FINITE AND LOCALLY SOLVABLE PERIODIC GROUPS WITH GIVEN INTERSECTIONS OF CERTAIN SUBGROUPS

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Abstract. *Let G be a group and p be a prime. We say that two subgroups H, K are incident if either $H \cap K = H$ or $H \cap K = K$. A group G is an IC_p-group if, for any finite non-incident subgroups H, K of G, a p-Sylow subgroup of $H \cap K$ is cyclic.*

In this paper we give a complete classification of solvable and locally solvable periodic IC_p-groups.

1.

Let G be a finite group, p a prime, $P \in Syl_p(G)$. We say that two subgroups F, H are incident if either $F \cap H = F$ or $F \cap H = H$, non-incident in the opposite case. A group G is said to be an IC_p-group if a Sylow p-subgroup of the intersection of any two non-incident subgroups is cyclic [B]. A group G is said to be an $IC - p$-group if the intersection of any two non-incident subgroups is cyclic whenever it is a p-group [Is]. Obviously $IC_p \subset IC - p$. In this note we obtain a complete classification of solvable finite and locally solvable periodic IC_p-groups; for the classification of non-solvable IC_2-groups see [B]. We feel that the description of solvable $IC - p$-groups is fairly difficult. As Isaacs noticed (see [Is]) generally an epimorphic image of an $IC - p$-group is not an $IC - p$-group.

We denote by $C(m)$ the cyclic group of order m, by $E(p^n)$ the elementary abelian group of order p^n, by $Q(2^n)$ the generalized quaternion group of order 2^n. Furthermore $Z(p^\infty)$ denotes the Prüfer p-group, $Q(2^\infty)$ the infinite quaternion group and $D(2^\infty)$ the infinite locally dihedral group. Let $A[B$ denote a semi-direct product with kernel B and complement A, and (A, B) a Frobenius group with kernel B and complement A.

2.

Finite solvable IC_p-groups. - In this section we consider only finite solvable groups. We prove the following

Main Theorem 1. *Let p be a prime divisor of the order of a finite solvable group G. Let $P \in Syl_p(G)$, $P < G$ and assume that P is not cyclic.*

Then G is an IC_p-group if and only if one of the following is true:

(a) $G = P \times C(q^n)$, where either $p = 2$, $P = Q(2^3)$ and $q \neq 2$, or $P = E(p^2)$ and $p \neq q$.

(b) $G = C(q^n)$ [P where either $P = Q(2^3)$ and $q = 3$, or $P = E(p^2)$ and $p \neq q$. Moreover, if $P = Q(2^3)$, then $G / Z(G) = A_4$, the alternating group of degree 4.

(c) $G = (C(q^n), E(p^3))$, and any subgroup of order qp^3 of G is minimal non-nilpotent.

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(d) $G = C(q^n)[P$, *where* P *is non-abelian of order* p^3 *and exponent* p, $p > 2, p \not\equiv 1 \pmod q$, $Z(G) = Z(P), G/Z(P)$ *is a Frobenius group with kernel* $P/Z(P)$, *and* $P/Z(P)$ *is a minimal normal subgroup of* $G/Z(P)$.

(e) $G = P[C(q^n)$, *where* $p = 2$, $P = Q(2^3)$ *and* $q \neq 2$.

(f) $G = (P, Q)$ *where* $p = 2$, $P = Q(2^3)$, Q *is a homocyclic* q*-group,* $q \neq 2$, *and* $\Omega_1(Q) = < x \in Q / x^q = 1 >$ *is a minimal normal subgroup of* G.

(g) $G = P[Q$, *where* $p = 2, P = Q(2^3)$, Q *is an extraspecial* q*-subgroup of exponent* $q, q \neq 2, Q/Z(Q)$ *is a minimal normal subgroup of* $G/Z(Q)$, *and* $G/Z(Q)$ *is a Frobenius group.*

(h) $G = C(p) \times (C(p), Q)$, *where* $P = E(p^2), Q$ *is a* q*-group,* $q \neq p$, *and all non-nilpotent subgroups of the group* $(C(p), Q)$ *have only one chief series whose* q*-factors have the same order* q^n, *where* n *is the order of* $q \pmod p$.

(i) $G = C(p) \times F$, *where* $F = C(p)[Q, Q$ *is an extraspecial* q*-group,* $q \neq p, P = E(p^2)$, $F/Z(Q)$ *is a Frobenius group and* $Q/Z(Q)$ *is a minimal normal subgroup of* $F/Z(Q)$.

This Theorem follows from a long chain of lemmas.

Lemma 1. (see [B]). *Let* G *be a* p*-group. Then* G *is an* IC_p*-group if and only if one of the following assertions holds:*

(a) G *is cyclic,*

(b) G *contains a cyclic subgroup of index* p,

(c) G *is of order* p^3 *and exponent* p.

Lemma 2. *Let* $P > 1$ *be a non-cyclic Sylow* p*-subgroup of a group* $G, P < G$, *and let* p *be the smallest prime divisor of* $|G|$. *Suppose that* G *satisfies the following condition:*

(*) If $P_1 < P$ *is a Sylow subgroup of a non-primary subgroup of* G, *then* P_1 *is cyclic.*

Then one of the following assertions holds:

(a) P *is extraspecial, and* $\exp P = p$ *for* $p > 2$,

(b) P *is elementary abelian.*

Proof. Suppose that $D = O_{p'}(G) > 1$. If $P_1 < P$ then $P_1 \in Syl_p(P_1D)$ and P_1 is cyclic by $(*)$. So all proper subgroups of P are cyclic; therefore either P is abelian of type (p,p) or $P = Q(2^3)$.

Suppose now that $D = 1$. Then by Frobenius normal p-complement Theorem there exists in G a minimal non-nilpotent subgroup H with a normal Sylow p-subgroup $P_1 > 1$. Now P_1 is not cyclic since p is the smallest prime divisor of $|G|$. Therefore we conclude from $(*)$ that $P_1 = P$. Since P is a normal Sylow p-subgroup of H it is special (i.e. either $P = E(p^n)$, or $Z(P) = P' = \Phi(P)$ is elementary abelian). Suppose that P is not abelian. If Q is a non-normal Sylow subgroup of H then $Z(P)$ is a Sylow p-subgroup of $Z(P)Q$ so $Z(P)$ is cyclic by $(*)$. Thus P is extraspecial. Assume that $p > 2$. Then P is a regular p-group. Thus $\Omega_1(P)$ is a noncyclic Sylow subgroup of $\Omega_1(P)Q$, so $P = \Omega_1(P)$ has exponent p.

Now we give a short proof of the main result from [Is].

Lemma 3. *Suppose that* G *is a non-primary IC-p-group,* $P \in Syl_p(G)$. *Then* P *is cyclic, quaternion, or* $|P| \leq p^3$ *and* $\exp P = p$.

Proof. By Lemma 1 we may assume that P is not cyclic, $|P| \geq p^3$ and P contains a cyclic subgroup of index p.

Suppose that $D = O_{p'}(G) > 1$. Then as in Lemma 2, all proper subgroups of P are cyclic, so $P = Q(2^3)$. Let now $D = 1$.

Suppose that p is the smallest prime divisor of $|G|$. Then G satisfies the condition $(*)$ of Lemma 2. Therefore P is either extraspecial or elementary abelian. By our hypothesis on P we get $|P| = p^3$. If P is not quaternion then $\exp P = p$ by Lemma 2.

Suppose that p is not the smallest prime divisor of $|G|$. Then $p > 2$ and P is regular. Therefore, by a result of Wielandt, $N_G(P) > P$. As above $\Omega_1(P) = P$ is of exponent p, $|P| = p^2$– a contradiction.

For the rest of this section we make the following

HYPOTHESIS: (i) G is a non-primary solvable IC_p-group.

(ii) p is a prime divisor of $|G|$.

(iii) P is a non-cyclic Sylow p-subgroup of G.

First we have

Lemma 4. *$G = PQ$ where $Q \in Syl_q(G)$, q a prime divisor of $|G|$.*

Proof. This is an easy corollary of Hall's Theorem on solvable groups.

Lemma 5. *If P is normal in G then $|P| < p^4$, and Q is cyclic.*

Proof. If Q_1, Q_2 are non-incident subgroups of Q then PQ_1, PQ_2 are non-incident and $P \leq PQ_1 \cap PQ_2$, a contradiction. Therefore Q is cyclic. The first assertion follows from Lemma 3.

If a Sylow p-subgroup P of a non-primary group G is normal in $G, G/P$ is a primary cyclic group, and either P is abelian of type (p,p) or $P = Q(2^3)$ and G satisfies the additional hypothesis in (*b*) of Main Theorem 1, then G is an IC_p-group. Therefore, if P is normal in G, we can assume $|P| = p^3$.

Lemma 6. *Suppose that P is elementary abelian of order p^3. Then $G = (Q,P)$. Moreover, if L/P is a subgroup of order q, then L is minimal non-nilpotent group.*

Proof. Obviously $O_q(G) = 1$. Since $C_G(O_p(G)) \leq O_p(G)$, then $P = O_p(G)$ is normal in G. By Lemma 5, Q is cyclic, and, by Maschke's Theorem, P is a minimal normal subgroup of G. Then $G = (Q,P)$, and any subgroup L of G with $P < L$ and $|L/P| = q$, is minimal non-nilpotent, as required.

Lemma 7. *Suppose that P is non-abelian of order p^3 and exponent p. Then $G = Q[P, Z(P) = Z(G), G/Z(G)$ is a Frobenius group. Moreover, if $S/Z(G)$ is a subgroup of order q in $G/Z(G)$, then S is a minimal non-nilpotent group.*

Proof. Obviously $O_q(G) = 1$. Then $D = O_p(G) > 1$. Suppose that $D < P$. Since $D = P \cap DQ$ then D is cyclic. Since $C_G(D) \leq D$, we get a contradiction. Thus $D = P$. Obviously $P/Z(P)$ is a minimal normal subgroup of $G/Z(P)$. Suppose that $Z(P) \neq Z(G)$, and set $C = C_G(Z(P))$. Then $|G/C|$ is a non-trivial power of q, and $p \not\equiv 1 \pmod q$. Take a minimal non-nilpotent subgroup $M/Z(P)$ of $G/Z(P)$. Then a Sylow p-subgroup M_p of M has order p^2 (since

$p \equiv 1 \pmod q$)) and it is not cyclic. Then M and P are non-incident, $M_p \leq M \cap P$ and G is not an IC_p-group. Thus $Z(P) = Z(G)$. Take a subgroup S/P of order q in G/P. Since $P/Z(P) = P/Z(S)$ is a minimal normal subgroup of $S/Z(P)$, then S is a minimal non-nilpotent group.

From Lemmas 5, 6, 7, it follows that, if P is normal in G, then G satisfies one of $(a), (b), (c), (d)$ of Main Theorem 1. Conversely, it is easy to see that if G satisfies one of $(a), (b), (c), (d)$, then G is an IC_p-group.

Now we study the case: P not normal in G. Then, by Lemmas 3, 6 and 7, either P is elementary abelian of order p^2, or $P = Q(2^3)$.

We investigate first the case $P = E(p^2)$.

Lemma 8. *Suppose that $P = E(p^2)$ is not normal in G. Then Q is normal in G.*

Proof. If $O_q(G) = 1$, then P is normal in G (since the p-length of G is equal to 1) - a contradiction. So assume that $Q \neq D = O_q(G) > 1$. By the above remark PD is normal in $G, PD < G$. Then $G = N_G(P)PD$, $N_G(P)$ and PD are non-incident, and $P \leq N_G(P) \cap PD$ is not cyclic - a contradiction.

Lemma 9. *Suppose that Q is normal in G and $|P| = p^2$. Then $G = C(p) \times C(p)[Q$.*

Proof. We have (see [G], Theorem 5.3.16, p. 188) $Q = < C_Q(x) | x \in P^\# >$.

Now, either there exist $x, y \in P^\#$ such that $C_Q(x), C_Q(y)$ are not incident, or $C_Q(x) = Q$ for some $x \in P^\#$. In the second case we have $G = <x> \times <y>[Q$. In the first case we have $P \leq PC_Q(x) \cap PC_Q(y)$, a contradiction.

Lemma 10. *Suppose that $P = E(p^2)$ is not normal in G and G does not contain a normal subgroup of order q. Then $G = <a> \times (<b>, Q)$, and all non-nilpotent subgroups of $(<b>, Q)$ have only one chief series, whose indices are p and q^n, where n is the order of $q (\mathrm{mod}\, p)$.*

Proof. By Lemmas 8, 9 we have $G = <a> \times (<b>[Q)$. By assumption any two subgroups of $F = <b>[Q$ that contain $<b>$ are incident. Set $q^n = |Q/\Phi(Q)|$. If $n = 1$ then Q is cyclic, and the result is obvious. Hence assume that $n > 1$. By Maschke's Theorem $Q/\Phi(Q)$ is a minimal normal subgroup of $F/\Phi(Q)$. So n is the order of $q (\mathrm{mod}\, p)$. If q is not an index of a chief series of F, then all q-indices of this series are equal to q^n. Suppose that R/T is a chief factor of F of order q. Without loss of generality we may assume that T is a minimal normal subgroup of F. By assumption $|T| > 1$. Since $T \leq Z(Q)$, then R is abelian. By Maschke's Theorem R is not elementary abelian. Then $\Phi(R)$ is a normal subgroup of order q of F - a contradiction. Thus all q-indices of a chief series of F are equal to q^n. Now we prove that F has only one chief series. It suffices to prove that F contains only one minimal normal q-subgroup. This is obvious since any two subgroups of F that contain $<b>$ are incident. Finally F is a Frobenius group.

Notice that every group that satisfies the conditions of Lemma 10 is an IC_p-group.

Lemma 11. *Let $P = E(p^2)$ be not normal in G, and let G contain a normal subgroup of order q. Assume that Q is not cyclic. Then Q is extraspecial and $Q/Z(Q)$ is a minimal normal subgroup of $G/Z(Q)$.*

Proof. Write $G = <a> \times F$, where a has order p and $F = <b>[Q$. Then F contains only one minimal normal q-subgroup, say R, and $|R| = q$. By Maschke's Theorem R is the only normal elementary abelian q-subgroup of F. Therefore any abelian normal q-subgroup of F is cyclic. In particular $Z(\Phi(Q))$ is cyclic, and $\Phi(Q)$ is cyclic by a result of Hobby (see [H]). Also $Q / \Phi(Q)$ is a minimal normal subgroup of $F / \Phi(Q)$, and $F / \Phi(Q)$ is not nilpotent. First we prove that $|\Phi(Q)| = q$.

If the subgroups $\Phi(Q)$ and $\Omega_1(Q)$ are not incident, then $P\Phi(Q)$ and $P\Omega_1(Q)$ are not incident, and $P \leq P\Phi(Q) \cap P\Omega_1(Q)$ is not cyclic, a contradiction. Then either $\Omega_1(Q) \leq \Phi(Q)$, or $\Phi(Q) < \Omega_1(Q)$ and $\Omega_1(Q) = Q$, since $Q / \Phi(Q)$ is a minimal normal subgroup of $G / \Phi(Q)$.

Now, if $q > 2$, then Q is regular, and $\Omega_1(Q)$ has exponent q. Futhermore $|\Omega_1(Q)| > q$, since Q is not cyclic. Then $\Omega_1(Q)$ is not contained in $\Phi(Q)$, and $Q = \Omega_1(Q)$ has exponent q, so that $|\Phi(Q)| = q$.

Now assume that $q = 2$. The subgroup $\Phi(Q)P$ is nilpotent, thus $P \leq C_G(\Phi(Q))$. Write C the critical subgroup of Q (see [G], p. 185). Arguing as before, either $C \leq \Phi(Q)$, or $\Phi(Q) < C$ and $C = Q$. But C is not contained in $\Phi(Q)$, otherwise P acts trivially on Q, then $C = Q$ and Q has class at most 2. Now, if $|\Omega_1(Q)| = 2$ then $Q = Q(2^3)$ and $|\Phi(Q)| = 2$. If $Q = \Omega_1(Q)$, then $[a,b]^2 = 1$ for any $a, b \in Q$ of order 2, and $Q / \Omega_1(\Phi(Q))$ is abelian. Then $\Omega_1(\Phi(Q)) = \Phi(Q)$, and again $|\Phi(Q)| = 2$.

Thus $|\Phi(Q)| = q$. Suppose that $|Z(Q)| > q$. Then $Z(Q) = Q$, since $Q \backslash \Phi(Q)$ is a minimal normal subgroup of $G \backslash \Phi(Q)$ - a contradiction. Thus $Z(Q)$ has order q and Q is extraspecial.

From Lemmas 10, 11 it follows that, if $P = E(p^2)$ is not normal in G, then G satisfies either (h) or (i) of Main Theorem 1, and, conversely, if G satisfies (h) or (i), then G is an IC_p-group.

It remains to consider the case $P = Q(2^3)$ not normal in G.

Lemma 12. *If $P = Q(2^3)$ is not normal in G then Q is normal in G.*

Proof. Set $D = O_q(G)$. Assume that $D < Q$. Suppose that $D = 1$. Then $L = O_2(G) < P$ and $C_G(L) \leq L$. So $|L| = 4$, G / L has a subgroup H / L of index 2, and H has a characteristic subgroup Q_0 of index 4. Then $Q_0 = Q$ is normal in G - a contradiction. Thus $D > 1$. Then by the above argument PD is normal in G and P is not normal in PD. Now $PDN_G(P) = G, P \leq PD \cap N_G(P)$ and $PD, N_G(P)$ are not incident - a contradiction.

Lemma 13. *Suppose that $P = Q(2^3)$ is not normal in G. Then one of the following is true:*

(a) $G = (P, Q)$, where Q is homocyclic, and $Q / \Phi(Q)$ is a minimal normal subgroup of $G / \Phi(Q)$.

(b) $G = P[Q$ where Q is cyclic.

(c) $G = P[Q$ where Q is extraspecial, exp $Q = q, G / Z(Q)$ is a Frobenius group and $Q / Z(Q)$ is a minimal normal subgroup of $G / Z(Q)$.

Proof. By Lemma 12 we get $G = P[Q$. Set $<z> = Z(P), T = C_Q(z)$.

Suppose that $T = 1$. Then $G = (P, Q)$ and Q is abelian. Let $R = \Omega_1(Q)$. Then by Maschke's Theorem R is a minimal normal subgroup of G. Let exp $Q = q^n$. Suppose that Q is not homocyclic. Then obviously $n > 1$. Set $M = <x^k | x \in Q>$, $k = p^{n-1}$. Then $R = M \times M_1$ where M_1 is a normal subgroup of G by Maschke's Theorem, and $M < R$ since Q is not homocyclic, a contradiction since R is a minimal normal subgroup of G. Thus Q is homocyclic and (a) holds.

Suppose that $T > 1$. Then $C_G(z) / < z >$ is an IC_2-group with a non-cyclic Sylow 2-subgroup $P / < z >$ of order 4. We have $C_G(z) = PT$. If PT is nilpotent then T is cyclic. Suppose that PT is not nilpotent. Then PT is supersolvable and T is cyclic by Lemma 11. Thus in any case T is cyclic. If $T = Q$, then (b) holds. Assume that $T < Q$. Then Q is noncyclic. If Q is abelian, then by Maschke's Theorem $P\Omega_1(Q)$ is not an IC_2-group - a contradiction. So Q is not abelian. Now Q / Q' is not cyclic. So by the above remark G / Q' is a Frobenius group. Since $p = 2$ then G / Q'' is not a Frobenius group. Set $\bar{G} = G / Q''$. Then $\Omega_1(\bar{Q}')$ is a minimal normal subgroup of $\bar{G}$ by Maschke's Theorem.

In particular $\Omega_1(\bar{Q}') \leq Z(\bar{Q})$. Hence $\Omega_1(\bar{T})$ is normal in $\bar{G}$, and $\Omega_1(\bar{Q}') \leq \bar{T}$. This shows that Q' is cyclic. Then $Q' \leq T$.

Moreover Q is regular since $q > 2$. Hence $H = \Omega_1(Q)$ is of exponent q. If $|T| > q$, then PT and PH are not incident, $P \leq PT \cap PH$, a contradiction. Thus $|T| = q$ and $T = Q'$. Now Q / T is a minimal normal subgroup of G / T, hence $Z(Q) = T$ and Q is extraspecial. Also $H = Q$ and $\exp Q = q$.

The Main Theorem 1 follows from Lemmas 1-13.

3.

Locally solvable IC_p-groups. Let G be a locally solvable torsion group. We say that G is an IC_p-group if for any finite subgroups H, K of G, either H and K are incident, or a Sylow p-subgroup of $H \cap K$ is cyclic.

We prove in this section the following

Main Theorem 2. *Let G be a locally solvable infinite periodic group. Then G is IC_p-group if and only if one of the following holds:*

(a) Every Sylow p-subgroup of G is either cyclic or quasi-cyclic.

(b) $G = Z(p^\infty) \times C(p)$.

(c) $p = 2$ and $G = D(2^\infty)$.

(d) $p = 2$ and $G = Q(2^\infty)$.

(e) $G = P \times Z(q^\infty)$, where $p \neq q$, and P is either quaternion or elementary abelian of order p^2.

(f) $p = 2$ and $G = P[Q$, where $P = Q(2^3), Q$ is the direct product of finitely many copies of $Z(q^\infty), q \neq 2$, and $\Omega_1(Q)$ is the only minimal normal subgroup of G.

(g) $G = (< a >[Q) \times < b >$, where $|< a >| = |< b >| = p$ and Q is as in (f) with $q \neq p$.

We start with the following

Lemma 14. *Let G be a locally finite infinite p-group. Then G is an IC_p-group if and only if one of the following holds:*

(a) $G = Z(p^\infty)$.

(b) $G = Q(2^\infty)$.

(c) $G = Z(p^\infty) \times C(p)$.

(d) $G = D(2^\infty)$.

Proof. If G has only one subgroup of order p, then G is either quaternion or locally cyclic and one of (a) or (b) holds. Then we can assume that there exists $H \leq G$, with H elementary abelian of order p^2.

If M is a finite subgroup of G, then $d(M) \leq 3$ by Lemma 1, where $d(M)$ is the minimal number of generators of M. Hence G has finite rank and there exists a normal subgroup A in G with G/A finite and $A = A_1 \times \ldots \times A_n, A_i \cong Z(p^\infty)$, $n \leq 3$ (see for example [R1], Part 2, Corollary 2, p. 38). Thus $n = 1$, otherwise with $a \in A_1, b \in A_2$, $|<a>| = |<b>| = p^2$, we get that $<a> \times <b>$ is an IC_p-group, a contradiction.

Furthermore $G = AH$, otherwise for any $x \in G - AH$, $a \in A$ with $|<a>| > |<x,H>|$, we have $H \leq <x,H> \cap <a,H>$, and $<a,H>$ and $<x,H>$ non-incident.

Finally $A \cap H \neq 1$, otherwise, with $b \in H$, $b \neq 1$, $a \in A$, $|<a>| = p^2$, we get $<a,b> \cap <a^p,H> \geq <a^p,b> = <a^p> \times <b>$, with $<a,b>$ and $<a^p,H>$ non-incident.

Thus $G = <c>[A$, with $A \cong Z(p^\infty)$, $|<c>| = p$, and the result follows (see for example [R1], Part 1, Lemma 3.28, p. 83).

Proof of Main Theorem 2. If G is a p-group, then the result follows from Lemma 14. So assume that G has an element x of order q^α, where $q \neq p$ is a prime. We can also assume that $P \in Syl_p(G)$ is neither cyclic nor $Z(p^\infty)$. Then a Sylow p-subgroup of every non-primary finitely generated subgroup of G is cyclic, or quaternion, or of order p^3 and exponent p (Lemma 3).

If P is infinite, then $P = Z(p^\infty)<b>$, by Lemma 14, and if $a \in Z(p^\infty)$, $|<a>| = p^3$, then $<a,b,x>$ is a finite IC_p-group, a contradiction, since $|<a,b>| \geq p^4$ and $<a,b>$ is not cyclic (Lemma 3). Thus P is finite, and or $P = Q(2^3)$, or $|P| = p^3$ and $\exp P = p$, or P is elementary abelian of order p^2.

If P is normal in G, then it is easy to see that G/P is a locally cyclic q-group. Then G/P has not proper non-trivial subgroups of finite index, and $G = PC_G(P)$ since $G/C_G(P)$ is finite. Hence $G = P \times Z(q^\infty)$, and it is easy to see that (e) holds. Therefore we can assume that P is not normal in G.

Let $a \in G - N_G(P)$. Then $H = <a,P,x>$ is a finite solvable IC_p-group and P is not normal in H. Thus by Main Theorem 1 either $P = Q(2^3)$ or P is elementary abelian of order p^2. Moreover by the same Theorem $|\pi(F)| \leq 2$ for any finite subgroup of G, so that G is a π-group where $\pi = \pi(G) = \{p,q\}$.

By [Z], there exists an infinite abelian subgroup A of G normalized by P, and obviously we can assume that A is a q-group. For any $y \in \Omega_1(A) - \{1\}$, we have $|<y,P>| \leq q^{|P|}|P|$, and if y is such that $<y,P>$ has maximal order, we get easily that $z \in <y,P>$ for all $z \in \Omega_1(A)$. Hence $d(\Omega_1(A)) \leq |P|$ and (see for example [R2], p. 107) $A = A_1 \times \ldots \times A_n$, where A_i is either cyclic or quasi-cyclic. If A_i is cyclic, for any i, then A is finite, a contradiction. Thus we may assume $A_1 \cong Z(p^\infty)$. Let $g \in G$, and write $d = |<g,P>|$. If $a \in A_1$, $|<a>| > d$, we have $P \leq <a,P> \cap <g,P>$, and $<a,P>$ is not contained in $<g,P>$. Hence $<g,P> < <a,P>$, and $G = AP = P[A$. Now, using Main Theorem 1, it is easy to see that either (f) holds (if $P = Q(2^3)$) or (g) holds (if $P = <a> \times <b>$, $|<a>| = |<b>| = p$).

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