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Near-embeddings of the affine plane with 9 points into Desarguesian projective and affine planes

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Abstract. Let $\mathcal{C}$ be obtained from the affine plane with 9 points by removing at most 4 lines. We describe the embeddings of these configurations $\mathcal{C}$ into Desarguesian planes.

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1 Introduction

The affine plane $\mathcal{A}$ of order 3 (with 9 points and 12 lines) is usually represented in the rational (or real) plane as follows: 8 lines of $\mathcal{A}$ are given by the sides of a parallelogram, its diagonals and the two medians of opposite sides, and two pairs of parallel lines are represented by curves (see Figure 1).

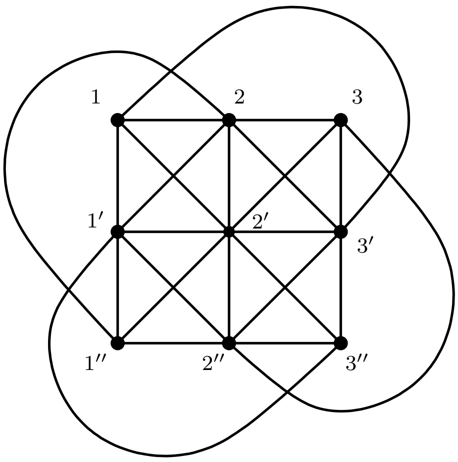

Figure 1.

This suggests the following generalization, which we call a *near-embedding* of $\mathcal{A}$. Consider a configuration $\mathcal{C}$ obtained from $\mathcal{A}$ by deleting a set $\mathcal{D}$ of $d \leq$

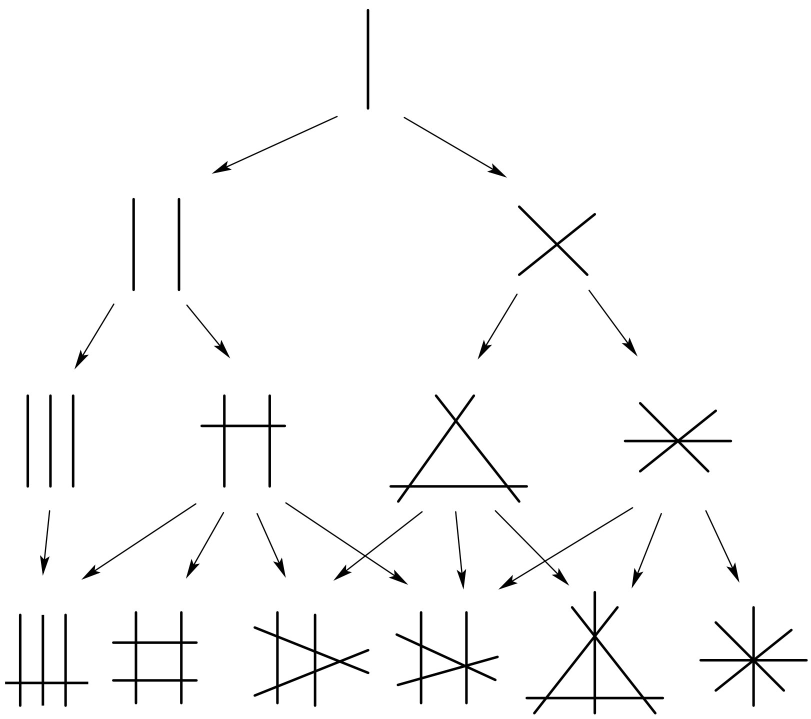

Figure 2.

$4 = 12 - 8$ lines, and by keeping the 9 points of $\mathcal{A}$. We call d the *defect*, $\mathcal{D}$ the *defect set*, and each line in $\mathcal{D}$ a *defect line*, and we study the embeddability of $\mathcal{C}$ into a Desarguesian projective (or affine) plane $\mathcal{P}$; such an embedding consists of injective maps of the point set and the line set of $\mathcal{C}$ into the corresponding sets of $\mathcal{P}$ such that incidence of points and lines holds in $\mathcal{C}$ if, and only if, the images are incident. The points of $\mathcal{C}$ are identified with their images in $\mathcal{P}$, and the lines of $\mathcal{C}$ are written as (unordered) triples ABC of their points.

Let K be the skew field which coordinatizes $\mathcal{P}$. For cardinality reasons, the case $|K| = 2$ is impossible, so we can assume that $|K| \geq 3$. In the case of characteristic 2, the projective plane of order 4 plays a special role: in Section 5 we obtain an interesting decomposition of that plane.

The various types of defect sets $\mathcal{D} \neq \emptyset$ are shown in Figure 2 (in contrast to affine planes of order > 3, there exists in $\mathcal{A}$ no set of 4 lines such that any 3 of them are not confluent and any 2 of them are not parallel).

The embeddability of the corresponding configuration $\mathcal{C}$ is inherited in the direction of the arrows, and hence the non-embeddability in the converse direction.

Since the automorphism group of $\mathcal{A}$ acts transitively on the triangles, all defect sets of the same type are isomorphic, and hence also the corresponding configurations $\mathcal{C}$ are isomorphic.

2 Confluent defect lines

First we study the type in the lower right corner of Figure 2, where $\mathcal{D}$ consists of 4 confluent lines. We introduce homogeneous coordinates (x, y, z) in $\mathcal{P}$ such that the points 3, $3'$, 2, $2'$ in Figure 1 have the coordinate triples $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 1)$. Without loss of generality we may assume that

$$\mathcal{D} = \{\, 11'1'', 1''2''3'', 1''2'3, 1''23' \,\} .$$

Since $3''$ is on $33'$ and $2''$ on $22'$, the points $3''$ and $2''$ have the coordinate triples $(1, a, 0)$ and $(1, 1, b)$ with $a, b \in K \setminus \{0, 1\}$. The lines 23, $2'3''$ and $2''3'$ pass through 1 and have the equations $y = 0$, $ax - y + (1 - a)z = 0$ and $bx - z = 0$. This yields the coordinate triple $(1, 0, b)$ for the point 1 and the condition

$$a + b = ab \; . \tag{1}$$

The lines $2'3'$, $2''3$ and $23''$ through $1'$ have the equations $x = z$, $by - z = 0$ and $ax - y = 0$. Therefore $1'$ has the coordinate triple $(1, a, 1)$, and a, b satisfy

$$ba = 1 \; . \tag{2}$$

Thus $b = a^{-1}$, and a, b commute, hence it suffices to consider a (commutative) subfield of K (i.e. an embedding into a pappian plane).

If we choose the point $1''$, which is not yet determined in this embedding, to be the intersection point of $23'$ and $2'3$, then $1''$ has the coordinate triple $(0, 1, 1)$. The equations (1) and (2) imply that $1''$ is also on $11'$ and on $2''3''$; thus we obtain an embedding of defect 0. This yields the embeddability of all types at the right edge of Figure 2, that is, of all types with confluent defect lines. According to (1) and (2), this embeddability is equivalent to the existence of $a \in K \setminus \{0, 1\}$ with

$$a^2 - a + 1 = 0 \; . \tag{3}$$

If the characteristic of K is not 2 (i.e. if $2 = 1 + 1 \neq 0$), then (3) is equivalent to $(a - 2^{-1})^2 = -3 \cdot 4^{-1}$, hence the embeddability condition says that -3 is a square in K. (This condition has been derived already in [1], [3], [5]; for $K = \mathbb{C}$ see also [6, p. 86], where $\mathcal{A}$ is described as the configuration of the 9 points of inflection of a non-singular cubic curve.)

In characteristic 2, equation (3) says that $\{\, 0, 1, a, a^{-1} \,\}$ is the field GF(4), hence embeddability means that GF(4) is a subfield of K. In particular, we obtain the known result that the affine plane of order 3 can be embedded into the projective plane of order 4. In this embedding, each parallel class of $\mathcal{A}$ gives a triangle in $\mathcal{P}$, which can be checked easily for the lines 123, $1'2'3'$, $1''2''3''$. The 4 triangles obtained in this fashion are disjoint, since non-parallel lines of $\mathcal{A}$ intersect in $\mathcal{A}$, hence not in any point of these 4 triangles. As $21 = 9 + 12$, these 4 triangles consist of all the points of $\mathcal{P} \setminus \mathcal{A}$.

3 Triangles

Now we study the case where the defect lines form a triangle, so we may assume that

$$\mathcal{D} = \{\, 11'1'', 1''2''3'', 12'3'' \,\} . \tag{4}$$

As in Section 1 we determine a, b, obtaining again equation (2) and the coordinate triple $(1, a, 1)$ for the point $1'$. The point 1 is the intersection point of 23 and $2''3'$ and has the coordinate triple $(1, 0, a^{-1})$, as in Section 1, and $1''$ is the intersection point of $23'$ and $2'3$ and has the coordinate triple $(0, 1, 1)$. This covers all 9 lines of $\mathcal{C}$, with the only condition that $a \in K \setminus \{0, 1\}$. Thus the 4 types where $\mathcal{D}$ contains the sides of a triangle are embeddable; again it suffices to consider the (commutative) subfield of K generated by a.

For the case (4), the configuration $\mathcal{C}$ is easier to survey in Figure 3, where the letters A, B, C etc. indicate the situation of inscribed triangles. The triples required for the representation of $\mathcal{A}$ are drawn by dashed curves.

The 9 collinear triples

$$\begin{array}{ccc} BA'C, & CB'A, & AC'B, \\ B'A''C', & C'B''A', & A'C''B', \\ AA''A', & BB''B', & CC''C' \end{array} \tag{5}$$

give rise to the following configurational proposition, which holds in every pappian affine plane: Denote by (M_i) the assertion that for each triple in the i-th row of (5), the second point of the triple is the midpoint of the other two points of the triple; then the assertions (M_i) with $i = 1, 2, 3$ are equivalent.

This can be proved easily, with vectors or with ratios (in characteristic 2 the proposition holds trivially, due to the non-existence of midpoints).

If C', B', A' lie between A, B and A, C and B, C, respectively, then (M_3) says that the triangles $A'BC'$, $C'AB'$, $B'CA'$ and $A'B'C'$ have the same area. With the above proposition we infer that (M_1) holds; in [4] this has been proved in a different way.

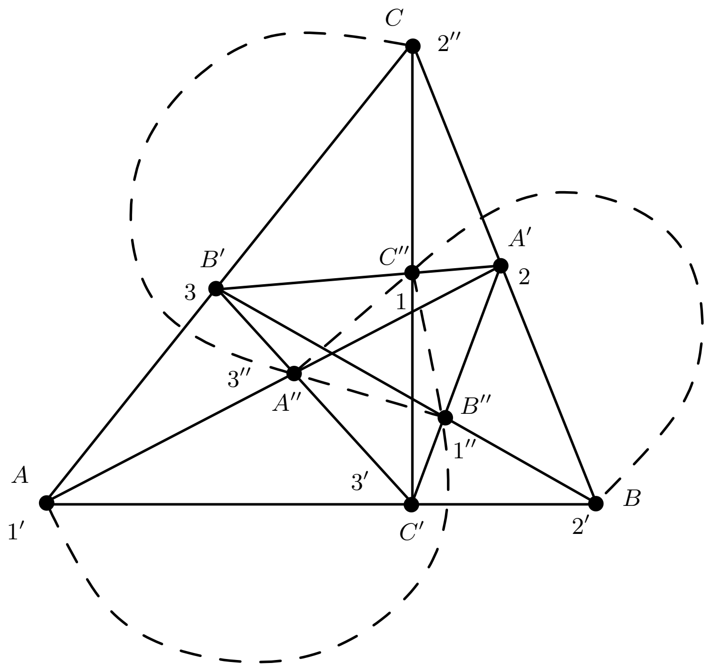

Figure 3.

4 Parallel defect lines

It remains to consider those types where $\mathcal{D}$ contains a pair of parallel lines (two of these types in the last row of Figure 2 have been considered already in Section 2). First let

$$\mathcal{D} = \{\, 11'1'', 22'2''\,\}\ . \tag{6}$$

With the notation of Section 1 and with $(1, a, 0)$ as the coordinate triple of $3''$, we obtain the coordinate triple $(a-1, 0, a)$ for the intersection point 1 of 23 and $2'3''$. Since $2''$ is on $13'$ (with the equation $ax + (1-a)z = 0$), this point has a coordinate triple $(a-1, c, a)$ with $c \in K \setminus \{0\}$. As in Section 1, the intersection point $1'$ of $2'3'$ and $23''$ has the coordinate triple $(1, a, 1)$; however, since $2''3$ now has the equation $ay - cz = 0$, we obtain the condition

$$a^2 = c\ . \tag{7}$$

The intersection point $1''$ of $23'$ and $2'3$ has the coordinate triple $(0, 1, 1)$. Using (7) we obtain $(a-1, c, a) - (1, a, 0)\cdot(a-1) = (0, a, a)$, which gives the yet missing collinearity of $1''2''3''$.

Each choice of $a \in K\setminus\{0,1\}$ (which is possible as $|K| \geq 3$) determines $c = a^2$ and yields an embedding of $\mathcal{C}$; again it suffices to consider the (commutative)

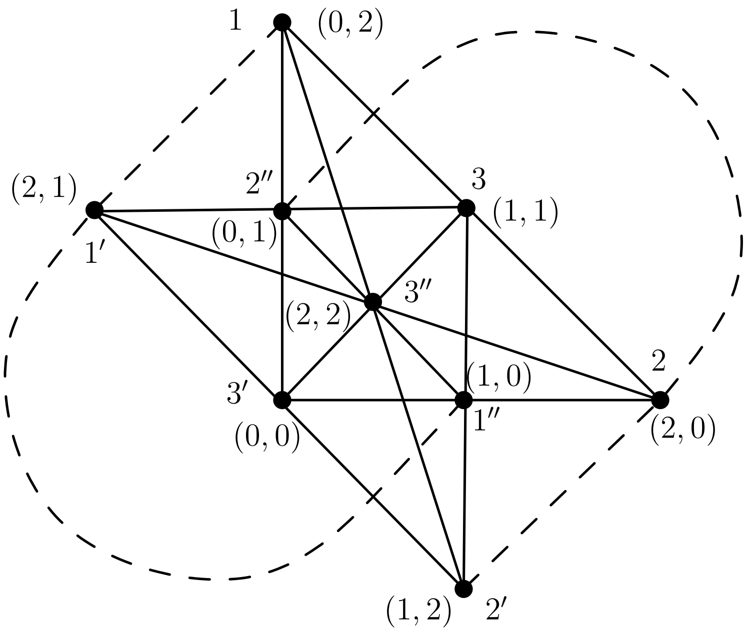

Figure 4.

subfield of K generated by a. According to the arrows in Figure 2, the same applies to all the other types where $\mathcal{D}$ contains a pair of parallel lines.

In summary, we have the following result.

1 Theorem. *If all defect lines are confluent, then a near-embedding of $\mathcal{A}$ into the projective plane $\mathcal{P}$ over K exists if, and only if, -3 is a square in K and, if K has characteristic* 2, *the field* GF(4) *is a subfield of K. For all other types of defect sets, a near-embedding is possible whenever $|K| \geq 3$.*

2 Remark. The configuration $\mathcal{C}$ with (6) can be regarded as a Pappos hexagon $12'31'23'$ with the carrier lines 123, $1'2'3'$, the line $1''2''3''$ of diagonal points, and the additional line $33'3''$, but also as the Pascal hexagon $12'1''21'2''$ with the Pascal line $33'3''$ and the additional lines 123, $1'2'3'$, $1''2''3''$ (each of these 3 additional collinearities is implied by the other two). $\mathcal{C}$ is represented in Figure 4 in a similar way as $\mathcal{A}$ in Figure 1, with the coordinate pairs of the 9 points in a coordinate system of $\mathcal{A}$ as the affine plane over GF(3).

3 Remark. If $\mathcal{D}$ consists of 3 parallel lines, then $\mathcal{C}$ is a 9_3-configuration; these configurations have been classified in [2, p. 107]. We have a 9_3-configuration of type I, i.e. a Pappos configuration ($\mathcal{C}$ as in Remark 1 without $33'3''$); this may also be inferred from the fact that type II contains no triple of points which are mutually not joined, and type III has just one such triple. Adding this triple as a line to a 9_3-configuration of type III yields a configuration which has 3 points of order 4 and 6 points of order 3, like the configuration $\mathcal{C}$ in Remark 1, without being isomorphic to that configuration. The 9_3-configuration of type III differs from the configuration belonging to (4) (which is not a 9_3-configuration) as follows: in (5), one has to replace the second row by $A'B''C''$, $A''B'C''$, $A''B''C'$.

5 Affine near-embeddings

If not only an embedding of $\mathcal{C}$ into a projective plane $\mathcal{P}$ is required, but rather an embedding into an affine plane obtained by removing a line (and its points) from $\mathcal{P}$, then one needs a line of $\mathcal{P}$ which contains none of the 9 points of $\mathcal{A}$. If $|K| > 4$, such a line always exists, as we show now. For two parallel lines of $\mathcal{A}$, let S be their intersection point in $\mathcal{P}$, which does not belong to $\mathcal{A}$. Projecting the 9 points of $\mathcal{A}$ (with projection center S) into some line g not passing through S yields at most 5 points. Since $|K| \geq 5$, the line g contains at least 6 points, hence also a point P which is distinct from the points of the $\mathcal{A}$-projection. Then the line SP contains no point of $\mathcal{A}$. If $|K| = 3$, then $\mathcal{A}$ itself can be taken as the embedding affine plane. Therefore, if $|K| \neq 4$, then projective embeddability implies affine embeddability.

It remains to consider the case $K = \mathrm{GF}(4)$. Here each $a \in K \setminus \{0, 1\}$ satisfies

$$a^2 + a + 1 = 0, \quad a^3 = 1 \tag{8}$$

and $K = \{\, 0, 1, a, a^2 \,\}$. At most one of the 4 lines of $\mathcal{A}$ passing through a point P can be a non-collinear triple of points in $\mathcal{P}$; indeed, such a triple yields in $\mathcal{P}$ two joining lines with P, so that the existence of two such lines (through P) in $\mathcal{P}$ would lead to at least $2+2+2 = 6$ lines through P. Hence two lines of $\mathcal{A}$ which are not collinear as point triples in $\mathcal{P}$ are necessarily parallel in $\mathcal{A}$; therefore there are at most 3 such lines. Using the notation of Figure 1, we may assume that they are among $11'1''$, $22'2''$, $33'3''$. We choose a homogeneous coordinate system as in Section 1, with the coordinate triples for $3, 3', 2, 2'$ as in Section 1. The point 1 on 23 has then a coordinate triple $(a, 0, 1)$ with $a \in K \setminus \{0, 1\}$. The lines $23'$ and $2'3$ have the equations $x = 0$ and $y = z$, thus their intersection point $1''$ has the coordinate triple $(0, 1, 1)$. The line $12'$ has the equation $x+!(a-1)y-az = 0$; therefore, in view of (8) and the equation $1+1 = 0$, the point $3''$ on this line has a coordinate triple $(1, b, a^2(b+1)+b)$ with $b \in \{a, a^2\}$. The lines $13'$ and $1''3''$ with the equations $x - az = 0$ and $a^2(b-1)x + y - z = 0$ yield the coordinate triple $(a, b, 1)$ for the point $2''$, and the lines $23''$ and $2'3'$ with the equations $bx - y = 0$ and $x = z$ yield the coordinate triple $(1, b, 1)$ for $1'$. In view of the equation $(1, b, 1) - (a, b, 1) = (1 - a, 0, 0)$ also $1'2''3$ are collinear. Thus we have an embedding of the configuration obtained from $\mathcal{A}$ by removing the lines $11'1''$, $22'2''$, $33'3''$.

In the case $b = a$ we obtain the coordinate triples derived in Section 1 for a complete embedding of $\mathcal{A}$. Since each point P of $\mathcal{A}$ carries exactly one line of $\mathcal{P}$ which meets $\mathcal{A}$ only in P, there are $12+9 = 21$ lines of $\mathcal{P}$ which contain a point of $\mathcal{A}$, and in view of $21 = 4^2+4+1$ these are all the lines of $\mathcal{P}$. Hence this case does not allow affine embeddability. (The non-existence of an affine embedding of $\mathcal{A}$ is also a consequence of the fact that GF(3) is not a subfield of GF(4).)

In the case $b = a^2$, we use (8) and the equation $1+1 = 0$ to prove that each of the triples $11'1''$, $22'2''$, $33'3''$ is not collinear. Each of the 9 points of $\mathcal{C}$, which is now a Pappos configuration, carries two $\mathcal{P}$-lines with exactly two $\mathcal{C}$-points and three $\mathcal{P}$-lines with exactly three $\mathcal{C}$-points, hence no $\mathcal{P}$-line with exactly one $\mathcal{C}$-point. Therefore precisely $18 = 9+9$ $\mathcal{P}$-lines meet $\mathcal{C}$, hence $3 = 21-18$ $\mathcal{P}$-lines contain no point of $\mathcal{C}$. We conclude that $\mathcal{C}$ can be embedded into an affine plane of order 4.

Thus we have the following result: Only the two types on the left edge of Figure 2 with 3 parallel defect lines lead to embeddability into the affine plane of order 4; for all other types one has affine embeddability only for $|K| \neq 4$, provided that one has projective embeddability.

6 The projective plane of order 4

The three lines without $\mathcal{C}$-points appearing in Section 4 for $|K| = 4, b = a^2$ have the equations

$$x + ay + z = 0, \quad ax + ay + z = 0, \quad x + a^2y + z = 0 \ ,$$

as one shows easily using (8) and the equation $1+1 = 0$. The intersection point U of the first two lines has the coordinate triple $(0, a^2, 1)$, the intersection point V of the last two lines has the coordinate triple $(a, 1, 1)$, and the intersection point W of the first and the last line has the coordinate triple $(1, 0, 1)$. The three lines $12'3''$, $1''23'$, $1'2''3$ are confluent in U, the three lines $1''2'3$, $12''3'$, $1'23''$ are confluent in V, and 123, $1'2'3'$, $1''2''3''$ are confluent in W. The line VW meets those three lines through U in their fifth points 4, $4'$, $4''$ with the coordinate triples $(1, a, 0)$, $(0, 1, a^2)$, $(a^2, a^2, 1)$. Analogously, replacing U by V or W, we obtain the points 5, $5'$, $5''$ on WU with the coordinate triples $(a^2, 1, 1)$, $(a, a, 1)$, $(1, a^2, 0)$ and the points 6, $6'$, $6''$ on UV with the coordinate triples $(1, 0, a)$, $(1, a, 1)$, $(1, 1, 0)$. The lines $11'$, $1'1''$, $1''1$, $22'$, $2'2''$, $2''2$, ! $33'$, $3'3''$ and $3''3$ carry the (yet missing) point triples $4'56''$, $45'6$, $4''5''6'$, $4''5'6''$, $4'5''6$, $456'$, $45''6''$, $4''56$ and $4'5'6'$, respectively. Together with the triples $44'4''$, $55'5''$, $66'6''$ (which lie on VW, WU, UV, respectively) they form an affine plane $\mathcal{A}'$ of order 3. Each of the 21 lines of $\mathcal{P}$ belongs either to the Pappos configuration $\mathcal{C}$ or to the affine plane $\mathcal{A}'$, and each of the 21 points of $\mathcal{P}$ belongs either to $\mathcal{C}$ or to $\mathcal{A}'$ or to $\{U, V, W\}$. This proves the following result.

4 Theorem. *The projective plane of order* 4 *can be decomposed into a Pappos configuration, an affine plane of order* 3 *and a triangle, in such a way that the point sets are disjoint* but the line sets are not completely disjoint, since the sides of the triangle are lines of the affine plane.

As the above construction shows, the Pappos configuration $\mathcal{C}$ determines the affine plane $\mathcal{A}'$ and the triangle $\{U, V, W\}$. In fact, $\{U, V, W\}$ determines $\mathcal{A}'$: the lines of $\mathcal{A}'$ are the lines of $\mathcal{P}$ which pass through one of the points U, V, W. In contrast, $\mathcal{A}'$ has 4 parallel classes and yields therefore 4 possibilities for $\{U, V, W\}$ (see the remark at the end of Section 1).

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