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On functions with strongly δ-semiclosed graphs

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Abstract. In 1997, Park et al. [5] offered a new notion called δ-semiopen sets which are stronger than semi-open sets but weaker than δ-open sets. It is the aim of this paper to introduce and study some properties of functions with strongly δ-semiclosed graphs by utilizing δ-semiopen sets and the δ-semi-closure operator.

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1 Introduction and preliminaries

In what follows (X, τ) and (Y, σ) (or X and Y) denote topological spaces. Let A be a subset of X. We denote the interior, the closure and the complement of a set A by $Int(A)$, $Cl(A)$ and $X \backslash A$ or A^c, respectively.

Levine [3] defined semiopen sets which are weaker than open sets in topological spaces. After Levine's semiopen sets, mathematicians gave in several papers different and interesting new modifications of open sets as well as generalized open sets. In 1968, Veličko [6] introduced δ-open sets, which are stronger than open sets, in order to investigate the characterization of H-closed spaces. In 1997, Park et al. [5] introduced the notion of δ-semiopen sets which are stronger than semiopen sets but weaker than δ-open sets and investigated the relationships between several types of these open sets.

A subset A of a topological space X is said to be δ-semiopen [5] (resp.

semiopen [3]) set if there exists a δ-open (resp. open) set U of X such that $U \subset A \subset Cl(U)$. The complement of a δ-semiopen (resp. semiopen) set is called a δ-semiclosed (resp. semiclosed) set. A point $x \in X$ is called the δ-semicluster point of A if $A \cap U \neq \emptyset$ for every δ-semiopen set U of X containing x. The set of all δ-semicluster points of A is called the δ-semiclosure of A, denoted by $\delta Cl_S(A)$. We denote the collection of all δ-semiopen (resp. δ-semiclosed) sets by $\delta\mathrm{SO}(X)$ (resp. $\delta\mathrm{SC}(X)$). We set $\delta\mathrm{SO}(X,x) = \{U : x \in U \in \delta\mathrm{SO}(X)\}$ and $\delta\mathrm{SC}(X,x) = \{U : x \in U \in \delta\mathrm{SC}(X)\}$.

1 Lemma. *(Park et al. [5]) The intersection (resp. union) of arbitrary collection of δ-semiclosed (resp. δ-semiopen) sets in (X,τ) is δ-semiclosed (resp. δ-semiopen).*

2 Corollary. *Let A be a subset of a topological space (X,τ), then $\delta Cl_S(A) = \cap\{F \in \delta SC(X,\tau) : A \subset F\}$.*

3 Lemma. *(Park et al. [5]) Let A, B and A_i $(i \in I)$ be subsets of a space (X,τ), the following properties hold:*

(1) $A \subset \delta Cl_S(A)$.

(2) If $A \subset B$, then $\delta Cl_S(A) \subset \delta Cl_S(B)$.

(3) $\delta Cl_S(A)$ is δ-semiclosed.

(4) $\delta Cl_S(\delta Cl_S(A)) = \delta Cl_S(A)$.

(5) A is δ-semiclosed if and only $A = \delta Cl_S(A)$.

4 Corollary. *(Caldas et al. [1]) Let A_i $(i \in I)$ be subsets of a space (X,τ), the following properties hold:*

(1) $\delta Cl_S(\cap\{A_i : i \in I\}) \subset \cap\{\delta Cl_S(A_i) : i \in I\}$.

(2) $\delta Cl_S(\cup\{A_i : i \in I\}) \supset \cup\{\delta Cl_S(A_i) : i \in I\}$.

5 Definition. A topological space (X,τ) is called:

(1) δ-semi T_1 [1] if for any distinct pair of points x and y in X, there is a δ-semiopen U in X containing x but not y and δ-semiopen V in X containing y but not x,

(2) δ-semi T_2 [1] if for any distinct pair of points x and y in X, there exist $U \in \delta SO(X,x)$ and $V \in \delta SO(X,y)$ such that $U \cap V = \emptyset$.

Recall that a function $f : X \to Y$ is said to be:

(1) δ-semicontinuous [1] if for each $x \in X$ and each $V \in \delta SO(Y, f(x))$, there exists $U \in \delta SO(X, x)$ such that $f(U) \subset V$, equivalently if the inverse image of each δ-semiopen set is δ-semiopen,

(2) quasi δ-semicontinuous [1] if for each $x \in X$ and each $V \in \delta SO(Y, f(x))$, there exists $U \in \delta SO(X, x)$ such that $f(U) \subset \delta Cl_S(V)$.

2 Strongly δ-semiclosed graphs

Let $f : (X, \tau) \to (Y, \sigma)$ be any function, then the subset $G(f) = \{(x, f(x)) : x \in X\}$ of the product space $(X \times Y, \tau \times \sigma)$ is called the graph of f [2].

6 Definition. A function $f : X \to Y$ has a strongly δ-semiclosed graph if for each $(x, y) \in (X \times Y) \backslash G(f)$, there exist $U \in \delta SO(X, x)$ and $V \in \delta SO(Y, y)$ such that $[U \times \delta Cl_S(V)] \cap G(f) = \emptyset$.

7 Lemma. *A function $f : X \to Y$, has a strongly δ-semiclosed graph if for each $(x, y) \in (X \times Y) \backslash G(f)$, there exist $U \in \delta SO(X, x)$ and $V \in \delta SO(Y, y)$ such that $f(U) \cap \delta Cl_S(V) = \emptyset$.*

8 Theorem. *If $f : X \to Y$ is a function with strongly δ-semiclosed graph, then for each $x \in X$, $f(x) = \cap\{\delta Cl_S(f(U)) : U \in \delta SO(X, x)\}$.*

PROOF. Suppose the theorem is false. Then there exists a point $y \in Y$ such that $y \neq f(x)$ and $y \in \cap\{\delta Cl_S(f(U)) : U \in \delta SO(X, x)\}$. This implies that $y \in \delta Cl_S(f(U))$ for every $U \in \delta SO(X, x)$. So $V \cap f(U) \neq \emptyset$ for every $V \in \delta SO(Y, y)$. This, in its turn, indicates that $\delta Cl_S(V) \cap f(U) \supset V \cap f(U) \neq \emptyset$ which contradicts the hypothesis that f is a function with a strongly δ-semiclosed graph. Hence the theorem holds. $\boxed{QED}$

9 Theorem. *If $f : X \to Y$ is δ-semicontinuous and Y is δ-semi T_2, then $G(f)$ is strongly δ-semiclosed.*

PROOF. Let $(x, y) \in (X \times Y) \backslash G(f)$. The δ-semi T_2-ness of Y gives the existence of a set $V \in \delta SO(Y, y)$ such that $f(x) \notin \delta Cl_S(V)$.
Now $Y \backslash \delta Cl_S(V) \in \delta SO(Y, f(x))$. Therefore, by the δ-semicontinuity of f there exists $U \in \delta SO(X, x)$ such that $f(U) \subset Y \backslash \delta Cl_S(V)$. Consequently, $f(U) \cap \delta Cl_S(V) = \emptyset$ and therefore $G(f)$ is strongly δ-semiclosed. $\boxed{QED}$

10 Theorem. *If $f : X \to Y$ is surjective and has a strongly δ-semiclosed graph $G(f)$, then Y is both δ-semi T_2 and δ-semi T_1.*

PROOF. Let y_1, y_2 $(y_1 \neq y_2) \in Y$. The surjectivity of f gives a $x_1 \in X$ such that $f(x_1) = y_1$. Now $(x_1, y_2) \in (X \times Y) \backslash G(f)$. The strong δ-semiclosedness of $G(f)$ provides $U \in \delta SO(X, x_1)$ and $V \in \delta SO(Y, y_2)$ such that $f(U) \cap \delta Cl_S(V) = \emptyset$, whence one infers that $y_1 \notin \delta Cl_S(V)$. This means that there exists $W \in$

$\delta SO(Y, y_1)$ such that $W \cap V = \emptyset$. So, Y is δ-semi T_2 and δ-semi T_2-ness always guarantees δ-semi T_1-ness. Hence Y is δ-semi T_1. $\boxed{QED}$

11 Theorem. *A space X is δ-semi T_2 if and only if the identity function* $\mathrm{id} : X \to X$ *has a strongly δ-semiclosed graph $G(\mathrm{id})$.*

Proof. Necessity. Let Y is δ-semi T_2. Since the identity function $\mathrm{id} : X \to X$ is δ-semicontinuous, it follows from Theorem 9 that $G(\mathrm{id})$ is strongly δ-semiclosed.

Sufficiency. Let $G(\mathrm{id})$ be strong δ-semiclosed graph. Then the surjectivity of id and strongly δ-semiclosedness of $G(\mathrm{id})$ together imply, by Theorem 10, that X is δ-semi T_2. $\boxed{QED}$

12 Theorem. *If $f : X \to Y$ is an injection and $G(f)$ is strongly δ-semiclosed, then X is δ-semi T_1.*

Proof. Since f is injective, for any pair of distinct points $x_1, x_2 \in X$, $f(x_1) \neq f(x_2)$. Then $(x_1, f(x_2)) \in (X \times Y)\backslash G(f)$. Since $G(f)$ is strongly δ-semiclosed, there exist $U \in \delta SO(X, x_1)$ and $V \in \delta SO(Y, f(x_2))$ such that $f(U) \cap \delta Cl_S(V) = \emptyset$. Therefore $x_2 \notin U$. Pursuing the same reasoning as before we obtain a set $W \in \delta SO(X, x_2)$ such that $x_1 \notin W$. Hence Y is δ-semi T_1. $\boxed{QED}$

13 Theorem. *If $f : X \to Y$ is a bijective function with a strongly δ-semiclosed graph, then both X and Y are δ-semi T_1.*

Proof. The proof is an immediate consequence of Theorems 10 and 12. $\boxed{QED}$

14 Theorem. *If a quasi δ-semicontinuous function $f : X \to Y$ is an injection with a strongly δ-semiclosed graph $G(f)$, then X is δ-semi T_2.*

Proof. Since f is injective, for any pair of distinct points $x_1, x_2 \in X$, $f(x_1) \neq f(x_2)$. Therefore $(x_1, f(x_2)) \in (X \times Y)\backslash G(f)$. Since $G(f)$ is strongly δ-semiclosed, there exist $U \in \delta SO(X, x_1)$ and $V \in \delta SO(Y, f(x_2))$ such that $f(U) \cap \delta Cl_S(V) = \emptyset$, hence one obtains $U \cap f^{-1}(\delta Cl_S(V)) = \emptyset$. Consequently, $f^{-1}(\delta Cl_S(V)) \subset X\backslash U$. Since f is quasi δ-semicontinuous, it is so at x_2. Then there exists $W \in \delta SO(X, x_2)$ such that $f(W) \subset \delta Cl_S(V)$. From this and the foregoing follow that $W \subset f^{-1}(\delta Cl_S(V)) \subset X\backslash U$, hence one infers that $W \cap U = \emptyset$. Thus for the pair of distinct points $x_1, x_2 \in X$, there exist $U \in \delta SO(X, x_1)$ and $W \in \delta SO(X, x_2)$ such that $W \cap U = \emptyset$. This guarantees the δ-semi T_2-ness of X. $\boxed{QED}$

15 Corollary. *If a δ-semicontinuous function $f : X \to Y$ is an injection with a strongly δ-semiclosed graph, then X is δ-semi T_2.*

Proof. The proof follows from Theorem 14 and the fact that every δ-semicontinuous is quasi δ-semicontinuous. $\boxed{QED}$

16 Theorem. *If $f : X \to Y$ is quasi δ-semicontinuous bijective with strongly δ-semiclosed graph, then both X and Y are δ-semi T_2.*

Proof. The proof follows from Theorem 14 and 10. $\boxed{QED}$

For the rest of this article we shall assume the Property $\triangle$: $\delta SO(X)$ closed under finite intersections.

17 Definition. X is called strongly $\triangle$-closed (resp. A subset A of X is said to be strongly $\triangle$-closed relative to X), if every δ-semiopen cover of X (resp. if every cover of A by δ-semiopen sets) has a finite subfamily such that the union of their δ-semiclosures covers X (resp. has a finite subfamily such that the union of their δ-semiclosures covers X).

18 Lemma. *Every δ-semiclopen subset of a strongly $\triangle$-closed space X is strongly $\triangle$-closed relative to X.*

Proof. Let B be any δ-semiclopen subset of a strongly $\triangle$-closed space X. Let $\{O_\lambda : \lambda \in \Omega\}$ be any cover of B by δ-semiopen sets in X. Then the family $F = \{O_\lambda : \lambda \in \Omega\} \cup \{X \backslash B\}$ is a cover of X by δ-semiopen sets in X. Because of strongly $\triangle$-closedness of X there exists a finite subfamily $F^* = \{O_{\lambda_i} : 1 \leq i \leq n\} \cup \{X \backslash B\}$ of F such that the union of their δ-semiclosures covers X. So, because of δ-semiclopenness of B we now infer that the family $\{\delta Cl_S(O_{\lambda_i}) : 1 \leq i \leq n\}$ covers B and hence B is strongly $\triangle$-closed relative to X. $\boxed{QED}$

19 Lemma. *The δ-semiclosure of every δ-semiopen set is δ-semiopen.*

Proof. Every regular open set is δ-open and every δ-open set is δ-semiopen. Thus, every regular closed set is δ-semiclosed. Now let A be any δ-semiopen set. There exists a δ-open set U such that $U \subset A \subset Cl(U)$. Hence, we have $U \subset \delta Cl_S(U) \subset \delta Cl_S(A) \subset \delta Cl_S(Cl(U)) = Cl(U)$ since $Cl(U)$ is regular closed. Therefore, $\delta Cl_S(A)$ is δ-semiopen. $\boxed{QED}$

20 Theorem. *Let $f : X \to Y$ be a function. If Y is a strongly $\triangle$-closed, δ-semi-T_2 space and $G(f)$ is strongly δ-semiclosed, then f is quasi-δ-semicontinuous.*

Proof. Let $x \in X$ and $V \in \delta SO(Y, f(x))$. Take any $y \in Y \backslash \delta Cl_S(V)$. Then $(x, y) \in (X \times Y) \backslash G(f)$. Now the strong δ-semiclosedness of $G(f)$ induces the existence of $U_y(x) \in \delta SO(X, x)$ and $V_y \in \delta SO(Y, y)$ such that $f(U_y(x)) \cap \delta Cl_S(V_y) = \emptyset. \dots$ (*).
The δ-semi-T_2-ness of Y implies the existence of $V_y \in \delta SO(Y, y)$ such that $f(x) \notin \delta Cl_S(V_y)$. Now by Lemma 19 induces the δ-semiclopenness of $\delta Cl_S(V)$ and hence $Y \backslash \delta Cl_S(V)$ is also δ-semiclopen. Now $\{V_y : y \in Y \backslash \delta Cl_S(V)\}$ is a cover of $Y \backslash \delta Cl_S(V)$ by δ-semiopen sets in Y. By Lemma 18, there exists a finite subfamily $\{V_{y_i} : 1 \leq i \leq n\}$ such that $Y \backslash \delta Cl_S(V) \subset \bigcup\limits_{i=1}^{n} \delta Cl_S(V_{y_i})$. Let $W =$

$\bigcap\limits_{i=1}^{n} U_{y_i}(x)$, where $U_{y_i}(x)$ are δ-semiopen sets in X satisfying (*). Since X enjoys the Property $\triangle$, $W \in \delta SO(X,x)$. Now $f(W) \cap (Y \backslash \delta Cl_S(V)) \subset f[\bigcap\limits_{i=1}^{n} U_{y_i}(x)] \cap (\bigcup\limits_{i=1}^{n} \delta Cl_S(V_{y_i})) = \bigcup\limits_{i=1}^{n} (f[U_{y_i}(x)] \cap \delta Cl_S(V_{y_i})) = \emptyset$, by (*). Therefore, $f(W) \subset \delta Cl_S(V)$ and this indicates that f is quasi-δ-semicontinuous. $\boxed{QED}$

21 Corollary. *If Y is a strongly $\triangle$-closed then a surjerction $f : X \to Y$ with a strongly δ-semiclosed graph is quasi-δ-semicontinuous.*

Proof. The proof follows from Theorem 10 and 19. $\boxed{QED}$

Noiri [4] showed that if $G(f)$ is strongly closed then f has the following property:
(P) For every set B quasi H–closed relative to Y, $f^{-1}(B)$ is a closed set of X.

Analogously, we have

22 Theorem. *A $f : X \to Y$ has a strongly δ-semiclosed graph $G(f)$, then f enjoys the following property:*

(P^*) For every set F which is strongly $\triangle$-closed relative to Y, $f^{-1}(F)$ is δ-semiclosed in X.

Proof. If possible let $f^{-1}(F)$ be not δ-semiclosed in X. Then there exists $x \in \delta Cl_S(f^{-1}(F)) \backslash f^{-1}(F)$. Let $y \in F$. Then $(x,y) \in (X \times Y) \backslash G(f)$. Strongly δ-semiclosedness of $G(f)$ gives the existence of $U_y(x) \in \delta SO(X,x)$ and $V_y \in \delta SO(Y,y)$ such that $f(U_y(x)) \cap \delta Cl_S(V_y) = \emptyset$. ...(*).
Clearly $\{V_y : y \in F\}$ is a cover of F by δ-semiopen sets in Y. The strong $\triangle$-closedness of F relative to Y guarantees the existence of δ-semiopen sets $V_{y_1}, V_{y_2}, \ldots, V_{y_n}$ in Y such that $F \subset \bigcup\limits_{i=1}^{n} \delta Cl_S(V_{y_i})$.
Let $U = \bigcap\limits_{i=1}^{n} U_{y_i}(x)$, where $U_{y_i}(x)$ are the δ-semiopen sets in X satisfying (*). Since X enjoys the Property $\triangle$, $U \in \delta SO(X,x)$.
Now $f(U) \cap F \subset f[\bigcap\limits_{i=1}^{n} U_{y_i}(x)] \cap (\bigcup\limits_{i=1}^{n} \delta Cl_S(V_{y_i})) = \bigcup\limits_{i=1}^{n} (f[U_{y_i}(x)] \cap \delta Cl_S(V_{y_i})) = \emptyset$.
But $x \in \delta Cl_S(f^{-1}(F))$. Therefore $U \cap f^{-1}(F) \neq \emptyset$ which contradicts to the above deduction. Hence the result is true. $\boxed{QED}$

3 Additional properties

23 Theorem. *A space X is δ-semi T_2 if and only if for every pair of points $x, y \in X$, $x \neq y$ there exist $U \in \delta SO(X,x)$, $V \in \delta SO(X,y)$ such that $\delta Cl_S(U) \cap \delta Cl_S(V) = \emptyset$.*

PROOF. NECESSITY. Suppose that X is δ-semi T_2. Let x and y be distinct points of X. There exist δ-semiopen sets U_x and U_y such that $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \emptyset$. Hence $\delta Cl_S(U_x) \cap \delta Cl_S(U_y) = \emptyset$ and by Lemma 19 $\delta Cl_S(U_x)$ is δ-semiopen. Therefore, we obtain $\delta Cl_S(U_x) \cap \delta Cl_S(U_y) = \emptyset$.
SUFFICIENCY. This is obvious. $\boxed{QED}$

24 Theorem. *If Y is δ-semi T_2 and $f : X \to Y$ is a quasi δ-semicontinuous injection, then X is δ-semi T_2.*

PROOF. Since f is injective, for any pair of distinct points $x_1, x_2 \in X$, $f(x_1) \neq f(x_2)$. By Theorem 23, the δ-semi T_2 property of Y indicates that there exist $V_i \in \delta SO(Y, f(x_i))$, $i = 1, 2$ such that $\delta Cl_S(V_1) \cap \delta Cl_S(V_2) = \emptyset$. Hence $f^{-1}(\delta Cl_S(V_1)) \cap f^{-1}(\delta Cl_S(V_2)) = \emptyset$. Since f is quasi δ-semicontinuous, there exists $U_i \in \delta SO(X, x_i)$, $i = 1, 2$ such that $f(U_i) \subset \delta Cl_S(V_i), i = 1, 2$. It, then follows that $U_i \subset f^{-1}(\delta Cl_S(V_i)), i = 1, 2$.
Hence $U_1 \cap U_2 \subset f^{-1}(\delta Cl_S(V_1)) \cap f^{-1}(\delta Cl_S(V_2)) = \emptyset$. This guarantees the δ-semi T_2-ness of X. $\boxed{QED}$

25 Definition. A function $f : X \to Y$ is called δ-semiopen if $f(A) \in \delta SO(Y)$ for all $A \in \delta SO(X)$.

26 Theorem. *If a bijection $f : X \to Y$ is δ-semiopen and X is δ-semi T_2, then $G(f)$ is strongly δ-semiclosed.*

PROOF. Let $(x, y) \in (X \times Y) \backslash G(f)$. Then $y \neq f(x)$. Since f is bijective, $x \neq f^{-1}(y)$. Since X is δ-semi T_2, there exist $U_x, U_y \in \delta SO(X)$ such that $x \in U_x$, $f^{-1}(y) \in U_y$ and $U_x \cap U_y = \emptyset$. Moreover f is δ-semiopen and bijective, therefore $f(x) \in f(U_x) \in \delta SO(Y)$, $y \in f(U_y) \in \delta SO(Y)$ and $f(U_x) \cap f(U_y) = \emptyset$. Hence $f(U_x) \cap \delta Cl_S(f(U_y)) = \emptyset$. This shows that $G(f)$ is strongly δ-semiclosed. $\boxed{QED}$

27 Theorem. *If $f : X \to Y$ is quasi δ-semicontinuous and Y is δ-semi T_2, then $G(f)$ is strongly δ-semiclosed.*

PROOF. Let $(x, y) \in (X \times Y) \backslash G(f)$. Then $y \neq f(x)$. Since Y is δ-semi T_2, by Theorem 23 there exist $V \in \delta SO(Y, y)$ and $W \in \delta SO(Y, f(x))$ such that $\delta Cl_S(V) \cap \delta Cl_S(W) = \emptyset$. Since f is quasi δ-semicontinuous, there exists $U \in \delta SO(X, x)$ such that $f(U) \subset \delta Cl_S(W)$. This, therefore, implies that $f(U) \cap \delta Cl_S(V) = \emptyset$. So by Lemma 7, $G(f)$ is strongly δ-semiclosed. $\boxed{QED}$

We close with the following questions which we were unable to answer:

Question 1. Is there any nontrivial example of a function with a strongly δ-semiclosed graph?

Question 2. Are there nontrivial examples of the notions introduced in Definition 17?

Question 3. Is there a nontrivial example of a δ-semiopen function?

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References

[1] M. Caldas, D. N. Georgiou, S. Jafari and T. Noiri: *More on δ-semiopen sets*, Note di Matematica, **22**(2) (2003/2004), 113–126.

[2] T. Husain: Topology and Maps, Plenum press, New York, 1977.

[3] N. Levine: *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly, **70**, (1963), 36–41.

[4] T. Noiri: *On functions with strongly closed graphs*, Acta Math., Acad. Sci. Hungar., **32**, (1978), 373–375.

[5] J. H. Park, B. Y. Lee and M. J. Son: *On δ-semiopen sets in topological space*, J. Indian Acad. Math, **19**, (1997), 1–4.

[6] N. V. Veličko: *H-closed topological spaces*, Mat. Sb., 70, (1966), 98–112; English transl., in Amer. Math. Soc. Transl., (2),**78**, (1968), 103–118.