COTYPE CONSTANTS AND INEQUALITIES BETWEEN SUMMING AND INTEGRAL NORMS OF FINITE RANK OPERATORS

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Abstract. *We establish inequalities between p-integral and absolutely r-summing norms of finite rank operators acting between Banach spaces in which the notion of cotype is considered. Several applications are given: (a) We get estimates, in this context, for the p-integral norm of the identity operator on n-dimensional Banach spaces and for some Banach-Mazur distance. (b) We obtain a sufficient condition for a kernel of weighted Besov type generates a nuclear operator.*

1. PRELIMINARIES

The identity operator on an n-dimensional Banach space E_n is denoted by I_n. We refer to [4] for definitions and main properties of the operator ideals $[\mathcal{M}_{r,s}\ ;\ \mu_{r,s}]$, $[\Pi_{p,q}\ ;\ \pi_{p,q}]$, $[\mathcal{I}_r\ ;\ i_r]$ and $[\mathcal{N}_s\ ;\ \nu_s]$ of (r,s)-mixing, absolutely (p,q)-summing, r-integral and s-nuclear operators, respectively. For $p = q$ we have the operator ideal $[\Pi_p\ ;\ \pi_p]$ of absolutely p-summing operators. We shall freely make use of the results given there, omitting specific references.

We recall some definitions. Let $2 \leq q < \infty$. A Banach space E is said to be of (Rademacher) cotype q if there exists a constant k such that

$$\left(\sum_{i=1}^{n} ||x_i||^q\right)^{1/q} \quad \leq \quad k\int_0^1 ||\sum_{i=1}^{n} r_i(t)x_i||dt$$

for all finite families of elements $x_1, \ldots, x_n \in E$, where r_i denotes the i^{th} Rademacher function. We put $K_q(E) := \inf k$.

Moreover, let E and F be two Banach spaces. The Banach-Mazur distance is defined by $d(E,F) := \inf ||T||||T^{-1}||$, where the infimum runs over all isomorphism $T : E \to F$.

We denote by l_p^n the Banach space of all n-dimensional scalar vectors $x = (\xi_1, \ldots, \xi_n)$ equipped with the norm

$$||x||_p := \left(\sum_{i=1}^{n} |\xi_i|^p\right)^{1/p} \quad , \quad 1 \leq p < \infty.$$

If $1 \leq p \leq \infty$, then the dual exponent p' is defined by

$$1/p + 1/p' = 1.$$

In the sequel $c, c_1, c_2, \ldots$, are positive constants which may depend on certain exponents, but not on operators, Banach spaces, or natural numbers.

2. THE RESULTS

We start our considerations with two inequelities of Lewis-type. For examples we refer to [1] and [2].

Theorem 1. *Let* $n = 1, 2, \dots$.

(i) Let $1 < p < q' \leq 2$. *Let F be a Banach space of cotype q. Then*

$$i_p(T : F \to E_n) \leq cK_q(F)n^{1/2-1/q}\pi_2(T : F \to E_n).$$

(ii) Let $2 \leq q < s < \infty$. *Then for every Banach space F one has*

$$i_2(T : F \to E_n) \leq cK_q(E_n)n^{1/2-1/q}\pi_s(T : F \to E_n).$$

Proof. (*i*) By Tomczak-Jaegerman [7, p. 150], given an operator $T : E_n \to F$, we have

$$\pi_2(T : E_n \to F) \leq c_1 n^{1/2-1/q}\pi_{q,2}(T : E_n \to F).$$

Using $T = I_F T$ with I_F the identity operator on F, by a multiplication formula of [6] we get

$$\pi_{q,2}(T : E_n \to F) \leq c_2 K_q(F)\pi_{p'}(T : E_n \to F)$$

for $2 \leq q < p' < \infty$.

Combining the above inequalities we arrive at

$$\pi_2(T : E_n \to F) \leq cK_q(F)^{1/2-1/q}\pi_{p'}(T : E_n \to F)$$

for $2 \leq q < p' < \infty$, and by duality we obtain

$$i_p(T : F \to E_n) \leq cK_q(F)n^{1/2-1/q}i_2(T : F \to E_n),$$

concluding the proof of (*i*).

(*ii*) Given an operator $T : E_n \to F$, we have the factorization $T = JT_0$:

$$E_n \xrightarrow{T_0} T(E_n) \xrightarrow{J} F$$

where T_0 is the astriction of T and J the natural injection. Let $1 < p < q' \leq 2$. From (*i*) we obtain

$$\begin{aligned} i_p(T : E_n \to F) &\leq i_p(T_0 : E_n \to T(E_n)) \\ &\leq cK_q(E_n)n^{1/2-1/q}\pi_2(T_0 : E_n \to T(E_n)) \\ &= cK_q(E_n)n^{1/2-1/q}\pi_2(T : E_n \to F). \end{aligned}$$

Hence, for $2 \leq q < s < \infty$, we arrive again by duality at

$$i_2(T : F \to E_n) \leq cK_q(E_n)n^{1/2-1/q}\pi_s(T : F \to E_n),$$

completing the proof.

Now we give some estimates in connection with the local theory of Banach spaces. There is an extensive literature dealing estimates of this type; cf. [7].

Theorem 2. *Let* $n = 1, 2, \ldots$.

(i) If $1 < p < q' \leq 2$, *then* $i_p(I_n) \leq cK_q(E_n)n^{1/q'}$.

(ii) Let E_n *be an n-dimensional subspace of a Banach space E of cotype q. If* $1 < p < q' \leq 2$, *then there exists a proyection P from E onto* E_n *such that* $i_p(P : E \to E_n) \leq cK_q(E)n^{1/q'}$.

(iii) If $2 \leq q, \nu < \infty$, *then* $n^{1/q - 1/\nu} \leq cK_q(E_n)d(l_\nu^n, E_n)$.

Proof. To prove (i) we apply theorem $1(i)$ with $F = E_n$ and $\pi_2(I_n) = n^{1/2}$.

The statement (ii) also follows from Theorem $1(i)$ and the well-known result [3, p. 59] that there exists a proyection P with

$$\pi_2(P : E \to E_n) \leq n^{1/2}.$$

If $2 \leq v < r < \infty$, for the identity operator $I_n : l_v^n \to l_v^n$ we have

$$\pi_r(I_n) \leq i_r(I_n) \leq \mu_{r',1}(I_n)i_\infty(I_n) \leq c_1 n^{1/v},$$

and then

$$n = \nu_1(I_n) \leq \pi_r(I_n)\nu_{r'}(I_n) \leq c_1 n^{1/v}\nu_{r'}(I_n). \qquad (*)$$

Let $T : l_v^n \to E_n$ be any invertible operator. If $1 < p < \inf(\nu', q')$, from the above estimate (i) and $(*)$ we obtain

$$n^{1/'} \leq c_1\nu_p(I_n) \leq c_1||T^{-1}||||T||\nu_p(I_n : E_n \to E_n) \leq c_2||T^{-1}||||T||K_q(E_n)n^{1/q'}.$$

Hence $n^{1/q - 1/v} \leq c_2 K_q(E_n)||T^{-1}||||T||$, and taking the infimum over all invertible operators we get the part (iii) of the theorem.

Concerning the definition of the weighted Besov space $B_{p,u}^\sigma(\alpha_0, \alpha_1)$ the reader is referred to [6]. Given a kernel

$$K \epsilon [B_{p,u}^\sigma(\alpha_0, \alpha_1), B_{q,v}^\tau(\beta_0, \beta_1)],$$

the rule

$$T_K : g'(\eta) \to f(\xi) = \int K(\xi, \eta)g'(\eta)d\eta$$

defines an operator from $B_{q,v}^\tau(\beta_0, \beta_1)'$ into $B_{p,u}^\sigma(\alpha_0, \alpha_1)$. Let us now suppose that the embedding operator I from $B_{p,u}^\sigma(\alpha_0, \alpha_1)$ into $B_{q,v}^\tau(\beta_0, \beta_1)'$ exists. Then the operator IT_K acts in $B_{q,v}^\tau(\beta_0, \beta_1)'$. We assume that

$$\alpha := \alpha_0 - \alpha_1 + \sigma \geq 0 \qquad \beta := \beta_0 - \beta_1 + \tau \geq 0$$

$$\omega := \alpha_0 + \beta_0 + 1/p + 1/q - 1 > 0, \qquad \sigma > 0, \tau > 0$$

and we consider (see [6]) the corresponding regular parameter configuration.

We refer to [3, (2)] for definitions and fundamental properties of the operator ideals $\mathcal{L}^{(a)}_{r,w}$ and $(\Pi_p)^{(a)}_{r,w}$, consisting of operators of approximation type $l_{r,w}$ and of Π_p-approximation type $l_{r,w}$, respectively.

Theorem 3. *Let*

$$\rho = \rho(\alpha_0 - \alpha_1 + \sigma, \beta_0 - \beta_1 + \tau, \alpha_0 + \beta_0 + 1/p + 1/q - 1, \sigma, \tau) > 0,$$

$1 \leq u, p < \infty, 1/p + 1/q < 1$ *and* $1/\rho < q \leq 2$. *If*

$$K \in [B^{\sigma}_{p,u}(\alpha_0, \alpha_1), B^{\tau}_{q,v}(\beta_0, \beta_1)],$$

then in the regular case

$$IT_K \in (\Pi_2)^{(a)}_{2,1}(B^{\tau}_{q,v}(\beta_0, \beta_1)', B^{\tau}_{q,v}(\beta_0, \beta_1)').$$

Proof. From [6] we know that

$$T_K \in \Pi_r(B^{\tau}_{q,v}(\beta_0, \beta_1)', B^{\sigma}_{p,u}(\alpha_0, \alpha_1))$$

with $r = \max(p, u)$ and also that the embedding I admits a factorization $I = SR$

$$B^{\sigma}_{p,u}(\alpha_0, \alpha_1) \xrightarrow{R} E \xrightarrow{S} B^{\tau}_{q,v}(\beta_0, \beta_1)'$$

where E is a sequence space of cotype q' and $R \in \mathcal{L}^{(a)}_{s,\infty}(B^{\sigma}_{p,u}(\alpha_0, \alpha_1), E)$ with $1/s = \rho$. If $t > \max(q', r)$, using theorem 1(ii) and the embedding theorem in [5] we obtain

$$RT_K \in \mathcal{L}^{(\dashv)}_{\text{II},1} \circ \Pi_{\sqcup} \mathcal{B}^{\tau}_{\text{II},\sqsubseteq}(\beta_0, \beta_1)', \mathcal{E})$$

$$\subseteq (\Pi_t)^{(a)}_{q,1}(B^{\tau}_{q,v}(\beta_0, \beta_1)', E) \subseteq (\Pi_2)^{(a)}_{2,1}(B^{\tau}_{q,v}(\beta_0, \beta_1)', E).$$

This proves the assertion.

Finally, we recall that various criteria of nuclearity were established by W.F. Stinespring and others; cf. [3]. Since

$$\nu_1(T : E \to F) \leq n^{1/2}\pi_2(T : E \to F) \quad \text{whenever} \quad \dim T(E) \leq n$$

(with E and F arbitrary Banach spaces), the embedding theorem in [3, p. 102] yields $(\Pi_2)^{(a)}_{2,1} \subseteq \mathcal{N}_1$. Hence the preceding operator IT_K is nuclear. The behavior of the Weyl numbers of IT_K was investigated in [6].

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