Chapter 4

Isochoric deformations of compressible materials

Rubbers and elastomers are highly deformable solids, which have the remarkable property of preserving their volume through any deformation. This permanent isochoricity, incorporated mathematically into the equations of continuum mechanics through the concept of internal constraint of incompressibility (see (1.39) or (1.40)), has led to the discovery of several exact solutions in isotropic finite elasticity, most notably to the controllable or universal solutions of Rivlin and co-workers (e.g. Rivlin [106]).

Subsequently, Ericksen [33] examined the problem of finding all such solutions. He found that there are no controllable finite deformations in isotropic compressible elasticity, except for homogeneous deformations [34]. The impact of that result on the theory of nonlinear elasticity was quite important, and for many years there has been "the false impression that the only deformations possible in an elastic body are the universal deformations" (see [25]). However, around the same time as the publication of Ericksen's result, there was considerable activity in trying to find solutions for nonlinear elastic materials using the semi-inverse method. A summary of these earlier results may be found in the monograph by Green and Adkins [49].

Even though for homogeneous isotropic incompressible nonlinearly elastic solids, the simplified kinematics arising from the constraint of no volume change has facilitated the analytic solution of a wide variety of boundary-value problems, the situation is quite different for compressible materials. Firstly, the absence of the isochoric constraint leads to more complicated kinematics. Secondly, since the only controllable deformations are the homogeneous deformations, the discussion of inhomogeneous deformations has to be confined to a particular strain energy function or class of strain energy functions. Nevertheless, some progress has been made in recent years in the development of analytic forms for the deformation and in the solution of boundary value problems. One strategy to find some exact solutions for compressible elastic materials consists in taking inspiration from isochoric solutions for incompressible materials, and to seek similar solutions in compressible elastic materials. However, it is obvious that the isochoric deformations of an incompressible elastic body have differents loads than the isochoric deformations of a compressible elastic body, because they will in general produce changes in volume when applied in compressible materials. A review of isochoric problems can be found in [60].

4.1 Pure torsion

In the incompressible isotropic theory of nonlinear elasticity, the problem of finite torsion was first considered by Rivlin [106, 107, 108], while relevant experimental issues were discussed in [105, 112]. Rivlin showed that finite torsion is a universal deformation (see Family 3 of universal solution in (3.53) when $A = C = F = 1,\ B = E = 0$). By virtue of the constraint of zero volume change, the deformation is that of *pure torsion* so that there is no extension in the radial direction and the cross-section of the cylinder remains circular. We know, by Ericksen result [34], that finite torsion is not sustainable, however, in all compressible isotropic elastic materials. In fact the deformation here is more complicated and in general, there will be some radial extension, see Polignone and Horgan [97] and Kirkinis and Ogden [70]. Those two articles present the torsional problem for the strain energy written either in terms of the principal invariants I_1, I_2, I_3, or in terms of the principal stretches $\lambda_1, \lambda_2, \lambda_3$ of the left Cauchy stress tensor $\boldsymbol{B}$, or in terms of the principal invariants i_1, i_2, i_3 of $\boldsymbol{V}$. Polignone and Horgan [97] obtain a necessary condition for a pure torsion to be possible without imposing the zero traction on the lateral surface. Kirkinis and Ogden [70] find new necessary and sufficient conditions on the strain energy function for pure torsion with zero traction on the lateral surface of the cylinder.

The torsion problem in the compressible case is discussed in other important works as well. For example it is discussed from both theoretical and experimental viewpoints in [48] or in [49], where a formula is derived for the couple required to maintain the deformation in respect of an arbitrary (isotropic) strain energy function. Slight compressibility effects are investigated in [38], using the general theory of small deformations superimposed on a large deformation for the Blatz-Ko material model and for the Levinson-Burgess material in [78]. Currie and Hayes [25] determined constitutive relations for which pure torsion is sustainable and proposed a general class of materials, which includes the Hadamard material. The Blatz-Ko material for foam polyurethane elastomers has been studied recently in respect of pure torsion by various authors (see, for example, [8, 9, 20]). Loss of ellipticity for this material model during a pure torsional deformation was examined by Horgan and Polignone [99].

4.1.1 Formulation of the torsion problem

Let us consider the torsional deformation of an elastic solid circular cylinder of radius A due to applied twisting moments at its ends,

$$r = r(R), \qquad \theta = \Theta + \tau Z, \qquad z = Z, \tag{4.1}$$

where (R, Θ, Z) and (r, θ, z) are the cylindrical coordinates in the reference and in the current configurations, respectively, $\mathrm{d}r/\mathrm{d}R > 0$, and the constant $\tau > 0$ is the *twist* per unit undeformed length. Let us consider the strain energy function in terms of the first three principal invariants of $\boldsymbol{B}$, $W = \bar{W}(I_1, I_2, I_3)$. The deformation gradient tensor $\boldsymbol{F}$ for (4.1) is given by

$$\begin{bmatrix} r' & 0 & 0 \\ 0 & r/R & \tau r \\ 0 & 0 & 1 \end{bmatrix}, \tag{4.2}$$

and the physical components of $\boldsymbol{B}$ and its inverse $\boldsymbol{B}^{-1}$ are given by

$$\begin{bmatrix} r'^2 & 0 & 0 \\ 0 & r^2/R^2 + \tau^2 r^2 & \tau r \\ 0 & \tau r & 1 \end{bmatrix}, \quad \begin{bmatrix} r'^{-2} & 0 & 0 \\ 0 & R^2/r^2 & -\tau R^2/r \\ 0 & -\tau R^2/r & 1 + \tau^2 R^2 \end{bmatrix}, \tag{4.3}$$

respectively. The first three principal strain invariants are

$$\begin{aligned} I_1 &= 1 + r'^2 + \frac{r^2}{R^2} + \tau^2 r^2, \\ I_2 &= \frac{r^2}{R^2} + r'^2 + \frac{r'^2 r^2}{R^2} + \tau^2 r'^2 r^2, \\ I_3 &= \frac{r'^2 r^2}{R^2}. \end{aligned} \tag{4.4}$$

Substituting (4.3) into (1.36), we obtain the physical components of the Cauchy stress

$$\begin{aligned} T_{rr} &= \beta_0 + \beta_1 r'^2 + \beta_{-1} r'^{-2}, \\ T_{\theta\theta} &= \beta_0 + \beta_1 \left(\frac{r^2}{R^2} + \tau^2 r^2 \right) + \beta_{-1} \frac{R^2}{r^2}, \\ T_{zz} &= \beta_0 + \beta_1 + \beta_{-1}(1 + \tau^2 R^2), \\ T_{\theta z} &= \beta_1(\tau r) - \frac{\tau R^2}{r} \beta_{-1}, \\ T_{r\theta} &= 0, \quad T_{rz} = 0. \end{aligned} \tag{4.5}$$

In the present case, the equilibrium equations, in absence of body forces, div$\boldsymbol{T} = \boldsymbol{0}$, reduce to the single equation

$$\frac{\partial T_{rr}}{\partial r} + \frac{1}{r} (T_{rr} - T_{\theta\theta}) = 0. \tag{4.6}$$

Since $r = r(R)$, it is possible to consider the stress as a function of the reference co-ordinate R, i.e., $\boldsymbol{T} = \boldsymbol{T}(R)$ instead of $\boldsymbol{T} = \boldsymbol{T}(r)$. In this case the chain rule gives

$$\frac{\partial T_{rr}}{\partial R} + \frac{r'}{r} (T_{rr} - T_{\theta\theta}) = 0. \tag{4.7}$$

By $(4.5)_{1,2}$, (1.38) and (4.4) we obtain a single ordinary nonlinear equation for $r(R)$,

$$\frac{\mathrm{d}}{\mathrm{d}R} \left[\frac{Rr'}{r}(\bar{W}_1 + \bar{W}_2) + \frac{rr'}{R}(\bar{W}_3 + \bar{W}_2) + \tau^2 Rr'r\bar{W}_2 \right] + \left(\frac{Rr'^2}{r^2} - \frac{1}{R} \right) (\bar{W}_1 + \bar{W}_2) - \tau^2 R \bar{W}_1 = 0, \tag{4.8}$$

where the $\bar{W}_i$ $(i = 1, 2, 3)$ are evaluated at (4.4). For a solid circular cylinder of initial radius A subjected to end torques only, the boundary conditions of traction-free lateral surface are satisfied when

$$T_{rr}(A) = 0, \tag{4.9}$$

since $T_{r\theta} = T_{rz} \equiv 0$ by $(4.5)_5$. In addition, to ensure that $\boldsymbol{F}$ is bounded, we impose the following regularity condition

$$r(R) = \mathcal{O}(R) \text{ as } R \to 0. \tag{4.10}$$

Thus the two-point boundary value problem consists in solving (4.8) for $r(R)$ on $0 < R < A$ subject to the conditions (4.9) and (4.10).

The same problem has been written by Kirkinis and Ogden [70] in terms of the principal stretches in Eulerian principal axes[1] but in a more general form than here. These authors consider the case of torsional deformation superimposed on a uniform extension,

$$r = r(R), \qquad \theta = \Theta + \lambda_z \tau Z, \qquad z = \lambda_z Z, \tag{4.11}$$

where (R, Θ, Z) and (r, θ, z) are the cylindrical coordinates in the reference and in the current configurations, respectively, $\mathrm{d}r/\mathrm{d}R > 0$, the constant $\tau > 0$ is the twist per unit undeformed length, and λ_z is the uniform axial stretch. Here we consider the strain energy function in terms of the principal stretches, $W = \hat{W}(\lambda_1, \lambda_2, \lambda_3)$. The deformation gradient tensor $\boldsymbol{F}$ for (4.11) has components,

$$\begin{bmatrix} r' & 0 & 0 \\ 0 & r/R & \lambda_z \tau r \\ 0 & 0 & \lambda_z \end{bmatrix}, \tag{4.12}$$

and the physical components of $\boldsymbol{B}$ and its inverse $\boldsymbol{B}^{-1}$ are given by

$$\begin{bmatrix} r'^2 & 0 & 0 \\ 0 & r^2/R^2 + \lambda_z^2 \tau^2 r^2 & \lambda_z^2 \tau r \\ 0 & \lambda_z^2 \tau r & \lambda_z^2 \end{bmatrix}, \tag{4.13}$$

$$\begin{bmatrix} r'^{-2} & 0 & 0 \\ 0 & R^2/r^2 & -\tau R^2/r \\ 0 & -\tau R^2/r & 1/\lambda_z^2 + \tau^2 R^2 \end{bmatrix}, \tag{4.14}$$

respectively. Let $\boldsymbol{\mu}^i$, $i = 1, 2, 3$, be the unit Eulerian principal axes associated with this deformation. We see that $\boldsymbol{e}_r$ is the Eulerian principal axis associated with the principal stretch $\boldsymbol{\mu}^1$ and hence

$$\lambda_1 = r'. \tag{4.15}$$

We may express the remaining two principal directions in terms of the cylinder polar axes $\boldsymbol{e}_\theta$, $\boldsymbol{e}_z$. Thus, we write

$$\boldsymbol{\mu}^2 = \cos\phi \boldsymbol{e}_\theta + \sin\phi \boldsymbol{e}_z, \quad \boldsymbol{\mu}^3 = -\sin\phi \boldsymbol{e}_\theta + \cos\phi \boldsymbol{e}_z, \tag{4.16}$$

[1]Ogden [95], Section 5.2.5, writes the same problem using the Lagrangian principal axes.

where ϕ defines the orientation of the axes $\boldsymbol{\mu}^2,\ \boldsymbol{\mu}^3$ relative to $\boldsymbol{e}_\theta,\ \boldsymbol{e}_z$. By defining the following rotation matrix

$$\boldsymbol{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}, \tag{4.17}$$

considering the left stretch tensor $\boldsymbol{V}$ in $(1.10)_2$, and by comparing $\boldsymbol{V}^2$ and $\boldsymbol{B}$ as in $(1.11)_1$, we obtain the connections

$$\begin{aligned} &\lambda_2^2\cos^2\phi + \lambda_3^2\sin^2\phi = \frac{r^2}{R^2} + \lambda_z^2\tau^2 r^2, \\ &\lambda_2^2\sin^2\phi + \lambda_3^2\cos^2\phi = \lambda_z^2, \\ &(\lambda_2^2 - \lambda_3^2)\sin\phi\cos\phi = \lambda_z^2\tau r, \end{aligned} \tag{4.18}$$

from which we deduce that

$$\begin{aligned} &\lambda_2^2 + \lambda_3^2 = \frac{r^2}{R^2} + \lambda_z^2\tau^2 r^2 + \lambda_z^2, \\ &(\lambda_2^2 - \lambda_z^2)(\lambda_z^2 - \lambda_3^2) = \lambda_z^4\tau^2 r^2, \\ &\lambda_2\lambda_3 = \frac{\lambda_z r}{R}. \end{aligned} \tag{4.19}$$

Further, we obtain the explicit expression for ϕ as

$$\cos 2\phi = \frac{\lambda_2^2 + \lambda_3^2 - 2\lambda_z^2}{\lambda_2^2 - \lambda_3^2}. \tag{4.20}$$

Since the Cauchy stress tensor $\boldsymbol{T}$ is coaxial in the isotropic case with the left Cauchy-Green strain tensor $\boldsymbol{B}$, we may express it in terms of its principal stresses $T_1,\ T_2,\ T_3$ through

$$\begin{aligned} T_{rr} &= T_1, & T_{\theta z} &= (T_2 - T_3)\cos\phi\sin\phi, \\ T_{\theta\theta} &= T_2\cos^2\phi + T_3\sin^2\phi, & T_{zz} &= T_2\sin^2\phi + T_3\cos^2\phi. \end{aligned} \tag{4.21}$$

By (4.20) and (4.21) we obtain the following connection

$$\lambda_z^2\tau r(T_{\theta\theta} - T_{zz}) = \left(\frac{r^2}{R^2} + \lambda_z^2\tau^2 r^2 - \lambda_z^2\right)T_{\theta z}. \tag{4.22}$$

The principal stresses are given by (1.42) and we use (4.21) to obtain from equation (4.7)

$$\frac{\mathrm{d}}{\mathrm{d}R}(R\hat{W}_1) = \frac{\lambda_z}{\lambda_2\lambda_3}\frac{\lambda_2(\lambda_2^2 - \lambda_z^2)\hat{W}_2 - \lambda_3(\lambda_3^2 - \lambda_z^2)\hat{W}_3}{\lambda_2^2 - \lambda_3^2}, \tag{4.23}$$

where $\hat{W}_i = \partial\hat{W}/\partial\lambda_i$, $(i, = 1, 2, 3)$ are evaluated at the values given by (4.15) and (4.19). In this case the boundary condition to be satisfied is

$$T_{rr}(A) = T_1(A) = 0. \tag{4.24}$$

4.1.2 Pure torsion: necessary and sufficient condition

On setting $r = R$ in equation (4.8), Polignone and Horgan [97] obtain a necessary condition on $\bar{W}$ for pure torsion to be possible. From (4.4), for pure torsional deformation ($r = R$), we obtain

$$I_1 = I_2 = 3 + \tau^2 R^2, \qquad I_3 = 1, \tag{4.25}$$

and so the deformation is *isochoric*, and by (4.8), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}R}\left[\bar{W}_1 + \bar{W}_3 + (2 + \tau^2 R^2)\bar{W}_2\right] - \tau^2 R \bar{W}_1 = 0. \tag{4.26}$$

On employing the chain rule, (4.26) may be written as

$$2(3+\tau^2 R^2)\bar{W}_{21} + 2(2+\tau^2 R^2)\bar{W}_{22} + 2(\bar{W}_{31} + \bar{W}_{32} + \bar{W}_{11}) + 2\bar{W}_2 - \bar{W}_1 = 0, \tag{4.27}$$

where $\bar{W}_{ij} = \partial^2 \bar{W}/(\partial I_i \partial I_j)$ $(i, j = 1, 2, 3)$ are evaluated at the values (4.25). The condition (4.27) is therefore a necessary condition on $\bar{W}$ for pure torsion to be possible (an equivalent condition was obtained by Currie and Hayes [25]).

Kirkinis and Ogden [70], on setting $r = \lambda_z^{-1/2} R$ in order to have isochoric deformation for the torsion superimposed on uniform extension, obtain from (4.15) and $(4.19)_{(1,3)}$

$$\lambda_1 = \lambda_z^{-1/2}, \quad \lambda_2\lambda_3 = \lambda_z^{1/2}, \quad \lambda_2^2 + \lambda_3^2 = \lambda_z^2 + \lambda_z^{-1} + \lambda_z \tau^2 R^2. \tag{4.28}$$

In terms of the stretches, the equations (4.21) are given by

$$\begin{aligned} T_{rr} &= \lambda_z^{-1/2}\hat{W}_1, \quad T_{\theta z} = \sqrt{(\lambda_2^2 - \lambda_z^2)(\lambda_z^2 - \lambda_3^2)}\frac{\lambda_2\hat{W}_2 - \lambda_3\hat{W}_3}{\lambda_2^2 - \lambda_3^2}, \\ T_{\theta\theta} &= \frac{(\lambda_2^2 - \lambda_z^2)\lambda_2\hat{W}_2 - (\lambda_3^2 - \lambda_z^2)\lambda_3\hat{W}_3}{\lambda_2^2 - \lambda_3^2}, \\ T_{zz} &= \frac{(\lambda_z^2 - \lambda_3^2)\lambda_2\hat{W}_2 + (\lambda_2^2 - \lambda_z^2)\lambda_3\hat{W}_3}{\lambda_2^2 - \lambda_3^2}. \end{aligned} \tag{4.29}$$

On use of (4.28), the equilibrium equation (4.23) specializes to

$$(\lambda_2^2 + \lambda_3^2 - \lambda_z^2 - \lambda_z^{-1})\frac{\lambda_2\hat{W}_{12} - \lambda_3\hat{W}_{13}}{\lambda_2^2 - \lambda_3^2} + \hat{W}_1 = \lambda_z^{1/2}\frac{(\lambda_2^2 - \lambda_z^2)\lambda_2\hat{W}_2 - (\lambda_3^2 - \lambda_z^2)\lambda_3\hat{W}_3}{\lambda_2^2 - \lambda_3^2}, \tag{4.30}$$

in which the derivatives of $\hat{W}$ are evaluated for (4.28). Equation (4.30) provides a necessary condition for the strain energy to admit the deformation considered and generalizes (4.27) for $\lambda_z = 1$ to the case $\lambda_z \neq 1$. When $\lambda_z = 1$, equation (4.28) reduces to

$$\gamma(\lambda\hat{W}_{12} - \lambda^{-1}\hat{W}_{13}) + (\lambda + \lambda^{-1})\hat{W}_1 = \lambda^2\hat{W}_2 + \lambda^{-2}\hat{W}_3, \tag{4.31}$$

where γ is defined by

$$\gamma = \lambda - \lambda^{-1} = \tau R, \tag{4.32}$$

setting $\lambda_2 = \lambda,\ \lambda_3 = \lambda^{-1}$. Equations (4.27) and (4.31) are clearly equivalent, but they do not guarantee that the zero-traction boundary condition on the lateral surface of the cylinder is satisfied and therefore, in general, appropriate radial tractions need to be supplied in order to maintain the deformation. By $(4.29)_1$, and since λ_z is constant, T_{rr} depends on the deformation only through the combination τR. On setting $T_{rr}(R) = T(\tau R)$, on the lateral surface, we then have $T_{rr}(A) = T(\tau A)$. Thus, the lateral traction vanish if

$$T_{rr}(A) = T(\tau A) = 0, \tag{4.33}$$

for all $\tau \geq 0$. From that condition follows

$$\frac{\mathrm{d}}{\mathrm{d}\tau} T_{rr}(A) = T'(\tau A)A = 0, \tag{4.34}$$

for all $\tau > 0$, and therefore, for any fixed $\tau > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}R} T_{rr}(R) = T'(\tau R)\tau = 0, \tag{4.35}$$

for all $0 < R < A$. Thus, $T_{rr}(R)$ is constant, and since it vanishes for $R = A$, $T_{rr} \equiv 0$ and from (4.6), it follows that $T_{\theta\theta} \equiv 0$ also. From $(4.29)_{(1,2)}$ we deduce that

$$\hat{W}_1 \equiv 0, \quad \lambda_2(\lambda_2^2 - \lambda_z^2)\hat{W}_2 - \lambda_3(\lambda_3^2 - \lambda_z^2)\hat{W}_3 = 0, \tag{4.36}$$

where the derivatives of $\hat{W}$ are evaluated for the stretches given by (4.28). The conditions (4.36) are necessary and sufficient for the strain energy function to admit the combined isochoric torsion and uniform extension with zero tractions on the lateral surface of the cylinder. To derive $(4.36)_2$, we used the inequality $\lambda_2 \neq \lambda_3$ (otherwise by (4.28) the trivial situation $\tau = 0,\ \lambda_z = 1,\ \lambda_2 = \lambda_3 = 1$ is verified). When $\lambda_z = 1$, the conditions (4.36) reduce to

$$\hat{W}_1 \equiv 0, \qquad \lambda^2\hat{W}_2 + \lambda^{-2}\hat{W}_3 = 0, \tag{4.37}$$

evaluated for $\lambda_2 = \lambda$, $\lambda_3 = \lambda^{-1}$, $\lambda - \lambda^{-1} = \tau R$. Conditions (4.31) are obviously implied by (4.37).

If the strain energy W is written in terms of the principal invariants i_1, i_2, i_3 (see (1.32)), then the Cauchy stress components for pure torsional deformation become

$$\begin{aligned}
T_{rr} &= \tilde{W}_1 + (i-1)\tilde{W}_2 + \tilde{W}_3, \quad T_{\theta z} = \frac{\gamma}{i-1}\left(\tilde{W}_1 + \tilde{W}_2\right), \\
T_{\theta\theta} &= \frac{1}{i-1}\left(\tilde{W}_1(\gamma^2+2) + (i-2)(i+1)\tilde{W}_2 + (i-1)\tilde{W}_3\right), \\
T_{zz} &= \frac{1}{i-1}\left(2\tilde{W}_1 + (i+1)\tilde{W}_2 + (i-1)\tilde{W}_3\right),
\end{aligned} \tag{4.38}$$

where $\tilde{W}_j$ $(j = 1, 2, 3)$ are the derivatives of $\tilde{W}$ with respect to i_1, i_2, i_3, respectively, and evaluated for

$$i = i_1 = i_2 = \lambda + \lambda^{-1} + 1, \quad \gamma = \lambda - \lambda^{-1}, \quad i_3 = 1. \tag{4.39}$$

Here, the equilibrium equation (4.27) and (4.31) become

$$(\tilde{W}_{11} + i\tilde{W}_{12} + (i-1)\tilde{W}_{22} + \tilde{W}_{31} + \tilde{W}_{32})(i+1) + i(\tilde{W}_2 - \tilde{W}_1) = 0. \tag{4.40}$$

When $T_{rr} = T_{\theta\theta} = 0$ we obtain, after some rearrangement, the necessary and sufficient conditions as

$$i\tilde{W}_1 + \tilde{W}_2 = 0, \qquad (i^2 - i - 1)\tilde{W}_1 - \tilde{W}_3 = 0, \tag{4.41}$$

where, here, $\tilde{W}$ depends on i_1, i_2, i_3 and the derivatives are evaluated for (4.39).

In the case of pure torsion, the resultant axial force N on any cross-section of the cylinder and the resultant moment M are related by

$$N = -\tau M, \tag{4.42}$$

independently of which strain energy function is used. Thus, (4.42) establishes another example of a *universal relation*[2]. To show the relation (4.42), we consider the definitions

$$N = 2\pi \int_0^A T_{zz} R \, \mathrm{d}R, \qquad M = 2\pi \int_0^A T_{\theta z} R^2 \, \mathrm{d}R, \tag{4.43}$$

where the integrals (which are independent of Z) are taken over any cross-section of the cylinder. The relationship (4.22) at $\lambda_z = 1,\ T_{\theta\theta} = 0,\ r = R$ reduces to

$$T_{zz} = -\tau R T_{\theta z}, \tag{4.44}$$

and from (4.43), the relation (4.42), therefore, holds.

4.1.3 Some examples

It has been shown by Beatty [8] and by Carroll and Horgan [21] that pure torsion is possible for the following Blatz-Ko material (2.40)

$$\bar{W}(I_1, I_2, I_3) = \frac{\mu}{2}\left(\frac{I_2}{I_3} + 2I_3^{1/2} - 5\right). \tag{4.45}$$

In fact, here the stress response equation takes the simple form

$$\boldsymbol{T} = \mu\left(\boldsymbol{I} - I_3^{-1/2}\boldsymbol{B}^{-1}\right), \tag{4.46}$$

and the equation (4.8) reduces to

$$3Rr^3 r'' - r^3 r' + R^3 r'^4 = 0, \tag{4.47}$$

where the prime refers to the ordinary derivative with respect to R. The equation (4.47) is a second-order nonlinear ordinary differential equation where the parameter τ does not appear. The components of the stress T_{rr} and $T_{\theta\theta}$ do not contain

[2] It may be compared to the universal relation $(3.29)_2$, because pure torsion is an example of locally simple shear of magnitude $\gamma = \tau R$ in the $(\boldsymbol{e}_\theta, \boldsymbol{e}_z)$ plane.

τ and it is clear that $r = R$ is a solution of (4.47) that verify (4.10). In this case, from (4.46), it is easy to see that

$$\begin{aligned} T_{rr} &= 0, & T_{r\theta} &= 0, \\ T_{\theta\theta} &= 0, & T_{rz} &= 0, \\ T_{zz} &= -\mu\tau^2 R^2, & T_{\theta z} &= \mu\tau R, \end{aligned} \tag{4.48}$$

and so the boundary free traction condition (4.9) is satisfied. For the general Blatz-Ko material (2.38), the necessary condition (4.26) holds if and only if $f = 0$. In this case, the traction free boundary condition is also satisfied.

Consider the Hadamard material (2.21): to ensure the normalization conditions (1.34) and (1.64), the arbitrary function H in (2.21) must satisfy

$$H(1) = 0, \qquad H'(1) + c_1 + 2c_2 = 0. \tag{4.49}$$

One can see that the necessary condition (4.27) is satisfied if and only if

$$2c_2 = c_1. \tag{4.50}$$

The stress components for the Hadamard material are given, by (1.36), as

$$\begin{aligned} T_{rr} &= 2c_2\tau^2 R^2, & T_{r\theta} &= 0, \\ T_{\theta\theta} &= 2(c_1 + c_2)\tau^2 R^2, & T_{\theta z} &= 2(c_1 + c_2)\tau R, \\ T_{zz} &= 0, & T_{rz} &= 0. \end{aligned} \tag{4.51}$$

Thus, Hadamard materials cannot sustain pure torsion ($2c_2 = c_1$) with traction free lateral surface, except in the degenerate case where $c_2 = 0$.

The authors in [70] and in [97] try to find a more general form of strain energy to sustain pure torsion[3] with the difference that Polignone and Horgan [97] do not impose free boundary condition. They try to obtain materials where pure torsion may be possible. For example they start by requiring that

$$\bar{W}_{21} + \bar{W}_{22} = 0, \tag{4.52}$$

where the derivatives are evaluated in (4.25). Condition (4.52) is a good device to eliminate the explicit dependence of the parameter τ in the equation (4.27). In fact the explicit term $\tau^2 R^2$ vanishes identically when (4.52) is assumed. Since functions of the form $P(I_1 - I_2, I_3)$ clearly satisfy (4.52), Polignone and Horgan consider the following general form of strain energy function,

$$\begin{aligned} W = \frac{\mu}{2}\big[&P(I_1 - I_2, I_3) + Q(I_1, I_3) + R(I_2)S(I_3) \\ &+ H_1(I_3)(I_1 - 3) + H_2(I_3)(I_2 - 3) + H_3(I_3)\big], \end{aligned} \tag{4.53}$$

where $\mu > 0$ is the infinitesimal shear modulus and P, Q, R, S, H_i $(i = 1, 2, 3)$ are sufficiently smooth functions. The strain energy function (4.53) satisfies (4.52) if and only if $R(I_2)$ is a linear function. Thus, we assume that

$$R(I_2) = k_1 I_2 + k_2, \tag{4.54}$$

[3]This is an important task from the mathematical point of view, but afterward one must establish whether the material described by the model obtained describes reality or if it remains only an idealization.

where k_1 and k_2 are arbitrary constants. On redefining $S(I_3)$ and $H(I_3)$ to include these constants, we rewrite (4.53) as

$$W = \frac{\mu}{2}\big[P(I_1 - I_2, I_3) + Q(I_1, I_3) + I_2 S(I_3) \\ + H_1(I_3)(I_1 - 3) + H_2(I_3)(I_2 - 3) + H_3(I_3)\big], \quad (4.55)$$

in order to satisfy (4.52). The normalization conditions (1.34) and (1.64) require that

$$P(0,1) + Q(3,1) + 3S(1) + H_3(1) = 0 \quad (4.56)$$

and

$$-P_1(0,1) + P_2(0,1) + \frac{\partial Q}{\partial I_1}(3,1) + \frac{\partial Q}{\partial I_3}(3,1) \\ + 3S'(1) + 2S(1) + H_1(1) + 2H_2(1) + H_3'(1) = 0, \quad (4.57)$$

where the subscripts 1 and 2 on P indicate the derivatives with respect to the first and second arguments, respectively. From (4.27) and (4.52), we ask that

$$2(\bar{W}_{21} + \bar{W}_{31} + \bar{W}_{32} + \bar{W}_{11}) + 2\bar{W}_2 - \bar{W}_1 = 0. \quad (4.58)$$

On substitution from (4.55) into (4.58), one finds that

$$-3P_1(0,1) + \left(2\frac{\partial^2 Q}{\partial I_1^2} + 2\frac{\partial^2 Q}{\partial I_1 \partial I_3} - \frac{\partial Q}{\partial I_1}\right)_{I_1 = 3+\tau^2 R^2,\ I_3 = 1} \\ + 2\left(S'(1) + S(1)\right) + 2H_1'(1) - H_1(1) + 2H_2'(1) + 2H_2(1) = 0. \quad (4.59)$$

Rather than describe the most general class of functions P, Q, S, H_i $(i = 1,2,3)$ for which (4.58) holds, Polignone and Horgan [97] indicate some possibilities. One possibility is to search for Q such that

$$2\frac{\partial^2 Q}{\partial I_1^2} + 2\frac{\partial^2 Q}{\partial I_1 \partial I_3} - \frac{\partial Q}{\partial I_1} = 0, \quad (4.60)$$

holds. One solution of (4.60) is given in the following form

$$Q(I_1, I_3) = \alpha e^{\beta(I_3 - 1)} e^{(1/2 - \beta)(I_1 - 3)}, \quad (4.61)$$

to within an arbitrary additive function of I_3, that we may include in $H_3(I_3)$. The parameters α and $\beta(\neq 1/2)$ are constants. To find a possible form for $S(I_3)$, one might seek S such that

$$S'(1) + S(1) = 0, \quad (4.62)$$

so that (4.59) is further simplified. Thus, it is possible to set

$$S = ke^{-(I_3 - 1)}, \quad (4.63)$$

where k is a constant. By the previous choices for Q and S, the condition (4.59) reduces to

$$-3P_1(0,1) + 2H_1'(1) - H_1(1) + 2H_2'(1) + 2H_2(1) = 0, \quad (4.64)$$

and the normalization condition (4.56) and (4.57) read as

$$P(0,1) + \alpha + 3k + H_3(1) = 0, \tag{4.65}$$
$$-P_1(0,1) + P_2(0,1) + \frac{\alpha}{2} - k + H_1(1) + 2H_2(1) + H_3'(1) = 0.$$

Thus, a material described by a strain energy function $\bar{W}$ of the form (4.55), with Q and S chosen as in (4.61) and (4.63), respectively, can sustain pure torsional deformations provided (4.64) holds. This is one possible way to generate a benchmark of possible strain energy functions. On every setting for the unknown function P, Q, S, H one could check afterwards if the strain energy function found satisfies the boundary condition as well.

An other investigation concerns the energy functions of the form

$$\tilde{W}(i_1, i_2, i_3) = f(i_1)h_1(i_3) + g(i_2)h_2(i_3) + h_3(i_3), \tag{4.66}$$

which is one generalization of the material described by classes I, II and III in (3.87), (3.89) and (3.91). Polignone and Horgan [97] show that the first two classes cannot sustain pure torsion (in general). Instead the third class verifies the condition (4.40) if and only if $a_3 = b_3$ but a uniformly distributed tensile loading would be required on the lateral surface of the cylinder.

4.2 Pure axial shear

The pure axial shear (also called telescopic shear) problem is a particular form of axisymmetric anti-plane shear. It has been introduced in Section (3.2.2) for the case of incompressible isotropic nonlinear elastic cylinder. Here, we are investigating when this isochoric deformation can be sustained for compressible homogeneous isotropic materials. In general, for an arbitrary compressible material, the cylinder will undergo both a radial $r(R)$ deformation, and an axial deformation $w(R)$. For an arbitrary incompressible, isotropic and homogeneous, hyperelastic material, Rivlin [107] has shown that the telescopic shear problem leads to a nonlinear ordinary differential equation for the radial displacement $w(R)$, whose solution may be obtained only upon specification of the strain energy function. Necessary and sufficient conditions on the form of the strain energy function for the compressible and incompressible cases, for which nontrivial states of anti-plane shear may be admissible, have been derived by Knowles [72, 74], but the mathematical structure used to derive the conditions for compressible material excludes the axisymmetric case. Here, our main references are Jiang and Beatty [64] and Polignone and Horgan [98], but some examples for telescopic shear in the compressible case are also described by Agarwal [3] and by Mioducowski and Haddow [82]. In [98], necessary conditions on the strain energy function W for pure axial shear to be possible are established by seeking solutions of the governing equations for which $r = R$. Two conditions on W are obtained in the form of a second-order and first-order nonlinear ordinary differential equation for axial displacement $w(R)$, whose solutions must be compatible. In [64], a single necessary and sufficient condition is obtained in order that the material may support pure axial shear, instead.

4.2.1 Formulation of the axial shear problem

Let us consider the axisymmetric finite axial shear deformation of an isotropic compressible nonlinearly elastic hollow circular cylinder with inner surface $R = A$ and outer surface $R = B$,

$$r = r(R), \qquad \theta = \Theta, \qquad z = Z + w(R), \tag{4.67}$$

where (R, Θ, Z) and (r, θ, z) are the cylindrical coordinates in the reference and in the current configurations, respectively, and $\mathrm{d}r/\mathrm{d}R > 0$. We set the inner surface $R = A$ to be bonded to a rigid cylinder so that

$$r(A) = A, \qquad w(A) = 0. \tag{4.68}$$

The deformation (4.67) may be achieved either by prescribing $r(R)$ and $w(R)$ on the outer surface $R = B$, or by applying a uniformly distributed axial shear traction to the outer surface of the cylinder and assuming that the radial traction is zero there,

$$T_{rr}(B) = 0, \qquad T_{rz}(B) = T_0, \tag{4.69}$$

where T_0 is a given constant. Let us assume that the cylinder is sufficiently long so that end effects are negligible and that the strain energy function is given in terms of the first three principal invariants of $\boldsymbol{B}$:

$$W = \bar{W}(I_1, I_2, I_3). \tag{4.70}$$

Corresponding to the deformation field (4.67), we have

$$\boldsymbol{F} = \begin{bmatrix} r' & 0 & 0 \\ 0 & r/R & 0 \\ w' & 0 & 1 \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} r'^2 & 0 & r'w' \\ 0 & r^2/R^2 & 0 \\ r'w' & 0 & 1 + w'^2 \end{bmatrix}, \tag{4.71}$$

$$\boldsymbol{B}^{-1} = \begin{bmatrix} (w'^2 + 1)/r'^2 & 0 & -w'/r' \\ 0 & R^2/r^2 & 0 \\ -w'/r' & 0 & 1 \end{bmatrix}, \tag{4.72}$$

where $r' = \mathrm{d}r/\mathrm{d}R$ and $w' \equiv \mathrm{d}w/\mathrm{d}R$. The first three principal invariants are given by

$$\begin{aligned} I_1 &= 1 + \frac{r^2}{R^2} + r'^2 + w'^2, \\ I_2 &= r'^2 + \frac{r^2}{R^2}\left(1 + r'^2 + w'^2\right), \\ I_3 &= r'^2 \frac{r^2}{R^2}. \end{aligned} \tag{4.73}$$

Substitution from $(4.71)_2$ and (4.72) into (1.36) yields the physical components of the Cauchy stress $\boldsymbol{T}$ as

$$\begin{aligned}
T_{rr} &= \beta_0 + \beta_1 r'^2 + \beta_{-1}\frac{w'^2+1}{r'^2},\\
T_{\theta\theta} &= \beta_0 + \beta_1\frac{r^2}{R^2} + \beta_{-1}\frac{R^2}{r^2},\\
T_{zz} &= \beta_0 + \beta_1(w'^2+1) + \beta_{-1},\\
T_{rz} &= \beta_1 r'w' - \beta_{-1}\frac{w'}{r'},\\
T_{r\theta} &= 0, \quad T_{\theta z} = 0.
\end{aligned} \tag{4.74}$$

Because $r = r(R)$, it is convenient to consider that $\boldsymbol{T} = \boldsymbol{T}(R)$. The equilibrium equations in the absence of body force, $\text{div}\boldsymbol{T} = \boldsymbol{0}$, for this deformation, reduce to the following two equations:

$$\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\left(T_{rr} - T_{\theta\theta}\right) = 0, \qquad \frac{\partial T_{rz}}{\partial r} + \frac{T_{rz}}{r} = 0. \tag{4.75}$$

We observe that equation $(4.75)_2$ can also be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}r}(rT_{rz}) = 0, \tag{4.76}$$

so that on integrating and using $(4.74)_4$, $(4.73)_3$ and (1.38), we arrive at the following first-order nonlinear ordinary differential equation

$$w'\left(RW_1 + \frac{r^2}{R}W_2\right) = K, \tag{4.77}$$

where K is a constant and W_1, W_2 are the derivatives of W with respect to I_1 and I_2, respectively, evaluated at the values (4.73). The constant K appearing in (4.77) can now be expressed in terms of T_0. In fact, by $(4.74)_4$, (4.77) and (4.69), we find that

$$K = \frac{r(B)T_0}{2}. \tag{4.78}$$

Equation $(4.75)_1$, after using a chain rule to differentiate with respect to R, may be written as the following second-order nonlinear ordinary differential equation

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}R}&\left(\frac{Rr'}{r}W_1 + \left(\frac{Rr'}{r} + \frac{rr'}{R}\right)W_2 + \frac{rr'}{R}W_3\right)\\
&+ W_1\left(\frac{Rr'^2}{r^2} - \frac{1}{R}\right) + W_2\left(\frac{Rr'^2}{r^2} - \frac{1}{R} - \frac{w'^2}{R}\right) = 0,
\end{aligned} \tag{4.79}$$

where again the derivatives W_i $(i = 1, 2, 3)$ are evaluated at the values (4.73). Equations (4.77) and (4.79) are a coupled pair of nonlinear ordinary differential equations for the unknowns functions $r(R)$ and $w(R)$.

4.2.2 Pure axial shear: necessary and sufficient conditions

Polignone and Horgan [98] obtain necessary conditions on the strain energy function for pure axial shear to be possible by setting $r = R$ in the equations (4.77) and (4.79). When $r = R$, first we know by (4.73) that

$$I_1 = I_2 = 3 + w'^2, \qquad I_3 = 1, \tag{4.80}$$

and so the deformation is isochoric. From (4.79), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}R}(W_1 + 2W_2 + W_3) - \frac{w'^2}{R}W_2 = 0. \tag{4.81}$$

Employing the chain rule, and setting $w' \neq 0$ to have a nontrivial solution, this last equation can be written as

$$2(W_{11} + 2W_{22} + 3W_{12} + W_{13} + W_{23})w'' - W_2\frac{w'}{R} = 0. \tag{4.82}$$

From (4.77), we obtain the necessary condition

$$(W_1 + W_2)w' = \frac{BT_0}{2R}. \tag{4.83}$$

In (4.82) and in (4.83) the derivatives are evaluated in (4.80). By differentiating both sides of (4.83) with respect to R and using the chain rule, we obtain

$$(W_1 + W_2)w'' = -\frac{BT_0}{2R^2} - 2(W_{11} + 2W_{12} + W_{22})w'^2w'', \tag{4.84}$$

where the derivatives are evaluated at values (4.80). When $r = R$, the corresponding stress components in (4.74) become

$$\begin{aligned}
T_{rr} &= \beta_0 + \beta_1 + \beta_{-1}(w'^2 + 1),\\
T_{\theta\theta} &= \beta_0 + \beta_1 + \beta_{-1},\\
T_{zz} &= \beta_0 + \beta_1(w'^2 + 1) + \beta_{-1},\\
T_{rz} &= (\beta_1 - \beta_{-1})w',\\
T_{r\theta} &= 0, \quad T_{\theta z} = 0,
\end{aligned} \tag{4.85}$$

where the β_i ($i = -1, 0, 1$) are evaluated at values (4.80). Setting $r = R$, the boundary condition $(4.68)_1$ is satisfied and so from $(4.68)_2$ and $(4.69)_1$, the remaining boundary conditions are

$$w(A) = 0, \quad (W_1 + 2W_2 + W_3)\,|_{I_1=I_2=3+w'^2,\ I_3=1,\ R=B} = 0, \tag{4.86}$$

respectively.

In [64], the problem is introduced in the reference configuration. From (1.21), the physical components of the first Piola-Kirchhoff stress tensor are given by

$$\begin{aligned}
&(T_R)_{RR} = (T_R)_{ZZ} = 2(W_1 + 2W_2 + W_3), \quad (T_R)_{RZ} = -2w'(W_2 + W_3),\\
&(T_R)_{\Theta\Theta} = 2(W_1 + (2 + w'^2)W_2 + W_3), \quad (T_R)_{ZR} = 2w'(W_2 + W_2),\\
&(T_R)_{R\Theta} = (T_R)_{\Theta R} = (T_R)_{Z\Theta} = (T_R)_{\Theta Z} = 0,
\end{aligned} \tag{4.87}$$

where $W_i \equiv \partial W/\partial I_i$, $(i = 1, 2, 3)$ are evaluated at (4.80). The equilibrium equations, in the absence of body forces, reduce to the following radial and axial equilibrium equations

$$\begin{aligned} &R\frac{\mathrm{d}}{\mathrm{d}R}\left(W_1 + 2W_2 + W_3\right) = w'^2 W_2, \\ &\frac{\mathrm{d}}{\mathrm{d}R}\left(R(T_R)_{ZR}\right) = 0. \end{aligned} \tag{4.88}$$

Similarly to the definition in (3.24), after setting $w' = k$, we define the shear stress response function as

$$\tau(k) \equiv (T_R)_{ZR} = k\mu(k^2) \tag{4.89}$$

and the shear response function as

$$\mu(k^2) \equiv 2(W_1 + W_2). \tag{4.90}$$

We may therefore rewrite (4.88) as

$$\begin{aligned} &R\frac{\mathrm{d}}{\mathrm{d}R}\left(W_1 + 2W_2 + W_3\right) = k^2 W_2, \\ &\frac{\mathrm{d}}{\mathrm{d}R}\left(R\tau(k)\right) = 0. \end{aligned} \tag{4.91}$$

As in (3.26), by the empirical inequality (1.46), we know that

$$\mu(k^2) > 0, \quad \forall k. \tag{4.92}$$

We observe that the shear strain $k(R)$ vanishes identically if either the shear strain itself or its derivative $\mathrm{d}k/\mathrm{d}R$ vanishes at a single location in $[A, B]$. In fact by $(4.91)_2$ and (4.89), we obtain that

$$Rk\mu(k^2) = h, \tag{4.93}$$

where h is an integration constant. If $[A, B]$ contains the origin, the statement is trivial, because (4.92) holds. Thus, if there exist a point $0 \neq R_0 \in [A, B]$ such that $k(R_0) = 0$, by (4.92), it is necessary to have $h = 0$ and therefore $k \equiv 0$. If there exist a point $0 \neq R_1 \in [A, B]$ such that $\mathrm{d}k(R_1)/\mathrm{d}R = 0$, by differentiation of (4.93) with respect to R, we have

$$R\frac{\mathrm{d}k(R)}{\mathrm{d}R}\frac{\mathrm{d}}{\mathrm{d}k}[\tau(k(R))] = -\tau(k(R)), \tag{4.94}$$

which with the aid of (4.93), may be written as

$$\frac{\mathrm{d}k(R)}{\mathrm{d}R}\frac{\mathrm{d}}{\mathrm{d}k}[\tau(k(R))] = -\frac{h}{R^2}, \tag{4.95}$$

and since $\mathrm{d}k(R_1)/\mathrm{d}R = 0$ we therefore obtain the constant $h = 0$, deducing as in the previous case that $k \equiv 0$.

The necessary and sufficient condition for a compressible, isotropic and homogeneous, hyperelastic material to be capable of sustaining nontrivial, pure axial shear deformation whose strain energy function W satisfies (4.92) is the following condition

$$(W_1 + W_2)\left[W_{11} + I_1 W_{12} + (I_1 - 1)W_{22} + W_{13} + W_{23} + \tfrac{1}{2}W_2\right] = \\ (I_1 - 3)\left[W_1(W_{12} + W_{22}) - W_2(W_{11} + W_{21})\right], \quad (4.96)$$

for $I_1 = I_2 \geq 3$, $I_3 = 1$. By using (4.80), and recalling that the strain energy function W depends on the shear strain k only through the invariants I_1 and I_2, it follows that (4.96) admits the following representation

$$(W_1 + W_2)\frac{\mathrm{d}}{\mathrm{d}k}(W_1 + 2W_2 + W_3) = -kW_2\frac{\mathrm{d}}{\mathrm{d}k}\left[k(W_1 + W_2)\right]. \quad (4.97)$$

Recalling the definition of the shear stress response function in (4.89) and since we may suppose that k and its derivative with respect to R never vanishes (otherwise by previous consderations the only solution $w(R)$ would be a constant), we may rewrite (4.97) as

$$\tau(k)\frac{\mathrm{d}}{\mathrm{d}k}(W_1 + 2W_2 + W_3) = -k^2 W_2\frac{\mathrm{d}\tau(k)}{\mathrm{d}k}. \quad (4.98)$$

To prove sufficiency, we need to show that every solution of the equation $(4.91)_2$ also satisfies the radial equilibrium equation $(4.91)_1$ when the condition (4.92) and (4.98) are identically satisfied. Since the equilibrium equation $(4.91)_2$ may be written in the form (4.94), after substitution from (4.94) into (4.98), we obtain the equilibrium equation $(4.91)_1$ and sufficiency is hence shown.

To prove the necessary condition, we consider a solution $\bar{w}$ of both equations $(4.91)_1$ and $(4.91)_2$. Since the strain energy function depends on k trough the invariants I_1 and I_2, we may rewrite Equation $(4.91)_1$ as

$$R\frac{\mathrm{d}}{\mathrm{d}k}(W_1 + 2W_2 + W_3)\frac{\mathrm{d}k}{\mathrm{d}R} = k^2 W_2. \quad (4.99)$$

Because equation $(4.91)_2$ is equivalent to (4.94), we use (4.94) in (4.99) and we obtain

$$\tau(k)\frac{\mathrm{d}}{\mathrm{d}k}(W_1 + 2W_2 + W_3) = -k^2 W_2\frac{\mathrm{d}\tau(k)}{\mathrm{d}k}. \quad (4.100)$$

Thus (4.98) is obtained and the necessary condition is therefore proved. In order to attain this result, a division by μ was necessary, but we recall that this is always possible because (4.92) holds.

4.2.3 Some examples

Let us consider the Hadamard material (2.21). It follows from (4.90) that the shear response function for the Hadamard material (2.21), is

$$\mu(k^2) = 2(c_1 + c_2) > 0, \quad (4.101)$$

a constant, and that the shear stress response function (4.89) is

$$\tau(k) = 2k(c_1 + c_2), \tag{4.102}$$

a linear dependence of k. Here, in accordance with (4.96), it follows immediately that non-trivial, controllable, axial pure shear deformations are possible in every Hadamard material (2.21) for which

$$\frac{1}{2}(W_1 + W_2)W_2 = (c_1 + c_2)c_2 = 0. \tag{4.103}$$

Since $(c_1 + c_2) > 0$,

$$c_2 = 0 \tag{4.104}$$

is a necessary and sufficient condition for the pure axial shear to be controllable in an Hadamard material. When $c_2 = 0$ and $c_1 > 0$, from $(4.91)_2$ or (4.83) the out-of-plane displacement $w(R)$ is given by

$$w(R) = \frac{BT_0}{2c_1} \ln\left(\frac{R}{A}\right), \tag{4.105}$$

and from $(4.85)_{3,4}$, the nonzero stress components are

$$T_{zz} = \frac{B^2T_0^2}{2c_1R^2}, \quad T_{rz} = \frac{BT_0}{R}. \tag{4.106}$$

It is readily seen that (4.96) fails for a Blatz-Ko material (2.40), which is capable of sustaining pure torsion (see Section 4.1.3).

In searching for a more general class of material for which the pure axial shear is possible, Polignone and Horgan [98] require that the strain energy W satisfy the following condition

$$W_{11} + 2W_{12} + W_{22} = 0, \tag{4.107}$$

where the derivatives are evaluated in (4.80). By (4.84), and since we are assuming that $W_1 + W_2 > 0$, we obtain

$$w'' = -\frac{BT_0}{2R^2(W_1 + W_2)}. \tag{4.108}$$

On employing (4.107), (4.108), (4.83) in (4.82), we find that

$$2(W_{22} + W_{12} + W_{13} + W_{23}) + W_2 = 0. \tag{4.109}$$

Thus, Polinone and Horgan [98] start by considering the following form of the strain energy

$$W = \frac{\mu}{2}\left[P(I_1 - I_2, I_3)(I_1 - 3) + Q(I_1 - I_2, I_3)(I_2 - 3) + R(I_1 - I_2, I_3)\right] \tag{4.110}$$

where $\mu > 0$ is the infinitesimal shear modulus, and P, Q, R are sufficiently smooth functions. This form of strain energy function certainly verifies the conditions (4.107). The normalization conditions (1.34) and $(1.64)_1$ are satisfied by (4.110) if

$$R(0, 1) = 0, \tag{4.111}$$

and

$$P(0,1) + 2Q(0,1) - R_1(0,1) + R_2(0,1) = 0, \qquad (4.112)$$

where the subscripts $1, 2$ indicate the derivatives with respect to the first and second arguments, respectively. The task now is to find conditions on the functions P, Q, R such that condition (4.109) is satisfied. Hence, on substitution from (4.110) in (4.109), one finds that

$$\begin{aligned} 2\left[P_2(0,1) + Q_2(0,1) - (P_1(0,1) + Q_1(0,1))\right] \\ + Q(0,1) - R_1(0,1) - \left[P_1(0,1) + Q_1(0,1)\right] w'^2 = 0. \end{aligned} \qquad (4.113)$$

Since we are searching for w' as a varying function (otherwise (4.82) and (4.83) are not compatible), the condition (4.113) implies that

$$P_1(0,1) + Q_1(0,1) = 0 \qquad (4.114)$$

and

$$2P_2(0,1) + 2Q_2(0,1) + Q(0,1) - R_1(0,1) = 0. \qquad (4.115)$$

In order that $\mu > 0$, i.e. $W_1 + W_2 > 0$, it is necessary that

$$P(0,1) + Q(0,1) > 0. \qquad (4.116)$$

Thus a cylindrical tube composed of a material described by a strain energy function W of the form (4.110), with P, Q chosen so that (4.114) is satisfied, can sustain pure axial shear provided (4.115) and (4.116) hold. From (4.83) and the boundary condition $(4.68)_2$, we obtain the solution

$$w(R) = \frac{BT_0}{\mu\,(P(0,1) + Q(0,1))} \ln\left(\frac{R}{A}\right) \qquad (4.117)$$

for any W of the form (4.110). To satisfy the boundary condition $(4.69)_1$, by (4.86), (4.112) and (4.114), one finds that

$$P_2(0,1) + Q_2(0,1) = 0. \qquad (4.118)$$

Combining (4.118) and (4.115), we obtain

$$Q(0,1) - R_1(0,1) = 0. \qquad (4.119)$$

Thus, in summary, provided that (4.114), (4.116), (4.118), and (4.119) hold, any compressible material described by (4.110) allows pure axial shear of the tube arising from a uniform shear traction applied to its outer surface, with the radial traction vanishing there. By application also of (4.112), from (4.85), we find that the only nonzero stresses are then

$$\begin{aligned} T_{zz} &= \frac{B^2 T_0^2}{\mu\,(P(0,1) + Q(0,1))\,R^2}, \\ T_{rz} &= \frac{BT_0}{R}. \end{aligned} \qquad (4.120)$$

In the same way as for the pure torsion deformation, here, it is possible to give some explicit example of (4.110) when P, Q, R are chosen. Polignone and Horgan [98] give some examples. One of these is the following strain energy,

$$W = \frac{\mu}{2}\gamma\Big[e^{\alpha(I_1-I_2)}e^{\beta(I_3-1)}(I_1-3) + e^{-\alpha(I_1-I_2)}e^{-\beta(I_3-1)}(I_2-3) + e^{(I_1-I_2)}e^{-2(I_3-1)} - 1\Big],$$

with $\alpha \neq 0,\ \beta \neq 0,\ \gamma > 0$ arbitrary constants.

4.3 Some other meaningful isochoric deformations

A third isochoric deformation for compressible materials that has been investigated in a similar fashion is that of *azimuthal shear* (or circular shear) of a cylindrical tube,

$$r = R, \qquad \theta = \Theta + g(R), \qquad z = Z, \tag{4.121}$$

where (R, Θ, Z) and (r, θ, z) are the cylindrical coordinates in the reference and in the current configurations, respectively, and the inner surface of the tube is bonded to a rigid cylinder. The deformation may be achieved either by applying a uniformly distributed azimuthal shear traction on the outer surface together with zero radial traction or by subjecting the outer surface to a prescribed angular displacement, with zero radial displacement. For compressible materials, we know by Ericksen's result [34] that azimuthal shear is not a universal solution and that in general, it is accompained by a radial deformation. These axisymmetric fields are governed by a coupled pair of nonlinear ordinary differential equations, one of which is second-order and the other first-order. Azimuthal shear, therefore, cannot be sustained by all compressible materials, unless certain auxiliary conditions on the strain energy function are satisfied. That problem has been examined by Beatty and Jiang [10], Haughton [52], Jiang and Ogden [66] and Polignone and Horgan [100].

The *generalized azimuthal shear* is an isochoric deformation of the form

$$r = R, \qquad \theta = \Theta + g(R, Z), \qquad z = Z, \tag{4.122}$$

where (R, Θ, Z) and (r, θ, z) are the cylindrical coordinates in the reference and in the current configurations, respectively. This deformation (or its $Z-$independent specialization) may also appear under the names of *circular* or *rotational* shear. For compressible materials, that problem has been investigated by Kirkinis and Tsai [71].

The isochoric deformation consisting of the composition of the shearing deformation (4.67) (with $r(R) = R$) and (4.121) is called *helical shear* and it is described by

$$r = R, \qquad \theta = \Theta + g(R), \qquad z = Z + w(R), \tag{4.123}$$

where (R, Θ, Z) and (r, θ, z) are the cylindrical coordinates in the reference and in the current configurations, respectively. This last problem has been examined by Beatty and Jiang [11].

4.4 Nearly isochoric deformations for compressible materials

In the previous sections we have seen an example of how a strategy of applying the semi-inverse method, while dealing with complex models, generalizes forms of solutions already known within the framework of a simpler theory. Motivated by the results obtained in the incompressible case, we have tried to understand what happens in the compressible case. By doing so, many important exact solutions for special classes of compressible elastic materials have been obtained. In [28], we emphasized that great care has to be exercised in using semi-inverse method in continuum mechanics to delineate classes of constitutive equations that admit a particular class of deformations and motions. Sometimes, the admissibility of a given deformation field is considered to delineate special classes of constitutive laws. We pointed out that the classes of constitutive equations thus identified from the standpoint that it may admit a type of deformation may lead to models that exhibit *physically unacceptable mechanical behavior*.

To illustrate the dangers inherent to merely turning the mathematical crank to determine classes of constitutive equations where a certain class of deformations are possible, we now consider the torsion of a cylindrical shaft (§4.4.1), the axisymmetric anti-plane shear of a cylindrical tube (§4.4.2), and then the propagation of transverse waves (§4.4.3) in a compressible nonlinear elastic material. We show that great care has to be exercised in appealing to the semi-inverse method. The first and third examples are extracted from our recent work [28]. In the first and second examples, we consider some static deformations with the help of which we can lay bare the confusion that has been created in seeking semi-inverse solutions. By considering torsional deformation and axial axisymmetric shear of a cylindrical shaft and tube, respectively, we discuss step by step the criticism concerning the mistakes that have been made as well as the possible errors that can be committed. Then in the third example, we consider the propagation of transverse bulk waves (primary motion), which, according to general nonlinear elasticity theory, must always be coupled to a longitudinal wave (secondary motion). Instead of considering what happens within the context of the linearized theory, a second-order theory and then the general nonlinear setting, we consider a top-to-bottom approach. We derive the general equations and, assuming that the amplitude of the displacements is of order ϵ, we show that at the first order we recover the results of the linearized theory and that at a higher order of approximation, we may have some insight into the coupling between the various modes of deformation. Here, the interesting point is the occurence of the phenomena of resonance between the primary and secondary fields.

Let us recall that it has been possible to determine the most general class of compressible materials for which pure torsion is a controllable deformation in the case of a circular solid cylinder. This means that for the constitutive equations that allow the deformation in question, the balance equations are satisfied for the pure torsion deformation. The next step is to ensure that the lateral surface of the circular cylinder is traction-free. Now, because simple torsion is an isochoric deformation, we have to ensure that the lateral boundary has to be traction-free while the volume remains constant. There is no reason to expect that this situation is automatically complied with in a compressible material. It is more natural to expect that when the lateral boundary of the cylinder is traction-free, the volume change has to be non-zero. In some sense, the behavior of a class of compressible materials such that pure torsion is controllable is *extraordinary.* We now investigate quantitatively the meaning of this sort of unusual possibility.

To make this claim quantitative, let us observe that any *idealized material* characterized by special mathematical properties cannot be clearly identified in the *real world.* That is, all mathematical models have to be viewed as approximations and one has to evaluate how well such models represent reality. We have to make some determination of what we will find acceptable in terms of an approximate answer. Such a determination cannot be totally subjective and one has to have some sort of agreement amongst those developing and using such models. Whether the criticism concerning the inapplicability of certain models is appropriate or otherwise needs to be judged by the modeller.

For example, let us suppose that we wish to consider the mathematical assumption that $W = W(I_2, I_3)$ only with regard to a specific body. This is exactly the constitutive assumption made by Blatz and Ko [8] in their celebrated model for foamed polyurethane elastomeric foams. It is imperative, when we make such an assumption, to check whether the experimental data backs the validity of the *mathematical* relationship

$$\partial W/\partial I_1 = 0. \tag{4.124}$$

Because, the first derivatives of the strain energy function are the mechanical quantities directly related to the stress, the relation (4.124) is indeed the correct way to check the constitutive assumption $W = W(I_2, I_3)$, for example in a biaxial experiment. It is clear that in the *real world*, our measurement in itself introduces an *uncertainty* with regard to the measured quantity, and that the accuracy of measurement is such that any measurement of the mechanical quantity $\partial W/\partial I_1$ to check the (4.124) will deliver a real number ϵ different from zero. It is not merely the prerogative of the modeller to say when ϵ is sufficiently small enough to be considered zero but, and as always, any theoretical assumption is an approximation and making such an approximation is an *art.* Roughly speaking, in a nonlinear theory, just because a certain quantity is small it does not follow that everything else connected with this quantity is or remains small. For this reason, we must be very careful in considering constitutive assumptions generated by purely mathematical arguments such as the ones arising from the semi-inverse method[4].

On the other hand, it is clear that approximations must be consistent and for the specific problem under consideration the following problem arises. If a given problem depends on various parameters α_i, $i = 1, ..., n$ and depends on a *small* parameter ϵ such that for $\epsilon = 0$ the secondary deformation may be ignored, then the small ϵ-approximation is consistent if for $\epsilon << 1$ the secondary field is neglible for any admissible value of the parameters α_i.

[4]We point out that this procedure is exactly the reverse of the constitutive assumption that comes out from a rigorous mathematical definition of some physical intuition. Notable examples of this last situation are the concept of frame indifference and material symmetry. In this case we start by the evidence provided by our observations in the real world and we then try to translate this into mathematics; in the former case we force mathematics to fit into the real world.

4.4.1 Nearly pure torsion of compressible cylinder

Let us consider a compressible cylinder of radius A subjected to the torsional deformation (4.1). We refer to the pure torsion of the cylindrical shaft as the "primary deformation", while by "secondary deformation" we mean the radial displacement $r(R)$. This means

$$\max_{R\in[0,A]}\left|\frac{r(R)}{R}-1\right|\approx O(\epsilon), \tag{4.125}$$

or $\sqrt{I_3}\approx 1$ for all $R\in[0,A]$ and for any other parameters (α_i for previous reference).

Now let us consider the classical Blatz-Ko material (2.40), with the strain energy function

$$W=\frac{\mu}{2}\left[\left(\frac{I_2}{I_3}-3\right)+2(\sqrt{I_3}-1)\right], \tag{4.126}$$

where μ is a constant, the initial shear modulus. This model is of the form $W=W(I_2,I_3)$ and it is well known (see Section 4.1.3) that for the class of materials described by the strain energy function given by (4.126), the isochoric simple torsion deformation is controllable.

Let us consider a more general strain energy function than (4.126), i.e.

$$W=k(I_1-3)+\frac{\mu}{2}\left[\left(\frac{I_2}{I_3}-3\right)+2(1-2k/\mu)(\sqrt{I_3}-1)\right], \tag{4.127}$$

where k and μ are constants. The strain energy function (4.127) differs from (4.126) by a term linear in I_1 and a *null-Lagrangian* term $\sqrt{I_3}$ (see Haughton [53]) such that the usual restrictions imposed by the normalization conditions are satisfied. Clearly as $k\to 0$ we recover (4.126) from (4.127).

The derivatives of the strain energy function (4.127) with respect to the invariants are

$$W_1=k,\qquad W_2=\frac{\mu}{2I_3},\qquad W_3=\frac{\mu}{2}\left(\frac{1-2k/\mu}{\sqrt{I_3}}-\frac{I_2}{I_3^2}\right). \tag{4.128}$$

Now it is possible to evaluate via a suitable experiment the magnitude of the parameter k and to decide if the assumption $W_1=0$ is reasonable on the basis of fitting the experimental data. If $k=0$, then the model (4.127) reduces to (4.126). Our point is that this model is so special that it is not possible to ensure that the predictions of the mechanical response are not in contradiction with the assumption $k=0$.

To make this point more quantitative, the next step is to introduce the dimensionless independent variable $\zeta=R/A\in[0,1]$, the dimensionless dependent variable

$$F(\zeta)=r/A \tag{4.129}$$

and the quantities

$$\hat{\tau}=A\tau,\quad \hat{k}=k/\mu. \tag{4.130}$$

The introduction of (4.128), evaluated for the specific deformation under consideration, in (4.8), leads to the equation

$$\hat{k}\left(\frac{\zeta F''}{F}+\frac{F'}{F}-\hat{\tau}^2\zeta-\frac{1}{\zeta}\right)+\frac{3}{2}\frac{\zeta F''}{FF'^4}+\frac{\zeta^3}{2F^4}-\frac{1}{2FF'^3}=0. \qquad (4.131)$$

(Here $F'=\mathrm{d}F/\mathrm{d}\zeta$). Moreover, from $(4.5)_1$, the dimensionless radial stress component associated with the deformation, for the model (4.127) is given by

$$\hat{T}_{\zeta\zeta}(\zeta)=1-2\hat{k}+2\hat{k}\frac{F'^2}{\sqrt{I_3}}-\frac{F'^{-2}}{\sqrt{I_3}}. \qquad (4.132)$$

Therefore, for a solid circular cylinder initially of radius A subjected to end torques only, the boundary value problem of interest here is given by equation (4.131), subject to the conditions $\hat{T}_{\zeta\zeta}(1)=0$ (i.e. $T_{rr}(A)=0$) and $F(\zeta)\to 0$ as $\zeta\to 0$. We point out that the isochoric solution $F(\zeta)=\zeta$ is controllable for the model (4.127) if and only if $k=0$ and in this case, $\hat{T}_{\zeta\zeta}(1)=0$.

It seems unlikely that one can obtain an explicit exact solution for equation (4.131), and even a numerical solution for the boundary value problem under investigation is not easy to obtain because the boundary condition on $\zeta=1$ is nonlinear and of mixed type. For this reason, we consider an approximate $\mathcal{O}(\hat{k})$ solution. A straighforward computation gives

$$F(\zeta)\approx\zeta+\hat{k}\frac{\hat{\tau}^2\zeta}{24}\left(2\zeta^2-5\right), \qquad (4.133)$$

and the $\mathcal{O}(\hat{k})$ volume approximation is

$$J\approx 1+\mathcal{V}(\hat{\tau}^2,\zeta)\hat{k}, \qquad (4.134)$$

where

$$\mathcal{V}(\hat{\tau}^2,\zeta)=\frac{(4\zeta^2-5)\hat{\tau}^2}{12}$$

is the local variation of volume at order $\hat{k}$. The maximum of this variation is

$$\left|\mathcal{V}(\hat{\tau}^2,0)\right|=\frac{5}{12}\hat{\tau}^2. \qquad (4.135)$$

Because equations (4.133) and (4.135) depend not only on $\hat{k}$ but also on τ^2, and because the two parameters are independent, it is clear that the approximation $\hat{k}=0$ may be not consistent.

Now, imagine that you are able to evaluate via an experiment the parameter $\hat{k}$ and that you discover that this parameter is small. It is clear that the experimentally determined number may be never small enough to justify the model corresponding to $\hat{k}=0$ and only the modeller can choose to set $\hat{k}=0$, or do otherwise. Our computation shows that such an assumption might be dangerous under certain circumstances. Indeed, while the limiting model for $\hat{k}\to 0$ predicts that during torsion the variation of volume is null, this is not always the case even for very small $\hat{k}$. To show this we generated pictures in 4.1, where two different coaxial cylinders are considered to describe the situation evoked. The external cylinder is represented in the picture by only its external surface through a circumference line of radius $R = 1$. It is the cylinder where no deformation occurs in the reference configuration. The dark circle stays in place for the cylinder in the current configuration, after a torsional deformation (4.1) is imposed. Now it can be appreciated, depending on the amount of torsion $\hat{\tau}$ imposed, how the radius reduces with the law (4.133) and consequent change of volume occurs. In Figures 4.1 a-b) the approximation value $\hat{k} = 0.05$ is considered, and the amounts of torsion are $\hat{\tau} = 2$ and $\hat{\tau} = 2.5$ respectively. When a small value for $\hat{k}$ is slightly increased to $\hat{k} = 0.1$, the reduction of the radius for the deformed cylinder is more appreciated. See Figures 4.1 c-d) where the parameters of the torsion are $\hat{\tau} = 2$ and $\hat{\tau} = 2.5$, respectively. Clearly, the use of the model (4.126) is fraught with danger because it is too special.

This situation is peculiar to all the constitutive models that are identified by enforcing special mechanical behaviors via purely mathematical properties, such as the controllability of isochoric deformations within the context of a theory to describe the response of compressible bodies.

4.4.2 Nearly pure axial shear of compressible tube

Let us consider a compressible tube of internal and external radii A and B, respectively, subjected to an axial axisymmetric shear deformation (4.67). Here, in order to search for pure axial shear deformation, we refer the out-displacement $w(R)$ as "primary deformation" while we refer to the radial displacement $r(R)$ as "secondary field". Similarly to the previous section, this means that

$$\max_{R\in[A,B]} \left| \frac{r(R)}{R} - 1 \right| \approx \mathcal{O}(\epsilon),$$

or $\sqrt{I_3} \approx 1$ for all $R \in [A, B]$ and for any other parameters. Now let us consider the classical Hadamard material (2.21), with strain energy function that we rewrite here as

$$W = c_1(I_1 - 3) + k(I_2 - 3) + H(I_3), \tag{4.136}$$

where $c_1 > 0$ and $k \geq 0$ are material constants. Clearly if $k = 0$, the model (4.136) satisfies the necessary and sufficient condition (4.104) and the material will be therefore capable of sustaining pure axial shear, for every function $H(I_3)$ satisfying the normalization conditions on the strain energy (1.34) and (1.64).

The derivatives of the strain energy function (4.136) with respect to the invariants are

$$W_1 = c_1, \quad W_2 = k, \quad W_3 = H'(I_3). \tag{4.137}$$

Now it is possible to evaluate, via a suitable experiment, the magnitude of the parameter k with respect to the parameter μ and to decide if the assumption $W_2 = 0$ is reasonable on the basis of fitting the experimental data. Let us consider the model (2.23) proposed by Levinson and Burgess [79] as special case of strain energy function (4.136),

$$W = \left(\frac{\mu}{2} - k\right)(I_1 - 3) + k(I_2 - 3) + \tfrac{1}{2}\left(-2k + \lambda + \mu\right)(I_3 - 1) - (\lambda + 2\mu)(\sqrt{I_3} - 1), \tag{4.138}$$

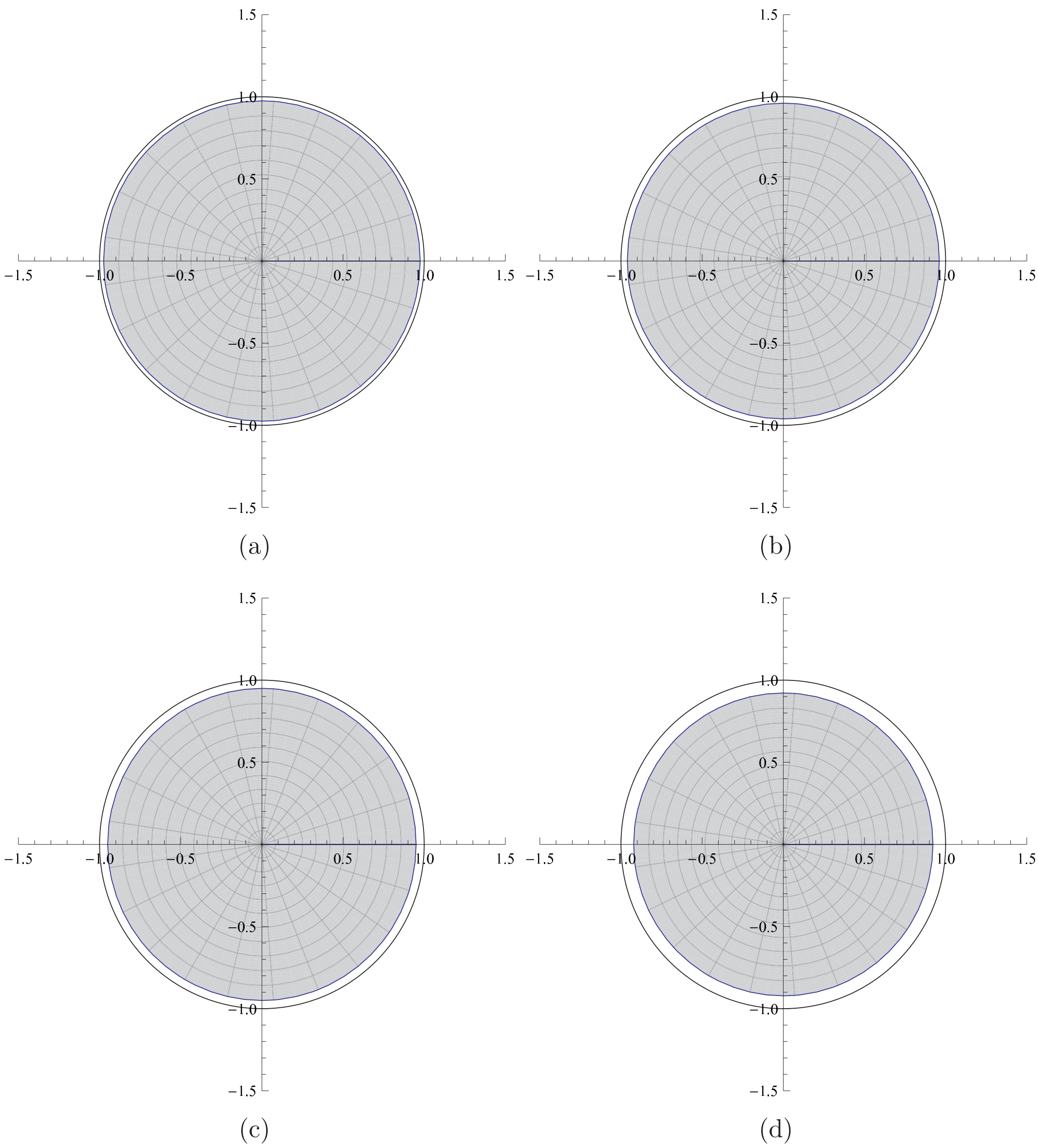

Figure 4.1: View of the transverse sections of two perturbed Blatz-Ko cylinder (4.127): the first one is a cylinder in the reference configuration when no deformation is applied (in the figure represents only its lateral surface through the circumference line of radius $R = 1$), and the second one is a cylinder in the current configuration when an amount of torsion is applied (in the figure, it is represented by a meshed cylinder of radius (4.133)) for a) $\hat{k} = 0.05,\ \hat{\tau} = 2.0$, b) $\hat{k} = 0.05,\ \hat{\tau} = 2.5$, c) $\hat{k} = 0.1,\ \hat{\tau} = 2.0$, d) $\hat{k} = 0.1,\ \hat{\tau} = 2.5$.

where λ and μ are the Lamé constants of linear elasticity. At $k = 0$ the model (4.138) reduces to

$$W = \frac{\mu}{2}(I_1 - 3) + \frac{1}{2}(\lambda + \mu)(I_3 - 1) - (\lambda + 2\mu)(\sqrt{I_3} - 1) \tag{4.139}$$

which satisfies the necessary and sufficient condition to sustain pure axial shear. In this last case ($k = 0$) it is easy to obtain the expression for the displacement

$w(R)$ and the stress field (see Section 4.2.3). The next step is to introduce the dimensionless independent variable $\zeta = R/B$, the dimensionless dependent variables

$$F(\zeta) = r/B, \quad \hat{w} = w/B, \tag{4.140}$$

and independent variables

$$\begin{aligned} \eta = A/B, \quad \hat{k} = k/\mu, \quad \hat{\lambda} = \lambda/\mu, \\ \hat{K} = K/(B\mu), \quad \hat{\boldsymbol{T}} = \boldsymbol{T}/\mu, \quad \hat{T}_0 = T_0/\mu, \end{aligned} \tag{4.141}$$

so that $\eta \leq \zeta \leq 1$. The derivatives of the strain energy (4.138) with respect to the invariants are

$$W_1 = \frac{\mu}{2} - k, \quad W_2 = k, \quad W_3 = \tfrac{1}{2}\left(-2k + \lambda + \mu\right) - \frac{\lambda + 2\mu}{2\sqrt{I_3}}. \tag{4.142}$$

The introduction of the dimensionless variables and of (4.142) evaluated for the specific deformation under consideration in (4.79), leads to the equation

$$\begin{aligned} &-2\hat{k}\zeta\hat{w}'^2 F + \zeta F(-1 + (\hat{\lambda}+1)F'^2) \\ &\qquad - (\hat{\lambda}+1)F^2(F' - \zeta F'') + \zeta^2(F' + \zeta F'') = 0, \end{aligned} \tag{4.143}$$

and in (4.77) leads to the equation

$$2k\hat{w}'[F^2 - \zeta^2] + \zeta(\zeta\hat{w}' - 2\hat{K}) = 0. \tag{4.144}$$

For $\hat{k} = 0$, we know that a solution for pure axial shear is $F(\zeta) = \zeta$. Here, we consider an approximation $\mathcal{O}(\hat{k})$ solution, in the spirit of the previous section. Let us assume that

$$F(\zeta) = \zeta + \hat{k}g(\zeta), \tag{4.145}$$

where g is an unknown dimensionless function of ζ. The problem is to solve both the equilibrium equations (4.143) and (4.144) for the unknowns F and $\hat{w}$ such that the following boundary conditions, equivalent to (4.68) and $(4.69)_1$,

$$g(\eta) = 0, \qquad \hat{w}(\eta) = 0, \qquad \hat{T}_{rr}(1) = 0, \tag{4.146}$$

are satisfied. Using (4.145) and $(4.146)_1$ we obtain (at first order) the boundary conditions that g must satisfy:

$$g(\eta) = 0, \quad (\lambda + 2)g'(1) + \lambda g(1) = 0. \tag{4.147}$$

The approximation equilibrium equations (4.143) and (4.144) (at first order) become

$$\zeta\left[(\hat{\lambda}+2)(g' + \zeta g'') - 2\hat{w}'^2\right] - (\hat{\lambda}+2)g = 0, \tag{4.148}$$

and

$$\tfrac{1}{2}\zeta\hat{w}' = \hat{K}, \tag{4.149}$$

respectively. The solution of (4.149) satisfying also the boundary condition $\hat{w}(\eta) = 0$ is given by

$$\hat{w}(\eta) = 2\hat{K}\ln\left(\frac{\zeta}{\eta}\right). \tag{4.150}$$

Using (4.150) in (4.148), the equilibrium equation in the unknown g is given by

$$(\hat{\lambda}+2)\zeta[\zeta(\zeta g''+g')-g]-8\hat{K}^2=0 \tag{4.151}$$

from which we obtain

$$g(\zeta)=\frac{1+\zeta^2}{2\zeta}d_1+\frac{\zeta^2-1}{2\zeta}d_2-2\hat{K}^2\frac{1+2\ln\zeta}{\left(\hat{\lambda}+2\right)\zeta}, \tag{4.152}$$

where d_1 and d_2 are integration constants obtained by the boundary conditions (4.147). From (4.78), making use of the solution (4.145), we obtain the value $\hat{K}$ as

$$\hat{K}=\frac{(\hat{\lambda}+1+\eta^2)\hat{T}_0}{(\hat{\lambda}+1+\eta^2)+\left[(\hat{\lambda}+1+\eta^2)[(\hat{\lambda}+1+\eta^2)+2\hat{k}\hat{T}_0^2(\eta^2-1-2\ln\eta)]\right]^{1/2}} \tag{4.153}$$

The $\mathcal{O}(\hat{k})$ volume change approximation is

$$J\approx 1+\hat{k}\hat{T}_0^2\frac{2\zeta^2\log\eta+(\hat{\lambda}+2)\zeta^2-1-\eta^2-\hat{\lambda}}{(\hat{\lambda}+2)(\hat{\lambda}+1+\eta^2)\zeta^2}. \tag{4.154}$$

It is interesting to study the behaviour of J when $\zeta\to\eta$, because it attains the maximum of this variation there,

$$J(\eta)\approx 1+\hat{k}\hat{T}_0^2\frac{2\eta^2\log\eta+(\hat{\lambda}+1)\eta^2-1-\hat{\lambda}}{(\hat{\lambda}+2)(\hat{\lambda}+1+\eta^2)\eta^2}. \tag{4.155}$$

Because $\eta<1$ is arbitrary, if we consider an approximation of $J(\eta)$ for small $\eta=\delta$, we obtain that

$$J(\delta)\approx 1-\hat{k}\frac{\hat{T}_0^2}{(\hat{\lambda}+2)\delta^2}. \tag{4.156}$$

Since (4.156) depends not only on $\hat{k}$ but also on the square of the traction $\hat{T}_0^2$, and because the two parameters are independent, it is clear that here as in the previous example, the approximation $\hat{k}=0$ may not be consistent. For example, if we are able to evaluate via an experiment the parameter $\hat{k}$ and we discover that this parameter is small, say $\hat{k}=0.01$, then we may in our upcoming numerical simulations take $A=B/10$, so that $\eta=0.1$, and assume $\hat{\lambda}=1$ of the same magnitude of μ. After these assumptions, the formula (4.156) becomes

$$J(\delta)\approx 1-\frac{1}{3}\hat{T}_0^2, \tag{4.157}$$

and cleary we can imagine that the isochoric assumption $J=1$ might be very dangerous when the magnitude of the traction $|\hat{T}_0|$ moves away from zero (see Figure (4.2)), because the dependence is quadratic[5].

[5] The formula (4.157) is a good approximation when small $\hat{k}$, δ and $\hat{T}_0$ are considered, to avoid zero or negative volume variation.

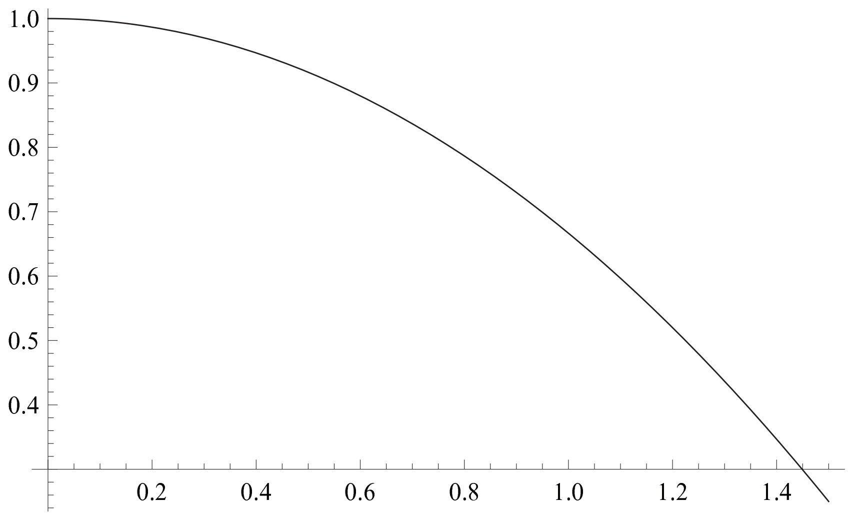

Figure 4.2: Plot of $J(\delta)$ when the assumption $\hat{k} = 0.01$, $\delta = 0.1$, $\hat{\lambda} = 1$ (see formula (4.157)) against $\hat{T}_0$ running from zero ($J = 1$) to 1.5 ($J \approx 0.25$).

4.4.3 Another example: transverse and longitudinal waves

Another important example emphasizing that if we ignore the full scope of the deformation, we may be misguided and we may miss real and interesting phenomena, is given by the propagation of longitudinal and transverse waves.

Introducing the Cartesian coordinates (X_1, X_2, X_3) in the undeformed configuration and the Cartesian coordinates (x_1, x_2, x_3) in the current configuration, we consider the motion given by

$$x_1 = u(X_1, t), \quad x_2 = X_2 + v(X_1, t), \quad x_3 = X_3, \tag{4.158}$$

where the longitudinal wave u and the transverse wave v must be determined from the balance equation. The principal invariants: I_1, I_2 and I_3, are given by

$$I_1 = 2 + u_{X_1}^2 + v_{X_1}^2, \quad I_2 = 1 + 2u_{X_1}^2 + v_{X_1}^2, \quad I_3 = u_{X_1}^2. \tag{4.159}$$

The equations of motion (1.24) in the absence of body forces, reduce to the two scalar equations

$$\begin{aligned} \rho_r \frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial X_1} \left[2 \left(W_1 + 2W_2 + W_3 \right) u_{X_1} \right], \\ \rho_r \frac{\partial^2 v}{\partial t^2} &= \frac{\partial}{\partial X_1} \left[2 \left(W_1 + W_2 \right) v_{X_1} \right]. \end{aligned} \tag{4.160}$$

Here the strain energy W is a function of $u_{X_1}^2$ and $v_{X_1}^2$.

We remark that in the linearized limit, (4.160) reduces to the classical *uncoupled* systems of linear wave equations (Atkin and Fox [4]).

If we consider the case $u(X_1, t) \equiv X_1$, equations (4.160) reduce, in the general case, to an overdetermined system of two differential equations in the single unknown v. Therefore it seems, at least at first sight, that it is not possible to ensure the existence of a transverse wave in the nonlinear theory for any material within the constitutive class (1.36). It is possible that for *special* classes of materials, this overdetermined system may have a solution. For example this is the case for Hadamard materials (2.21). In the case of Hadamard materials, because $u \equiv X_1$ and $I_3 = 1$, we find that (4.160) reduces

$$\rho_r \frac{\partial^2 v}{\partial t^2} = \mu \frac{\partial^2 v}{\partial X_1^2}. \tag{4.161}$$

In this case, the system is compatible and the transverse wave solution may be computed by solving a linear differential equation, as in the linearized theory of elasticity.

Now let us consider for the Hadamard material the case where the longitudinal wave $u(X_1, t)$ is of order ϵ, where $|\epsilon| << 1$. Then we consider the model (2.23),

$$H(I_3) = (\lambda + \mu)(I_3 - 1) - (\lambda + 2\mu)\left(\sqrt{I_3} - 1\right), \tag{4.162}$$

proposed by Levinson and Burgess [79]. Now equations (4.160) become

$$\rho_r \frac{\partial^2 u}{\partial t^2} = (\lambda + 2\mu)\frac{\partial^2 u}{\partial X_1^2}, \qquad \rho_r \frac{\partial^2 v}{\partial t^2} = \mu \frac{\partial^2 v}{\partial X_1^2}. \tag{4.163}$$

In this case we find that the equations are the same as in the linearized theory: they are uncoupled.

We take a further step and we consider a small coupling, i.e. we modify the constitutive equation (2.21) to be

$$\begin{aligned} W = c_1(I_1 - 3) + c_2(I_2 - 3) + (\lambda + \mu)(I_3 - 1) \\ - (\lambda + 2\mu)\left(\sqrt{I_3} - 1\right) + kI_3(I_1 - I_3 - 2), \end{aligned} \tag{4.164}$$

where k is the coupling parameter and

$$c_1 = \frac{1}{2}(\lambda + 2\mu - 4k), \qquad c_2 = \frac{1}{2}(2k - \lambda - \mu). \tag{4.165}$$

In this case we compute

$$\rho_r \frac{\partial^2 u}{\partial t^2} = (\lambda + 2\mu)\frac{\partial^2 u}{\partial X_1^2} + 2k\frac{\partial}{\partial X_1}\left(v_{X_1}^2 u_{X_1}\right), \tag{4.166}$$

and

$$\rho_r \frac{\partial^2 v}{\partial t^2} = (\mu - 2k)\frac{\partial^2 v}{\partial X_1^2} + 2k\frac{\partial}{\partial X_1}\left(u_{X_1}^2 v_{X_1}\right). \tag{4.167}$$

Clearly the term $\partial(u_{X_1}^2 v_{X_1})/\partial X_1$ in the right hand side of (4.167) may be (at least at first sight) ignored because the amplitude u is small. This means that we may consider the system of equations (4.166) and (4.167) as being decoupled. This is indeed a way to justify the Hadamard material (2.21), which is a model predicting an exact decoupling. As we have already remarked, any experimental determination of the coupling k may lead to k being *small* but never zero.

To make the idea rigorous, we must (at least) require that, given a set of boundary conditions (for example $u = v = 0$ at $X_1 = 0$ and L), the initial condition is such that $u(X_1, 0) \approx \mathcal{O}(\epsilon^n)$ with suitable $n \geq 1$ and that we have a suitable a priori bound on the solution such that for any time we ensure $u(X_1, t) \approx \mathcal{O}(\epsilon)$. Then, if this a priori bound exists, the initial conditions satisfy the requirements and when k is small it is possible to consider the transverse waves as being decoupled from the longitudinal waves.

The point is that it is clear from the structure of the equations that this bound cannot exist for all the admissible range of parameters. Let $k \approx \mathcal{O}(\epsilon)$. When the longitudinal motion is small, a better approximation than the linear one is to neglect the term $2k\partial_{X_1}\left(u_{X_1}^2 v_{X_1}\right)$ in (4.167) (which is $\mathcal{O}(\epsilon^3)$), but to maintain the coupling term in (4.166) (which is $\mathcal{O}(\epsilon^2)$). In this case (4.167) is a classical linear wave equation; introducing $c_T^2 = (\mu - 2k)/\rho_r$ this equation admits solutions of the usual form

$$v(X_1, t) = \sum_{n=1}^{\infty} \left[A_n \cos(k_n^T t) + B_n \sin(k_n^T t)\right] \sin(n\pi X_1/L),$$

where

$$k_n^T = n\pi c_T/L \tag{4.168}$$

is the transverse wave number of the $nth-$mode and A_n, B_n are integration constants such that the initial condition $u(X_1, 0) \approx \mathcal{O}(\epsilon^n)$ is verified. If we introduce this solution for $v(X_1, t)$ into (4.166) we obtain for $u(X_1, t)$ a linear but non-autonomous equation for which is possible to search for solutions in the form

$$u(X_1, t) = \sum_{n=1}^{\infty} \eta_n(t) \sin(n\pi X_1/L),$$

where $\eta_n(t)$ are unspecified functions of t. Using standard methods of nonlinear oscillations (Nayfeh and Mook [89]) we obtain a reduction of the equations to an infinite system of coupled ordinary differential equations in the unknowns η_n. These equations are non-autonomous and they display autoparametric resonance phenomena for some values of the various parameters. Therefore, an a priori bound is impossible. This means that it does not matter how small the longitudinal motions are, because after a certain time their amplitude cannot be neglected and a full coupling between transverse and longitudinal motions must be considered. Therefore, the Hadamard model is much too special to be considered as a reasonable idealization of real elastic bodies.

Phenomena of this kind are quite common in classical mechanics. For example in the framework of the elementary and classical theory for holonomic systems, it is well known that unstable normal modes may not contribute to the approximate linear theory. This happens for modes that are "latent" at the initial time. Nevertheless, the higher orders neglected in the Lagrangian can awaken these latent unstable modes, and bring the system away from equilibrium[6].

[6] A simple and clear example of a mechanical system displaying wake-up of latent modes is reported in page 133 of Biscari et al. [16].

Notes

Isochoric deformations play on important role in solid mechanics and here we have appealed to them to illustrate our thesis in the context of nonlinear elasticity. To simplify the exposition, we have only considered the theory of unconstrained nonlinear isotropic elasticity. But our remarks are completely general and apply (with some modifications) in general to the use of semi-inverse methods in continuum mechanics. For example Jiang and Beatty [65] find also necessary and sufficient conditions on the strain energy function for homogeneous and compressible, *anisotropic* hyperelastic materials to sustain controllable, axisymmetric helical shear deformations. Thus we think that one needs to exercise a great deal of prudence in ensuring that the results obtained by using the semi-inverse method make sense.