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On F-planar mappings of spaces with affine connections

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Abstract. In this paper we study F-planar mappings of n-dimensional or infinitely dimensional spaces with a torsion-free affine connection. These mappings are certain generalizations of geodesic and holomorphically projective mappings.
Here we make fundamental equations on F-planar mappings for dimensions $n > 2$ more precise.

Keywords: F-planar mapping, space with affine connections

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Introduction

In many papers geodesic mappings and their generalizations, like quasi-geodesic, holomorphically-projective, F-planar, 4-planar, mappings, were considered. One of the basic tasks was and is the derivation of the fundamental equations of these mappings. They were shown in the most various ways, see [1]-[7].

Unless otherwise specified, all spaces, connections and mappings under consideration are differentiable of a sufficiently high class. The dimension n of the spaces being considered is higher than *two*, as a rule. This fact is not specially stipulated. All spaces are assumed to be connected.

Here we show a method that simplifies and generalizes many of the results. Our results are valid also for infinite dimensional spaces with Banach bases ($n = \infty$).

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1 F-planar curves

We consider an n-dimensional ($n > 2$) or infinite dimensional ($n = \infty$) space A_n with a torsion-free affine connection ∇, and an affinor structure F, i.e. a tensor field of type $\binom{1}{1}$.

If $n = \infty$ we assume that A_n is locally homeomorphic to a Banach space E_∞. In connection with local studies we assume the existence of a coordinate neighbourhood U in the Euclidean space E_n, resp. $U \subset E_\infty$.

1 Definition (J. Mikeš, N.S. Sinyukov [4])**.** A curve ℓ, which is given by the equations

$$\ell = \ell(t), \quad \lambda(t) = d\ell(t)/dt \ (\neq 0), \quad t \in I \tag{1}$$

where t is a parameter, is called *F-planar*, if its tangent vector $\lambda(t_0)$, for any initial value t_0 of the parameter t, remains, under parallel translation along the curve ℓ, in the distribution generated by the vector functions λ and $F\lambda$ along ℓ.

In particular, if $F = \varrho\, I$ we obtain the definition of a geodesic parametrized by an arbitrary parameter, see [4]. Here ϱ is a function and I is the identity operator.

In accordance with this definition, ℓ is F-planar if and only if the following condition holds [4]:

$$\nabla_{\lambda(t)}\, \lambda(t) = \varrho_1(t)\, \lambda(t) + \varrho_2(t)\, F\lambda(t), \tag{2}$$

where ϱ_1 and ϱ_2 are some functions of the parameter t.

2 F-planar mappings between two spaces with affine connection

We suppose two spaces A_n and $\bar{A}_n$ with torsion-free affine connections ∇ and $\bar{\nabla}$, respectively. Affine structures F and $\bar{F}$ are defined on A_n, resp. $\bar{A}_n$.

2 Definition (J. Mikeš, N.S. Sinyukov [4])**.** A diffeomorphism $f\colon A_n \to \bar{A}_n$ between two manifolds with affine connections is called *F-planar* if any F-planar curve in A_n is mapped onto an $\bar{F}$-planar curve in $\bar{A}_n$.

Important convention. Due to the diffeomorphism f we always suppose that ∇, $\bar{\nabla}$, and the affinors F, $\bar{F}$ are defined on A_n. Moreover, we always identify a given curve $\ell\colon I \to A_n$ and its tangent vector function $\lambda(t)$ with their images $\bar{\ell} = f \circ \ell$ and $\bar{\lambda} = f_*(\lambda(t))$ in $\bar{A}_n$.

Two principially different cases are possible for the investigation:

$$\text{a)} \quad \bar{F} = a\, F + b\, I; \tag{3}$$

$$\text{b)} \quad \bar{F} \neq a\,F + b\,I, \tag{4}$$

a, b are some functions.

Naturally, case a) characterizes F-planar mappings *which preserve F-structures*. In case b) the structures of F and $\bar{F}$ are essentially distinct. The following holds.

3 Theorem. *An F-planar mapping f from A_n onto $\bar{A}_n$ preserve F-structures and is characterized by the following condition*

$$P(X, Y) = \psi(X)\,Y + \psi(Y)\,X + \varphi(X)\,FY + \varphi(Y)\,FX \tag{5}$$

for any vector fields X, Y, where $P \overset{def}{=} \bar{\nabla} - \nabla$ is the deformation tensor field of f, ψ, φ are some linear forms.

Let us recall that on each tangent space $T_x A_n$, $P(X, Y)$ is a symmetric bilinear mapping $T_x A_n \times T_x A_n \to T_x A_n$ and a tensor field of type $\binom{1}{2}$.

Theorem 3 was proved by J. Mikeš and N. S. Sinyukov [4] for finite dimension $n > 3$. Here we can show a more rational proof of this Theorem for $n > 3$ and also a proof for $n = 3$. We show a counter example for $n = 2$.

3 F-planar mappings which preserve F-structures

First we prove the following proposition

4 Theorem. *An F-planar mapping f from A_n onto $\bar{A}_n$ which preserves F-structures is characterized by condition* (5)*.*

In the sequel we shall need the following lemma:

5 Lemma. *Let V be an n-dimensional vector space, $Q\colon V \times V \to V$ be a symmetric bilinear mapping and $F\colon V \to V$ a linear mapping. If, for each vector $\lambda \in V$*

$$Q(\lambda, \lambda) = \varrho_1(\lambda)\,\lambda + \varrho_2(\lambda)\,F(\lambda) \tag{6}$$

holds, where $\varrho_1(\lambda)$, $\varrho_2(\lambda)$ are functions on V, then there are linear forms ψ and φ such that the condition

$$Q(X, Y) = \psi(X)\,Y + \psi(Y)\,X + \varphi(X)\,F(Y) + \varphi(Y)\,F(X) \tag{7}$$

holds for any $X, Y \in V$.

Proof. Formula (6) has the following coordinate expression

$$Q^h_{\alpha\beta}\lambda^\alpha\lambda^\beta = \varrho_1(\lambda)\,\lambda^h + \varrho_2(\lambda)\,F^h_\alpha\lambda^\alpha, \tag{8}$$

where $\lambda^i, F^h_i, Q^h_{ij}$ are the components of λ, F, Q.

By multiplying (8) with $\lambda^i F^j_\alpha \lambda^\alpha$ and antisymmetrizing the indices h, i and j we obtain

$$\left\{ Q^{[h}_{\alpha\beta}\delta^i_\gamma F^{j]}_\delta \right\} \lambda^\alpha\lambda^\beta\lambda^\gamma\lambda^\delta = 0, \tag{9}$$

where square brackets denote the alternation of indices. The term in curly brackets does not depend on λ and (9) holds for any vector $\lambda \in V$, therefore

$$Q^{[h}_{(\alpha\beta}\delta^i_\gamma F^{j]}_{\delta)} = 0 \tag{10}$$

holds, where the round brackets denote symmetrization of indices.

It is natural to assume that $F^h_i \neq a\,\delta^h_i$ with $a = \text{const}$. By virtue of this there exist some vectors ξ^h such that $\xi^\alpha F^h_\alpha \neq b\,\xi^h$, $b = \text{const}$. Introducing $P^h_i \overset{\text{def}}{=} P^h_{i\alpha}\xi^\alpha$, $P^h \overset{\text{def}}{=} P^h_\alpha\xi^\alpha$ and $F^h \overset{\text{def}}{=} F^h_\alpha\xi^\alpha$, we contract (10) with $\xi^\alpha\xi^\beta\xi^\gamma\xi^\delta$. Since $F^h \neq b\,\xi^h$, we obtain $P^h = 2a\,\xi^h + 2b\,F^h$, where a, b are certain constants. Contracting (10) with $\xi^\beta\xi^\gamma\xi^\delta$, and taking into account the precending, we have $P^h_i = a\,\delta^h_i + b\,F^h_i + a_i\,\xi^h + b_i\,F^h$, where a_i, b_i are some components of linear forms. Analogously, contracting (10) with $\xi^\gamma\xi^\delta$, we have

$$Q^h_{ij} = \psi_i\delta^h_j + \psi_j\delta^h_i + \varphi_i F^h_j + \varphi_j F^h_i + \xi^h a_{ij} + F^h b_{ij}, \tag{11}$$

where ψ_i, φ_i are components of a 1-form ψ, φ defined on V, and a_{ij}, b_{ij} are components of a symmetric 2-form defined on V.

In case that $a_{ij} = b_{ij} = 0$, evidently from (11) we obtain formula (7).

Now we will suppose that either $a_{ij} \neq 0$, or $b_{ij} \neq 0$. Since ξ^h and F^h are noncollinear, it is evident that

$$\xi^h a_{ij} + F^h b_{ij} \neq 0. \tag{12}$$

Formula (10) by virtue of (11) has the form

$$\Omega^{[hi}_{(\alpha\beta\gamma} F^{j]}_{\delta)} = 0, \tag{13}$$

where $\Omega^{hi}_{\alpha\beta\gamma} \overset{\text{def}}{=} (\xi^h a_{\alpha\beta} + F^h b_{\alpha\beta})\delta^i_\gamma - (\xi^i a_{\alpha\beta} + F^i b_{\alpha\beta})\delta^h_\gamma$. It is possible to show that there exists some vector ε^h for which $\Omega^{hi}_{\alpha\beta\gamma}\varepsilon^\alpha\varepsilon^\beta\varepsilon^\gamma \neq 0$, otherwise (12) would be violated.

Contracting (13) with $\varepsilon^\alpha\varepsilon^\beta\varepsilon^\gamma\varepsilon^\delta$, we have $F^h_\alpha\varepsilon^\alpha = a\,\xi^h + b\,F^h + c\,\varepsilon^h$, with a, b, c being constants. Analogously, contracting (13) with $\varepsilon^\beta\varepsilon^\gamma\varepsilon^\delta$, we obtain that F^h_i is represented in the following manner:

$$F^h_i = a\,\delta^h_i + a_i\,\xi^h + b_i\,F^h + c_i\,\varepsilon^h, \tag{14}$$

where a_i, b_i, c_i are components of 1-forms.

Formula (13) by virtue of (14) has the form

$$\omega^{[hi}_{(\alpha\beta\gamma}\delta^{j]}_{\delta)} = 0, \tag{15}$$

where

$$\omega^{hi}_{\alpha\beta\gamma} \stackrel{\text{def}}{=} \xi^{[h}F^{i]}(a_{(\alpha\beta}b_{\gamma)} - b_{(\alpha\beta}a_{\gamma)}) + \xi^{[h}\varepsilon^{i]}a_{(\alpha\beta}c_{\gamma)} + F^{[h}\varepsilon^{i]}b_{(\alpha\beta}c_{\gamma)}.$$

a) If $n > 3$ then $\omega^{hi}_{\alpha\beta\gamma} = 0$ follows from (13), and because ξ^h, F^h and ε^h are linear independent, we obtain $a_{(\alpha\beta}c_{\gamma)} = 0$ and $b_{(\alpha\beta}c_{\gamma)} = 0$. Therefore $c_i = 0$ and

$$F^h_i = a\,\delta^h_i + a_i\,\xi^h + b_i\,F^h. \tag{16}$$

b) If $n = 3$ the matrix F^h_i has always the previous form (16) while ξ^h, F^h and ε^h are not linear dependent.

Then formula (13) becomes (15), whereas $\omega^{hi}_{\alpha\beta\gamma} \stackrel{\text{def}}{=} \xi^{[h}F^{i]}(a_{(\alpha\beta}b_{\gamma)} - b_{(\alpha\beta}a_{\gamma)})$. For $n > 2$ it follows $\omega^{hi}_{\alpha\beta\gamma} = 0$ and consequently

$$a_{(\alpha\beta}b_{\gamma)} = b_{(\alpha\beta}a_{\gamma)}. \tag{17}$$

If a_α and b_α are linear indepedent, then from (17) we obtain

$$a_{ij} = a_{(i}\omega_{j)} \quad \text{and} \quad b_{ij} = b_{(i}\omega_{j)},$$

where ω_i are components of a 1-form. Afterwards it is possible to show that on the basis of (16) formula (11) assumes the following form

$$Q^h_{ij} = (\psi_i - a\omega_i)\delta^h_j + (\psi_j - a\omega_j)\delta^h_i + (\varphi_i + a\omega_i)F^h_j + (\varphi_j + a\omega_j)F^h_i,$$

i.e. formula (7) also holds.

Now there remains the case that a_α and b_α are linear depedent. For example, $b_\alpha = \alpha\,a_\alpha$, $\alpha \neq 0$. Then from (17) follows $b_{\alpha\beta} = \alpha\,a_{\alpha\beta}$. We denote $\Lambda^h = \xi^h + \alpha\,F^h$, $\omega_i = \psi_i + \alpha\,\varphi_i$, $\omega_{ij} = a_{ij} + a_{(i}\varphi_{j)}$, from (11) and (16) we obtain that Q^h_{ij} and F^h_i are represented by

$$Q^h_{ij} = \psi_i\delta^h_j + \psi_j\delta^h_i + \Lambda^h\omega_{ij} \quad \text{and} \quad F^h_i = a\delta^h_i + \Lambda^h a_i. \tag{18}$$

Then formula (8) appears in the following way

$$\Lambda^h\,(\omega_{\alpha\beta}\lambda^\alpha\lambda^\beta - \varrho_2(\lambda)\,a_\alpha\lambda^\alpha) = \lambda^h\,(\varrho_1(\lambda) + a\,\varrho_2(\lambda) - 2\psi_\alpha\lambda^\alpha).$$

From this it follows that

$$\omega_{\alpha\beta}\lambda^\alpha\lambda^\beta = \varrho_2(\lambda)\,a_\alpha\lambda^\alpha, \qquad \forall\lambda^h \neq \alpha\,\Lambda^h.$$

By simple analysis we obtain that $\omega_{ij} = a_{(i}\sigma_{j)}$, where σ_i are components of a 1-form.

Then due to (18) we have $Q^h_{ij} = (\psi_i - a\sigma_i)\delta^h_j + (\psi_j - a\sigma_j)\delta^h_i + \sigma_i F^h_j + \sigma_j F^h_i$. Evidently Lemma 5 is proved. $\boxed{QED}$

Proof of Theorem 4. It is obvious that geodesics are a special case of F-planar curves. Let a geodesic in A_n, which satisfies the equations (1) and $\nabla_\lambda\lambda = 0$, be mapped onto an F-planar curve in $\bar{A}_n$, which satisfies equations (1) and

$$\bar{\nabla}_\lambda\lambda = \bar{\varrho}_1(t)\,\lambda + \bar{\varrho}_2(t)F\lambda.$$

Here $\bar{\varrho}_1, \bar{\varrho}_2$ are functions of the parameter t.

Because the deformation tensor satisfies $P(\lambda,\lambda) = \bar{\nabla}_\lambda\lambda - \nabla_\lambda\lambda$, we have

$$P(\lambda(t),\lambda(t)) = \bar{\varrho}_1(t)\,\lambda + \bar{\varrho}_2(t)F\lambda.$$

It follows from the previous formula that in each point $x \in A_n$

$$P(\lambda,\lambda) = \varrho_1(\lambda)\,\lambda + \varrho_2(\lambda)F\lambda.$$

for each tangent vector $\lambda \in T_x$; $\varrho_1(\lambda), \varrho_2(\lambda)$ are functions dependent on λ.

Based on Lemma 5 it follows that there exist linear forms ψ and φ, for which formula (5) holds. $\boxed{QED}$

4 F-planar mappings which do not preserve F-structures

We now assume that the structures F and $\bar{F}$ are essentially distinct, i.e.

$$\bar{F}_i^h \neq a\delta_i^h + b\,F_i^h.$$

a) It is obvious, that geodesics are a special case of F-planar curves. Let a geodesic in A_n, which satisfies the equations (1) and $\nabla_\lambda\lambda = 0$, be mapped onto an $\bar{F}$-planar curve in $\bar{A}_n$, which satisfies the equations (1) and

$$\bar{\nabla}_\lambda\lambda = \bar{\varrho}_1(t)\,\lambda + \bar{\varrho}_2(t)\bar{F}\lambda.$$

Here $\bar{\varrho}_1, \bar{\varrho}_2$ are functions of the parameter t.

For the deformation tensor we have $P(\lambda(t),\lambda(t)) = \bar{\varrho}_1(t)\,\lambda + \bar{\varrho}_2(t)\bar{F}\lambda$. It follows from the previous formula that in each point $x \in A_n$

$$P(\lambda,\lambda) = \varrho_1(\lambda)\,\lambda + \varrho_2(\lambda)\bar{F}\lambda.$$

for each tangent vector $\lambda \in T_x$; $\varrho_1(\lambda), \varrho_2(\lambda)$ are functions dependent on λ.

Based on Lemma 5 it follows, that there exist linear forms ψ and φ, for which formula

$$P(X,Y) = \psi(X)\,Y + \psi(Y)\,X + \varphi(X)\,\bar{F}Y + \varphi(Y)\,\bar{F}X \tag{19}$$

holds.

b) Let a special F-planar curve in A_n, which satisfies the equations (1) and $\nabla_\lambda \lambda = F\lambda$, be mapped onto an $\bar{F}$-planar curve in $\bar{A}_n$, which satisfies the equations (1) and

$$\bar{\nabla}_\lambda \lambda = \bar{\varrho}_1(t)\,\lambda + \bar{\varrho}_2(t)\bar{F}\lambda.$$

Here $\bar{\varrho}_1, \bar{\varrho}_2$ are functions of the parameter t.

For the deformation tensor we have $P(\lambda(t), \lambda(t)) = F\lambda + \bar{\varrho}_1(t)\,\lambda + \bar{\varrho}_2(t)\bar{F}\lambda$. It follows from the previous formula that in each point $x \in A_n$

$$P(\lambda, \lambda) = F\lambda + \varrho_1(\lambda)\,\lambda + \varrho_2(\lambda)\bar{F}\lambda.$$

for each tangent vector $\lambda \in T_x$; $\varrho_1(\lambda), \varrho_2(\lambda)$ are functions dependent on λ.

Applying (19) we obtain

$$F\lambda = \tilde{\varrho}_1(\lambda)\,\lambda + \tilde{\varrho}_2(\lambda)\bar{F}\lambda.$$

Analyzing this expression like in Lemma 5 we convince ourselves that formula (3) holds. In this way we prove

6 Theorem. *Any F-planar mapping of a space with affine connection A_n onto $\bar{A}_n$ preserves F-structures.*

5 F-planar mappings for dimension n = 2

It is easy to see that for $n = 2$ Theorems 3 and 4 do not hold. If they would hold, the functions ϱ_1 and ϱ_2, appearing in (6), would be linear in λ.

In the case

$$F_i^h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

for example, these functions have the forms

$$\varrho_1(\lambda) = \frac{\lambda^1 P^1_{\alpha\beta}\lambda^\alpha\lambda^\beta + \lambda^2 P^2_{\alpha\beta}\lambda^\alpha\lambda^\beta}{(\lambda^1)^2 + (\lambda^2)^2} \quad \text{and} \quad \varrho_2(\lambda) = \frac{\lambda^1 P^2_{\alpha\beta}\lambda^\alpha\lambda^\beta - \lambda^2 P^1_{\alpha\beta}\lambda^\alpha\lambda^\beta}{(\lambda^1)^2 + (\lambda^2)^2},$$

which are not linear in general.

On the other hand an arbitrary diffeomorphism from A_2 onto $\bar{A}_2$ is an F-planar mapping with (6) being valid for the above functions ϱ_1 and ϱ_2.

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