidimensional) d.f.'s*

$$x \mapsto H(F_1(x), F_2(x)) \quad \textit{and} \quad y \mapsto H(G_1(y), G_2(y)).$$

Proof. For all $x, y \in \overline{\mathbb{R}}$, we have

$$\eta(A, B)(x, +\infty) = H(A(x, +\infty), B(x, +\infty)) = H(F_1(x), F_2(x)),$$

and, analogously,

$$\eta(A, B)(+\infty, y) = H(A(+\infty, y), B(+\infty, y)) = H(G_1(y), G_2(y)),$$

as claimed. □

Moreover, if H satisfies the assumptions of Theorem 7.2.1 and, then, it induces pointwise a binary operation η on Δ^2, it is entirely natural to ask whether anything may be said about the copula $\widetilde{C}$ of $\eta(A, B)$ for all A and B in Δ^2.

Proposition 7.3.2. *Under the above assumptions, if $F_1 = F_2 = F$, $G_1 = G_2 = G$ and H is idempotent, then $\widetilde{C}(x, y) = H(C(x, y), D(x, y))$.*

Proof. For every H in the Fréchet class $\Gamma(F, G)$, $(x, y) \mapsto H(A(x, y), B(x, y))$ is a bivariate d.f. with marginal d.f.'s given by

$$H(F(x), F(x)) = F(x) \qquad \text{and} \qquad H(G(y), G(y)) = G(y).$$

It follows that there exists a copula $\widetilde{C}$ such that

$$\widetilde{C}\left(F(x), G(y)\right) = H\left(A(x, y), B(x, y)\right) = H\left[C(F(x), G(y)), D(F(x), G(y))\right],$$

from which an argument similar to that used in the proof of Sklar's theorem ([114]) yields $\widetilde{C}(s, t) = H\left(C(s, t), D(s, t)\right)$ for all $s, t \in [0, 1]$. □

In general, when $F_1 \neq F_2$ and $G_1 \neq G_2$, the above result is not true.

Example 7.3.1. Let $H(x,y) = \lambda x + (1-\lambda)y$ be the weighted arithmetic mean and let $C = D = \Pi$ be the product copula, then, for $\lambda \in \,]0,1[$, we have

$$\begin{aligned} H(A(x,y), B(x,y)) &= \lambda F_1(x)G_1(y) + (1-\lambda)F_2(x)G_2(y) \\ &\neq [\lambda F_1(x) + (1-\lambda)F_2(x)]\,[\lambda G_1(y) + (1-\lambda)G_2(y)] \\ &= \Pi(H(F_1(x), F_2(x)), H(G_1(y), G_2(y))). \end{aligned}$$

We conclude this section with a remark on the convergence in Δ^2. Assume that $\{A_n\}$ and $\{B_n\}$ are two sequences of d.f.'s in Δ^2 that converge weakly to the d.f.'s A and B, respectively; in other words, if $C(A)$ and $C(B)$ are the dense subsets of $\overline{\mathbb{R}}^2$ formed by the points of continuity of A and B, respectively, then

$$\forall (x,y) \in C(A) \qquad \lim_{n\to+\infty} A_n(x,y) = A(x,y),$$

and

$$\forall (x,y) \in C(B) \qquad \lim_{n\to+\infty} B_n(x,y) = B(x,y).$$

The question naturally arises of whether, for $H \in \mathcal{P}$ that induces the operation η on Δ^2, the sequence of bivariate d.f.'s $\{\eta(A_n, B_n)\}$ converges weakly to $\eta(A,B)$. While we do not know a general answer to this question, the following result provides a useful sufficient condition.

Theorem 7.3.1. *Under the conditions just specified, if H is continuous in each place, then the sequence $\{\eta(A_n, B_n)\}_{n\in\mathbb{N}}$ converges weakly to $\eta(A,B)$.*

Proof. The set $C(A) \cap C(B)$ is dense in $\overline{\mathbb{R}}^2$. For every point (x,y) in $C(A) \cap C(B)$

$$A_n(x,y) \xrightarrow[n\to+\infty]{} A(x,y) \qquad \text{and} \qquad B_n(x,y) \xrightarrow[n\to+\infty]{} B(x,y).$$

In view of Lemma (7.2.1), we have

$$\begin{aligned} &|\eta(A_n, B_n)(x,y) - \eta(A,B)(x,y)| \\ &= |H(A_n(x,y), B_n(x,y)) - H(A(x,y), B(x,y))| \\ &\leq |H(A_n(x,y), 1) - H(A(x,y), 1)| + |H(1, B_n(x,y)) - H(1, B(x,y))|\,. \end{aligned}$$

The assertion now follows directly from the continuity of H. □

7.4 Remarks on the composition of copulas

Since every copula is also the restriction of a bivariate d.f. to the unit square, it is natural to study also induced pointwise binary operations on $\mathcal{C}$. Note that the function $H(x,y) = \lambda x + (1-\lambda)y$ induces pointwise a binary operation on $\mathcal{C}$, which is a convex set.

Proposition 7.4.1. *If* $H : [0,1]^2 \to [0,1]$ *induces pointwise a binary operation* ρ *on* $\mathcal{C}$*, then* H *is idempotent.*

Proof. Suppose that there exists a binary aggregation operator H that induces pointwise a binary operation ρ on $\mathcal{C}$, namely, for all A and B in $\mathcal{C}$,

$$\rho(A,B)(x,y) = H(A(x,y), B(x,y))$$

is a copula. It can be easily proved that $\rho(A,B)$ satisfies the boundary conditions (C1) if, and only if, $H(x,x) = x$ for all x in $[0,1]$. $\square$

In particular, *no copula induces pointwise a binary operation on* $\mathcal{C}$: in fact, M is the only idempotent copula but the minimum of two copulas need not be a copula (see Example 2.3.2).

Because the P–increasing property preserves the 2–increasing property, we have that, if H is a P–increasing and idempotent agop, then H induces pointwise a binary operation on copulas. However, this procedure is not useful in view of the following result.

Proposition 7.4.2. *Let* A *be a binary aggregation operator such that* $A(x,x) \geq x$ *for every* $x \in [0,1]$*. Then* A *is* P*–increasing if, and only if, there exists* $a \in [0,1]$ *such that* $A(x,y) = ax + (1-a)y$*.*

Proof. Let A be a P–increasing agop such that $A(x,x) \geq x$ for every $x \in [0,1]$. In particular, on account of Theorem 7.1.2, A is 2–increasing and its horizontal and vertical sections are convex. Set $a := A(1,0)$ and $b := A(0,1)$ and notice that $a+b \leq 1$.

In view of the 2–increasing property, for every $y \in [0,1]$ we have

$$A(0,y) + A(y,1) \geq A(y,y) + A(0,1) \geq y + b, \tag{7.7}$$

and, from the convexity of $y \mapsto A(0,y)$,

$$A(0,y) \leq yA(0,1) + (1-y)A(0,0) = by.$$

Therefore, connecting the two inequalities above, we obtain $A(y,1) \geq y + (1-y)b$. On the other hand, from the convexity of $y \mapsto A(y,1)$,

$$A(y,1) \leq yA(1,1) + (1-y)A(0,1) = y + (1-y)b,$$

viz. $A(y,1) = y + (1-y)b$. Analogously $A(1,y) = (1-a)y + a$.

From (7.7), it follows also that

$$A(0,y) \geq y + b - (1-b)y - b = by$$

and, because $A(0,y) \leq yA(0,1) = by$, we have $A(0,y) = by$. In the same manner, $A(x,0) = ax$.

Now, because A is 2–increasing, for every $y \geq x$, we have

$$A(x,y) \geq A(x,1) + A(y,y) - A(y,1) \geq (1-b)x + by$$

and

$$A(x,y) \leq A(x,1) + A(0,y) - b = (1-b)x + by,$$

viz. $A(x,y) = (1-b)x + by$. In the same manner, for every $x \geq y$, we obtain $A(x,y) = ax + (1-a)y$.

Finally, notice that

$$A(x,1/2) = \begin{cases} (1-b)x + b/2, & \text{if } x \leq 1/2; \\ ax + (1-a)/2, & \text{if } x > 1/2; \end{cases}$$

and, from the convexity of $x \mapsto A(x,1/2)$, we have

$$A\left(\frac{1}{2},\frac{1}{2}\right) \leq \frac{1}{2}A\left(0,\frac{1}{2}\right) + \frac{1}{2}A\left(1,\frac{1}{2}\right),$$

which is equivalent to $a + b \geq 1$. Therefore $a + b = 1$ and, for every $(x,y) \in [0,1]^2$, $A(x,y) = ax + (1-a)y$. □

Corollary 7.4.1. *Let A be a P–increasing agop. The following statements are equivalent:*

(a) *A is idempotent;*

(b) *there exists $a \in [0,1]$ such that $A(x,y) = ax + (1-a)y$.*

Thus, in the class of copulas, the characterization of induced pointwise operation is still an open problem.