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A study of some spaces related to β-open sets

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Abstract. The purpose of this paper is to introduce weakly β-R_0 spaces, β-US spaces and to obtain their basic properties. It is shown that weakly β-R_0 and β-US spaces are preserved under pre-β-closed injection and pre-β-closed bijection respectively.

Keywords: weakly semi-R_0 spaces, weakly β-R_0 spaces, β-US spaces

MSC 2000 classification: 54C10, 54D10

1 Introduction and preliminaries

In 1986, Monsef et al. [2] defined and studied β-closure of a set in topological spaces. In this paper, using β-closure, we introduce weakly β-R_0 spaces for non trivial topology, β-US spaces and obtained preservation theorems for weakly β-R_0 spaces and β-US spaces. Several characterizations and fundamental properties are obtained. Throughout the present paper, X and Y always mean topological spaces on which no separation axiom is assumed unless explicitly stated.

Let A be a subset of a space X. The closure of A and interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ respectively.

Here we recall the following definitions, which will be used often throughout the paper.

1 Definition. A subset A of a topological space X is said to be β-open [1] (resp. semi-open [7], α-open [11]) if $A \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$ (resp. $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$, $A \subseteq \mathrm{Int}(\mathrm{Cl}((\mathrm{Int}(A)))))$. The complement of a β-open (resp. semi-open) set is said to be β-closed [1](resp. semi-closed [5]) and β-closure [2] (resp. semi-closure [5]) of a set A, denoted by $\beta\,\mathrm{Cl}(A)$ (resp. $\mathrm{SCl}(A)$), is the intersection of all β-closed (resp. semi-closed) sets containing A. A is said to be β-clopen [13] if it is β-open as well as β-closed.

2 Weakly β-R_0 spaces

2 Definition. A topological space X is said to be *semi-*R_0 [8] if and only if for each $x \in X$, $\mathrm{SCl}(\{x\}) \subseteq G$, where G is an open subset of X.

3 Definition. A topological space X is said to be *weakly semi-*R_0 [4] if and only if $\cap \mathrm{SCl}(\{x\})\colon x \in X = \emptyset$.

4 Definition. A topological space X is said to be *weakly* β-R_0 if and only if $\cap \beta\,\mathrm{Cl}(\{x\})\colon x \in X = \emptyset$.

5 Theorem. *Every weakly semi-*R_0 *space is weakly* β-R_0.

Proof. The proof is obvious in view of the fact that

$$\cap \{\beta\,\mathrm{Cl}\,(\{x\}) : x \in X\} \subset \cap \{\mathrm{SCl}\,(\{x\}) : x \in X\}\,.$$

$\boxed{QED}$

The converse of the above theorem is not true as can be seen from the following example:

6 Example. Let $X = \{a, b, c\}$, $T = \{\emptyset, X, \{a, b\}\}$. It is evident that

$$\cap \{\beta\,\mathrm{Cl}\,(\{x\}) : x \in X\} = \emptyset \;\text{ but }\; \cap \{\mathrm{SCl}\,(\{x\}) : x \in X\} = \{c\} \neq \emptyset.$$

7 Theorem. *If a space* X *is weakly* β-R_0 *then for every space* Y*, the product space* $X \times Y$ *is weakly* β-R_0.

Proof. Using [2, Theorem 2.10], we have

$$\begin{aligned}
&\cap \{\beta\,\mathrm{Cl}\,(\{x, y\}) : (x, y) \in X \times Y\} \\
&\qquad \subseteq \cap \{\beta\,\mathrm{Cl}\,(\{x\}) \times \beta\,\mathrm{Cl}\,(\{y\}) : (x, y) \in X \times Y\} \\
&\qquad = \cap \{\beta\,\mathrm{Cl}\,(\{x\}) : x \in X\} \times \cap \{\beta\,\mathrm{Cl}\,(\{y\}) : y \in Y\} \subseteq \emptyset \times Y = \emptyset.
\end{aligned}$$

$\boxed{QED}$

8 Definition. A function $f\colon X \to Y$ is said to be *pre-*β*-closed* [9] if the image of each β-closed subset of X is β-closed in Y.

9 Theorem. *Let* $f\colon X \to Y$ *be a pre-*β*-closed injective function. If* X *is weakly* β-R_0*, then so is* Y.

Proof. Using [9, Theorem 3.3 (ii)], we have

$$\begin{aligned}
&\cap \{\beta\,\mathrm{Cl}\,(\{y\}) : y \in Y\} \subseteq \cap \{\beta\,\mathrm{Cl}\,(\{f(x)\}) : x \in X\} \\
&\qquad \subseteq f\,(\cap \{\beta\,\mathrm{Cl}\,(\{x\}) : x \in X\}) = f(\emptyset) = \emptyset.
\end{aligned}$$

$\boxed{QED}$

10 Definition. Let (X, T) be a topological space and let $x \in X$. The *β-Kernel* of x is denoted and defined by

$$\beta\text{-}\ker(x) = \cap \{U : U \text{ is } \beta\text{-open in } X \text{ and } x \in U\}.$$

11 Theorem. *A topological space X is weakly β-R_0 if and only if β-$\ker(x) \neq X$ for each $x \in X$.*

Proof. Let x_0 be a point of X such that β-$\ker(x_0) = X$. This means that x_0 does not belong to any proper β-open subset of X. Therefore x_0 belongs to every proper β-closed subset of X. Hence x_0 belongs to the β-closure of every singleton. Therefore $x_0 \in \cap\{\beta \operatorname{Cl}(\{x\}) : x \in X\} = \emptyset$, as by assumption X is weakly β-R_0. This is a contradiction.
Conversely, assume that β-$\ker(x) \neq X$ for each $x \in X$. If there is a point x_0 in X such that $x_0 \in \beta \operatorname{Cl}(\{x\})$ then, every β-open set containing x_0 must contain every point of X. Therefore, the unique β-open set containing x_0 is X. Hence β-$\ker(x_0) = X$, which is a contradiction. Thus X is weakly β-R_0. $\boxed{QED}$

3 β-US-spaces

The concept of US-space was introduced by Slepian [15] and was further studied by Cullen [6], Murudeshwar, Naimpally [10] and Wilansky [16]. A space X is said to be a US-*space* if every sequence in X converges to a unique point. In this section, we introduce a weaker concept namely the concept of β-US spaces by making use of β-open sets.

12 Definition. A sequence $\langle x_n \rangle$ is said to *s-converge* [3] to a point of x if $\langle x_n \rangle$ is eventually in every semi-open set containing x.

13 Definition. A sequence $\langle x_n \rangle$ is said to *β-converge* to a point x if $\langle x_n \rangle$ is eventually in every β-open set containing x.

14 Definition. A space X is said to be *semi*-US [3] if every sequence in X s-converges to a unique point. Every US-space is semi-US but not conversely [3, Example 1.5].

15 Definition. A space X is said to be β-US if every sequence in X β-converges to a unique point.

Clearly, every semi-US space is β-US but not conversely as can be seen from the following example:

16 Example. Let $X = \{a,b,c,d\}$ and let $T = \{\emptyset, X, \{a,b,c\}, \{b,c,d\}, \{b,c\}\}$ then (X, T) is β-US space but not semi-US space, not even T_1.

17 Definition. A space X is said to be β-T_1 (respectively, β-T_2) [9] if for any two distinct points x and y of X, there exist β-open sets U and V such that

$x \in U$ and $y \notin U$, $y \in V$ and $x \notin V$ (resp. $x \in U$,$y \in V$ and $U \cap V = \emptyset$).

18 Theorem. *Every β-*US *space is β-T_1.*

PROOF. Let X be β-US. Let x and y be two distinct points of X. Consider the sequence $\langle x_n \rangle$ where $x_n = x$ for every n. Clearly $\langle x_n \rangle$ β-converges to x. Also, since $x \neq y$ and X is β-US, $\langle x_n \rangle$ cannot β-converge to y. That is, there exists a β-open set V containing y but not x. Similarly if we consider the sequence $\langle y_n \rangle$ where $y_n = y$ for all n and proceeding as before, we get a β-open set U containing x but not y. Thus X is β-T_1. $\boxed{QED}$

The converse of the above theorem need not be true as can be seen from the following example:

19 Example. Let X be a countably infinite set and T be the cofinite topology on X. Then (X,T) is T_1, hence β-T_1 but not β-US since the every sequence of distinct points of X converges to more than one point.

20 Theorem. *Every β-T_2 space is β-*US.

PROOF. Let X be β-T_2 and let $\langle x_n \rangle$ be a sequence in X. If possible, suppose that $\langle x_n \rangle$ β-converges to two points x and y. That is, $\langle x_n \rangle$ is eventually in every β-open set containing x and also in every β-open set containing y. This is not possible since X is β-T_2. Thus $x = y$. $\boxed{QED}$

Converse of the above theorem need not be true as can be seen from the following example:

21 Example. Let X be an uncountable set and let T be the cocountable topology on X. Then (X,T) is β-US but not β-T_2. The above example also shows that a US-space need not be β-T_2.

22 Definition. *A* set B is said to be sequentially *β-closed* if every sequence in B β-converges to a point in B.

23 Theorem. *A space X is β-*US *if and only if the diagonal Δ is a sequentially β-closed subset of $X \times X$.*

PROOF. Let X be β-US. Let $\langle x_n{,}x_n \rangle$ be a sequence in Δ. Suppose that $\langle x_n{,}x_n \rangle$ β-converges to $\langle x, y \rangle$. That is, $\langle x_n \rangle$ β-converges to x and y. Therefore $x = y$. Hence Δ is sequentially β-closed. Conversely, let Δ be sequentially β-closed. Let a sequence $\langle x_n \rangle$ β-converges to x and y. Hence, $\langle x_n{,}x_n \rangle$ β-converges to $\langle x, y \rangle$. Since Δ is sequentially β-closed, $\langle x, y \rangle \in \Delta$, which means that $x = y$. $\boxed{QED}$

24 Definition. A subset Y of a space X is said to be *sequentially β-compact* if every sequence in Y has a subsequence which β-converges to a point in Y.

25 Theorem. *In a β-*US *space, every sequentially β-compact set is sequentially β-closed.*

Proof. Let X be β-US. Let Y be a sequentially β-compact subset of X. Let $\langle x_n \rangle$ be a sequence in Y. Suppose that $\langle x_n \rangle$ β-converges to a point in $X \backsim Y$. Let $\langle x_{np} \rangle$ be a subsequence of $\langle x_n \rangle$. Since Y is sequentially β-compact, there exists a subsequence $\langle x_{np} \rangle$ of $\langle x_n \rangle$ such that $\langle x_{np} \rangle$ β-converges to a point y in Y. Also, a subsequence of $\langle x_n \rangle$,$\langle x_{np} \rangle$ β-converges to $x \in X \backsim Y$. Since $\langle x_{np} \rangle$ is a sequence in β-US space X,$x = y$. Thus, Y is sequentially β-closed. $\boxed{QED}$

26 Theorem. *Product of arbitrary family of β-US spaces is β-US.*

Proof. Let $X = \prod\limits_{\lambda \in \Delta} X_\lambda$ where each X_λ is β-US. Let a sequence $\langle x_n \rangle$ in X β-converges to $x = (x_\lambda)$ and $y = (y_\lambda)$. Then the sequence $\langle x_{n\lambda} \rangle$ β-converges to x_λ and y_λ for each $\lambda \in \Delta$. For, suppose that there exists a $\mu \in \Delta$ such that $\langle x_{n\mu} \rangle$ does not β-converge to x_μ. Then there exists a τ_μ-β-open set U_μ containing x_μ such that $\langle x_{n\mu} \rangle$ is not eventually in U_μ. Consider the set $U = \prod\limits_{\lambda \neq \mu} X_\lambda \times U_\mu$. Then U is a β-open subset of X [2, Lemma 2.1] and $\langle x_n \rangle$ is not eventually in U which contradicts the fact that $\langle x_n \rangle$ β-converges to x. Thus we get $\langle x_{n\lambda} \rangle$ β-converges to x_λ and y_λ for each $\lambda \in \Delta$. Since X_λ is β-US, $x_\lambda = y_\lambda$ for each $\lambda \in \Delta$. Thus $x = y$. Hence, X is β-US. $\boxed{QED}$

27 Corollary. *Product of arbitrary family of β-T_2 space is β-T_2.*

28 Theorem. *Every α-open-subset of β-US space is β-US.*

Proof. Let $Y \subseteq X$ be an α-open set. Let $\langle x_n \rangle$ be a sequence in Y. Suppose that $\langle x_n \rangle$ β-converges to x and y in Y. We shall prove that $\langle x_n \rangle$ β-converges to x and y in X. Let U be any β-open subset of X containing x and V be any β-open subset of X containing y. Then $U \cap Y$ and $V \cap Y$ are β-open subsets of Y [1, Lemma 2.5]. Therefore $\langle x_n \rangle$ is eventually in $U \cap Y$ and $V \cap Y$ and hence in U and V. Since X is β-US, $x = y$. Hence Y is β-US. $\boxed{QED}$

29 Corollary. *Every α-open-subset of β-T_2 space is β-T_2.*

30 Remark. The subset $\{a, c\}$ considered in Example 6 is not β-T_2. This shows that an arbitrary subset of β-T_2 space need not be β-T_2.

4 Some functions

31 Theorem. *The image of a β-US space under a bijective pre-β-closed function is β-US.*

Proof. Let $f : X \to Y$ be a bijective and pre-β-closed function and let X be β-US. Let $\langle y_n \rangle$ be a sequence in Y. Suppose that $\langle y_n \rangle$ β-converges to two points y_1 and y_2. In that case we shall prove that the sequence $\langle f^{-1}(y_n) \rangle$ β-converges to $f^{-1}(y_1)$ and $f^{-1}(y_2)$. Let U be a β-open set containing $f^{-1}(y_1)$. Then $f(U)$ is a β-open set containing y_1 and hence $\langle y_n \rangle$ is eventually in $f(U)$. Therefore

$\langle f^{-1}(y_n)\rangle$ is eventually in U. Hence $\langle f^{-1}(y_n)\rangle$ β-converges to $f^{-1}(y_1)$. Similarly we can prove that $\langle f^{-1}(y_n)\rangle$ β-converges to $f^{-1}(y_2)$. This is not possible since X is β-US. Hence Y is β-US. $\boxed{QED}$

32 Corollary. *The image of a β-T_2 space under a bijective pre-β-closed function is β-T_2.*

33 Definition. A function $f\colon X \to Y$ is said to be totally β-continuous if the inverse image of every open subset of Y is a β-clopen subset of X.

34 Theorem. *A space X is β-T_2 if for any two distinct points x and y of X, there exist β-open sets U and V such $x \in U$,$y \in V$ and $\beta\,\mathrm{Cl}(U) \cap \beta\,\mathrm{Cl}(V) = \emptyset$.*

Proof. It is Lemma 2.3 of [14]. $\boxed{QED}$

35 Lemma. *Let $f\colon X \to Y$ be a totally β-continuous injection. If Y is T_0, then X is β-T_2.*

Proof. Let a and b be any pair of distinct points of X. Then $f(a) \neq f(b)$. Since Y is T_0, there exists an open set U containing $f(a)$ but not $f(b)$. Then $a \in f^{-1}(U)$ and $b \notin f^{-1}(U)$. As f is totally β-continuous, $f^{-1}(U)$ is a β-clopen subset of X. Also $a \in f^{-1}(U)$ and $b \in X \backsim f^{-1}(U)$. By Theorem 34, X is β-T_2. $\boxed{QED}$

36 Theorem. *Let $f\colon X \to Y$ be a totally β-continuous injection. If Y is T_0, then X is β-*US.

Proof. Immediate in view of Theorem 20 and Lemma 35. $\boxed{QED}$

37 Definition. A function $f\colon X \to Y$ is said to be

(i) *Sequentially β-continuous* at $x \in X$ if $f(x_n)$ β-converges to $f(x)$ whenever x_n is a sequence β-converges to x. If f is sequentially β-continuous for all x in X, then f is sequentially β-continuous.

(ii) *Sequentially nearly β-continuous* if for each x in X and each sequence x_n in X β-converging to x, there exists a subsequence x_{nk} of x_n such that $f(x_{nk})^\beta \to f(x)$

(iii) *Sequentially sub β-continuous* if for each x in X and each sequence x_n in X β-converging to x, there exists a subsequence x_{nk} of x_n and a point y in Y such that $f(x_{nk})^\beta \to y$.

(iv) *Sequentially β-compact preserving* if the image $f(C)$ of every sequentially β-compact set C of X is sequentially β-compact set in Y.

38 Theorem. *Let $f\colon X \to Y$ and $g\colon X \to Y$ be two sequentially β-continuous functions. If Y is β-*US*, then the set $E = \{x \mid f(x) = g(x)\}$ is sequentially β-closed.*

Proof. Suppose that Y is β-US and $\langle x_n \rangle$ is a sequence in E β-converging to x in X. By hypothesis f and g are sequentially β-continuous functions, so $f(x_n)^\beta \to f(x)$ and $g(x_n)^\beta \to g(x)$. Since $x_n \in E$ for each n and Y is β-US, $f(x) = g(x)$ and hence $x \in E$. This shows that the set E is sequentially β-closed. $\boxed{QED}$

39 Lemma. *Every function $f\colon X \to Y$ is sequentially sub β-continuous if Y is sequentially β-compact.*

Proof. Assume that $\langle x_n \rangle$ is a sequence in X β-converging to x in X. It follows that $\{f(x_n)\}$ is a sequence in Y. Since Y is sequentially β-compact, there exists a subsequence $\{f(x_{nk})\}$ of $\{f(x_n)\}$ β-converging to a point y in Y. Therefore $f\colon X \to Y$ is sequentially sub β-continuous. $\boxed{QED}$

40 Theorem. *Every sequentially nearly β-continuous function is sequentially β-compact preserving.*

Proof. Let $f\colon X \to Y$ be a sequentially nearly β-continuous function and C be any sequentially β-compact subset of X. We show that $f(C)$ is a sequentially β-compact set of Y. Assume that $\langle y_n \rangle$ be any sequence in $f(C)$. Then for each positive integer n, there exist a point x_n in C such that $f(x_n) = y_n$. Now $\langle x_n \rangle$ is a sequence in a sequentially β-compact set C. Thus there exists a subsequence $\langle x_{nk} \rangle$ of $\langle x_n \rangle$ β-converging to a point x in C. Since f is sequentially nearly β-continuous, there exists a subsequence $\langle x_j \rangle$ of $\langle x_{nk} \rangle$ such that $f(x_j)^\beta \to f(x)$. So there exists a subsequence $\langle y_i \rangle$ of $\langle y_n \rangle$ β-converging to $f(x)$ in $f(C)$. This implies that $f(C)$ is a sequentially β-compact set of Y. $\boxed{QED}$

41 Theorem. *Every sequentially β-compact preserving function is sequentially sub β-continuous.*

Proof. Suppose that $f\colon X \to Y$ is a sequentially β-compact preserving function. Let x be any point of X and x_n be any sequence in X β-converging to x. We denote the set $\{x_n \mid n = 1, 2, \dots\}$ by A and put $K = A \cup \{x\}$. Since x_n β-converges to x, K is sequentially β-compact, by hypothesis, f is sequentially β-compact preserving and hence $f(K)$ is a sequentially β-compact set of Y. Since $\{f(x_n)\}$ is a sequence in $f(K)$, there exists a subsequence $\{f(x_{nk})\}$ of $\{f(x_n)\}$ β-converging to a point $y \in f(K)$. This implies that f is sequentially sub β-continuous. $\boxed{QED}$

42 Theorem. *A function $f\colon X \to Y$ is a sequentially β-compact preserving function if and only if $f \mid K\colon K \to f(K)$ is sequentially sub β-continuous for each sequentially β-compact subset K of X.*

Proof. Necessity. Suppose that $f\colon X \to Y$ is a sequentially β-compact preserving function. Then $f(K)$ is sequentially β-compact in Y for each sequentially β-compact set K of X. Therefore by Lemma 39, $f \mid K\colon K \to f(K)$ is sequentially sub β-continuous.

Sufficiency. Let K be any sequentially β-compact set of X, we show that $f(K)$ is sequentially β-compact in Y. Let y_n be any sequence in $f(K)$. Then for each positive integer n, there exists a point $x_n \in K$ such that $f(x_n) = y_n$. Since $\langle x_n \rangle$ is a sequence in a sequentially β-compact set K, there exists a subsequence $\langle x_{nk} \rangle$ of $\langle x_n \rangle$ β-converging to a point $x \in K$. By hypothesis, $f \mid K \colon K \to f(K)$ is sequentially sub β-continuous and hence there exists a subsequence of the sequence $\langle y_{nk} \rangle$ β-converging to $y \in f(K)$. This implies that $f(K)$ is sequentially β-compact in Y. Thus, $f \colon X \to Y$ is sequentially β-compact preserving. $\boxed{QED}$

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