LIPSCHITZ STABILITY OF IMPULSIVE SYSTEMS OF DIFFERENTIAL-DIFFERENCE EQUATIONS

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Summary. *In the present paper an initial value problem for impulsive differential-difference equations is considered. The impulses take place at fixed moments. Sufficient conditions for Lipschitz stability of the zero solution of such equations are found.*

1. INTRODUCTION

The impulsive differential equations are adequate mathematical models of numerous processes and phenomena studied in biology, physics, chemistry, technology, medicine, etc. This is the main reason due to which in the recent years the theory of these equations has been developing very intensively (Bainov and Simeonov, 1989; Kulev and Bainov, 1991; Lakshmikantham, Bainov and Simeonov, 1989; Samoilenko and Perestyuk, 1987; Simeonov and Bainov, 1987). A natural generalization of the impulsive differential equations are the impulsive differential-difference equations. The processes whose mathematical models are the impulsive differential-difference equations, besides the change by jumps of their state, are characterized by a dependence of the process on its pre-history at each moment of time. In spite of the great possibilities for application, the theory of these equations is developing rather slowly due to a number of technical and theoretical difficulties (Bainov, Covachev and Stamova, 1994).

In the present paper Lipschitz stability of the solutions of systems of impulsive differential-difference equations is defined. For non-linear systems of differential equations without impulses this notion was introduced by Dannan and Elaydi (1986), and for impulsive systems of differential equations without delay by Kulev and Bainov (1991).

By means of a comparison equation and differential inequalities for piecewise continuous functions sufficient conditions for Lipschitz stability of the zero solution of an impulsive system of differential-difference equations are found.

2. PRELIMINARY NOTIONS AND DEFINITIONS

Let R^n be the n-dimensional Euclidean space with elements $x = col(x_1, \dots, x_n)$ and norm $|\cdot|$; $h > 0$; $\varphi_0 : [t_0 - h, t_0] \to R^n, t_0 \in R$.

Consider an initial value problem for the impulsive system of differential-difference equations

$$\begin{aligned} &\dot{x}(t) = f(t, x(t), x(t-h)), t \neq \tau_k, t > t_0, \\ &x(t) = \varphi_0(t), t \in [t_0 - h, t_0], \\ &\Delta x(\tau_k) = x(\tau_k + 0) - x(\tau_k - 0) = I_k(x(\tau_k)), k = 1, 2, \dots, \tau_k > t_0, \end{aligned} \tag{1}$$

where $f : (t_0, \infty) \times R^n \times R \to R^n, I_k : R^n \to R^n, k = 1, 2, \dots, t_0 \equiv \tau_0 < \tau_1 < \dots < \tau_k < \tau_{k+1} < \dots$.

Denote by $x(t) = x(t, t_0, y_0)$ the solution of problem (1) $I^+(t_0, \varphi_0)$ is the maximal interval of type $[t_0, \beta)$ in which the solution $x(t, t_0, \varphi_0)$ is defined; $\tau_\ell^h = \tau_\ell + h, \ell = 0, 1, 2, \ldots, \tilde{x}(t) = x(t-h), t > t_0$.

Let $\varphi_0 \in C[[t_0 - h, t_0], R^n]$.

Then the solution $x(t) = x(t, t_0, \varphi_0)$ of problem (1) is characterized in the following way:

1. For $t_0 - h \leq t \leq t_0$ the solution $x(t)$ coincides with the function φ_0.

2. Construct the sequence $\{t_i\}_{i=1}^{\infty}$ observing the following rules:

a) $\{t_i\}_{i=1}^{\infty} = \{\tau_k\}_{k=1}^{\infty} \cup \{\tau_\ell^h\}_{\ell=0}^{\infty}$.

b) The sequence $\{t_i\}_{i=1}^{\infty}$ is monotone increasing.

c) In general it is possible that $\{\tau_k\}_{k=1}^{\infty} \cap \{\tau_\ell^h\}_{\ell=0}^{\infty} \neq \phi$.

2.1. For $t_0 < t \leq t_1$ the solution of problem (1) coincides with the solution of the problem without impulses

$$\begin{cases} \dot{x}(t) = f(t, x(t), x(t-h)), t > t_0, \\ x(t) = \varphi_0(t), t \in [t_0 - h, t_0]. \end{cases}$$

2.2. For $t_i < t \leq t_{i+1}, i = 1, 2, \ldots$ one of the following three cases may occur:

a) If $t_i \in \{\tau_k\}_{k=1}^{\infty} \backslash \{\tau_\ell^h\}_{\ell=0}^{\infty}$ and $t_i = \tau_k$, then $x(t)$ coincides with the solution of the problem

$$\dot{y}(t) = f(t, y(t), x(t-h)) \tag{2}$$

$$y(\tau_k) = x(\tau_k) + I_k(x(\tau_k)). \tag{3}$$

b) If $t_i \in \{\tau_\ell^h\}_{\ell=0}^{\infty} \backslash \{\tau_k\}_{k=1}^{\infty}$, then the solution $x(t)$ coincides with the solution of the problem

$$y(t) = f(t, y(t), x(t-h+0)) \tag{4}$$

$$y(t_i) = x(t_i). \tag{5}$$

c) If $t_i \in \{\tau_k\}_{k=1}^{\infty} \cap \{\tau_\ell^h\}_{\ell=0}^{\infty}$ and $t_i = \tau_k$, then the solution $x(t)$ of problem (1) coincides with the solution of problem (4), (3).

3. The function $x(t)$ is piecewise continuous in $I^+(t_0, \varphi_0)$, continuous from the left at the points $\tau_1, \tau_2, \ldots \in I^+(t_0, \varphi_0)$ and $x(\tau_k + 0) = x(\tau_k) + I_k(x(\tau_k)), k = 1, 2, \ldots$.

Together with problem (1) we shall consider the problem

$$\begin{cases} \dot{u} = g(t, u), t \neq \tau_k, t > t_0, \\ u(t_0 + 0) = u_0, \\ u(\tau_k + 0) = G_k(u(_k)) \end{cases} \tag{6}$$

where: $g : (t_0, \infty) \times R_+ \to R_+, R_+ = [0, \infty), u_0 \in R_+, G_k : R_+ \to R_+, k = 1, 2, \ldots$

Introduce the following notation:

By $u(t) = u(t, t_0, u_0)$ denote the solution of problem (6); $I^+(t_0, u_0)$ is the maximal interval of type $[t_0, \omega)$ in which the solution $u(t) = u(t, t_0, u_0)$ is defined; $\mathcal{K}$ is the class of all continuous and strictly increasing functions $a : R_+ \to R_+$ such that $a(0) = 0$; $C_0 = C[[t_0 - h, t_0], R^n]$; $\|\varphi\| = \max_{S \in [t_0 - h, t_0]} |\varphi(S)|$ is the norm of the function $\varphi \in C_0$; $S(\rho) = \{x \in R^n : |x| < \rho\}$ $\rho > 0$; $\|A\|_1 = \sup_{|x|=1} |Ax|$ is the norm of the $(n \times n)$-matrix A.

We shall introduce definitions of two types of Lipschitz stability of the zero solution of problem (1) which are analogous to those introduced by Kulev and Bainov (1991) for impulsive systems of differential equations without delay.

Definition 1. *The zero solution of problem (1) is said to be:*
a) uniformly Lipschitz stable if

$$(\exists M > 0)(\exists \delta > 0)(\forall \varphi_0 \in C_0 : \|\varphi_0\| < \delta)$$

$$(\forall t > t_0) : |x(t; t_0, \varphi_0)| \leq M\|\varphi_0\|.$$

b) globally uniformly Lipschitz stable if

$$(\exists M > 0)(\forall \varphi_0 \in C_0)(\forall t > t_0) :$$

$$|x(t, t_0, \varphi_0)| \leq M\|\varphi_0\|.$$

Definition 2. *The solution $u^+ : I^+(t_0, u_0) \to R_+$ of problem (6) is said to be a maximal solution if any other solution of (6) $u : [t_0, \tilde{\omega}) \to R_+$ satisfies the inequality $u^+(t) \geq u(t)$ for $t \in I^+(t_0, u_0) \cap [t_0, \tilde{\omega})$.*

Definition 3. *(Kulev and Bainov, 1991). The zero solution of problem (6) is said to be:*
a) uniformly Lipschitz stable if

$$(\exists M > 0)(\exists \delta > 0)(\forall u_0 \in R_+ : u_0 < \delta)$$

$$(\forall t > t_0) : u^+(t; t_0, u_0) \leq M u_0.$$

b) globally uniformly Lipschitz stable if

$$(\exists M > 0)(\forall u_0 \in R_+)(\forall t > t_0) : u^+(t; t_0, u_0) \leq M u_0.$$

Introduce the following conditions:
H1. $t_0 \equiv \tau_0 < \tau_1 < \ldots < \tau_k < \tau_{k+1} < \ldots$
H2. $\lim_{k\to\infty} \tau_k = \infty$.
H3. The function $f(t, x, \tilde{x})$ is continuous in $(\tau_k, \tau_{k+1}] \times R^n \times R^n, k = 0, 1, 2, \ldots.$
H4. $f(t, 0, 0) = 0$ for $t \in (t_0, \infty)$.
H5. For any $(x, \tilde{x}) \in R^n \times R^n$ and any $k = 1, 2, \ldots$ there exists the limit

$$\lim_{\substack{(t, y, \tilde{y}) \to (t, x, \tilde{x}) \\ t > \tau_k}} f(t, y, \tilde{y}).$$

H6. The functions $I_k(x), k = 1, 2, \ldots$ are continuous in R^n.
H7. $I_k(0) = 0, k = 1, 2, \ldots.$
H8. $I^+(t_0, \varphi_0) = [t_0, \infty)$.
H9. The function $g(t, u)$ is continuous in $(\tau_k, \tau_{k+1}] \times R_+, k = 0, 1, 2, \ldots.$

H10. $g(t,0)=0, t\in(t_0,\infty)$.
H11. $G_k\in\mathcal{K}, \| = 1,2,\ldots$.
H12. For any $u\in R_+$ and any $k=1,2,\ldots$ there exists the limit

$$\lim_{\substack{(t,\nu)\to(t,u)\\ t>\tau_k}} g(t,\nu)$$

H13. $I^+(t_0,u_0)=[t_0,\infty)$.
In the proof of the main results we shall use the following lemmas:

Lemma 1. *(Bainov and Simeonov, 1989). Let the following conditions hold:*
1. Conditions H9, H11 and H13 are met.
2. The function $u^+ : I^+(t_0,u_0)\to R_+$ is the maximal solution of problem (6) and $u^+(\tau_k+0)\in R_+$ if $\tau_k\in I^+(t_0,u_0)$.
3. The function $m : I^+(t_0,u_0)\to R_+$ is piecewise continuous with points of discontinuity of the first kind $\tau_k, \tau_k\in I^+(t_0,u_0)$ at which it is continuous from the left and such that

$$m(\tau_k+0)\in R_+, \tau_k\in I^+(t_0,u_0),$$

$$m(t_0+0)\le u_0,$$

$$Dm(t)\le g(t,m(t)), t\in I^+(t_0,u_0)\backslash\{\tau_k\},$$

where $Dm(t)$ is an arbitrary Dini derivative of $m(t)$,

$$m(\tau_k+0)\le G_k(m(\tau_k)), \tau_k\in I^+(t_0,u_0).$$

Then for $t\in I^+(t_0,u_0)$ the following inequality is valid

$$m(t)\le u^+(t).$$

Lemma 2. *(Bainov and Simeonov, 1991). Let the following conditions hold:*
1. The functions $u, k : (t_0,\infty)\to R_+$ are piecewise continuous with points of discontinuity of the first kind τ_k at which they are continuous from the left.
2. $c_0 = const > 0, \beta_k = const \ge 0, k=1,2,\ldots$.
3. The function $\rho : R_+\to R_+$ is continuous, nondecreasing in R_+ and positive in $(0,\infty)$.
4. $G(u)=\int_{u_0}^{u}\frac{ds}{\rho(s)}, u\ge u_0>0$.
5. For $t\ge t_0$ the following inequality is valid

$$u(t)\le c_0+\int_{t_0}^{t}k(s)\rho(u(s))ds+\sum_{t_0<\tau_k<t}\beta_k u(\tau_k).$$

Then

$$u(t)\le G^{-1}\left[G\left(\prod_{t_0<\tau_k<t}(1+\beta_k)\right)+\int_{t_0}^{t}\left(\prod_{s<\tau_k<t}\frac{1+\beta_k}{\rho(1+\beta_k)}\right)k(s)ds\right], t\in[t_0,\infty).$$

Lemma 3. *(Bainov and Simeonov, 1992). Let the following conditions hold:*

1. Conditions 1-3 and condition 5 of Lemma 2 are met.

2. $G_k = \int_{c_k}^{u} \frac{ds}{\rho(s)}, k = 1,2,\ldots,$ *where* $c_k = (1+\beta_k)G_{k-1}^{-1}\left(\int_{\tau_{k-1}}^{\tau_k} k(s)ds\right)$.

3. $G_0 = \int_{u_0}^{u} \frac{ds}{\rho(s)}, u \geq u_0 > 0.$

Then

$$u(t) \leq G_k^{-1}(\int_{\tau_k}^{t} k(s)ds), \tau_k < t < \tau_{k+1}, k = 1,2,\ldots$$

3. MAIN RESULTS

Introduce the following conditions:

H14. $G_k : [0, \rho_o) \to [0, \rho), k = 1,2,\ldots$.

H15. For $x \in S(\rho)$ and any $k = 1,2,\ldots$ the following inequality is valid

$$|x + I_k(x)| \leq G_k(|x|).$$

H16. The zero solution of problem (6) is uniformly Lipschitz stable (globally uniformly Lipschitz stable).

Theorem 1. *Let the following conditions hold:*

1. Conditions H1-H16 are met.

2. For $(t, x, \tilde{x}) \in (t_0, \infty) \times S(\rho) \times S(\rho)$ *and for sufficiently small* $\sigma > 0$ *the inequality*

$$|x + \sigma f(t, x, \tilde{x})| \leq |x| + \sigma g(t, |x|) + \varepsilon(\sigma).$$

is valid, where $\frac{\varepsilon(\sigma)}{\sigma} \to 0$ *as* $\sigma \to 0$.

Then the zero solution of problem (1) is uniformly Lipschitz stable (globally uniformly Lipschitz stable).

Proof. Let $\rho^* = \min(\rho, \rho_0)$. From condition H16 it follows that there exist constants $M > 0$ and $\delta > 0 (M\delta < \rho^*)$ such that for $0 \leq u_0 < \delta$ and $t > t_0$ we have

$$u^+(t; t_0, u_0) \leq M u_0. \tag{7}$$

We shall prove that $|x(t, t_0, \varphi_0)| \leq M\|\varphi_0\|$ for $\|\varphi_0\| < \delta$ and $t > t_0$.

Suppose that this is not true. Then there exists a solution $x(t) = x(t, t_0, \varphi_0)$ of problem (1), $\|\varphi_0\| < \delta$ and $t^* \in (\tau_k, \tau_{k+1}]$ for some positive integer k such that

$$|x(t^*)| > M\|\varphi_0\| \text{ and } |x(t)| \leq M\|\varphi_0\| \text{ for } t_0 < t \leq \tau_k.$$

From condition H15 it follows that

$$|x(\tau_k + 0)| = |x(\tau_k) + I_k(x(\tau_k))| \leq G_k(|x(\tau_k)|) \leq$$

$$\leq G_k(M\|\varphi_0\|) < G_k(M\delta) \leq G_k(\rho^*) \leq \rho.$$

From the above estimate it follows that there exists $t^0, \tau_k < t^0 \leq t^*$, such that

$$M\|\varphi_0\| < |x(t^0)|\rho \quad \text{and} \quad |x(t)| < \rho, t_0 < t \leq t^0. \tag{8}$$

Introduce the notation $m(t) = |x(t)|$ and $u_0 = \|\varphi_0\|$. Since condition 2 of Theorem 1 is satisfied, then for $t \in (t_0, t^0], t \neq \tau_j, j = 1, 2, \ldots, k$, the following inequalities are valid

$$m'(t) = \lim_{\sigma\to 0} \frac{1}{\sigma}[|x(t+\sigma)| - |x(t)|] \leq$$

$$\leq \lim_{\sigma\to 0}[|x(t+\sigma)| + \sigma g(t, |x(t)|) + \varepsilon(\sigma) -$$

$$-|x(t) + \sigma f(t, x(t), x(t-h))|] \leq$$

$$\leq g(t, |x(t)|) + \lim_{\sigma\to 0} \frac{\varepsilon(\sigma)}{\sigma} +$$

$$+ \lim_{\sigma\to 0} |\frac{1}{\sigma}[x(t+\sigma) - x(t)] - f(t, x(t), x(t-h))| =$$

$$= g(t, |x(t)|) = g(t, m(t)).$$

From condition H15 for $j = 1, 2, \ldots, k$ we derive the inequalities

$$m(\tau_j + 0) = |x(\tau_j + 0)| = |x(\tau_j) + I_j(x(\tau_j))| \leq$$

$$\leq G_j(|x(\tau_j)|)$$

which imply the inequalities

$$m(\tau_j + 0) \leq G_j(m(\tau_j)), j = 1, 2, \ldots, k.$$

Since

$$m(t_0 + 0) = |x(t_0 + 0)| = |x(t_0)| \leq \|\varphi_0\| = u_0,$$

then from Lemma 1 there follows the estimate

$$|x(t)| = m(t) \leq u^+(t; t_0, u_0), t_0 < t \leq t^0. \tag{9}$$

From (7), (8) and (9) we are led to the inequalities

$$M\|\varphi_0\| < |x(t^0)| = m(t^0) \leq u^+(t^0; t_0, u_0) \leq Mu_0 = M\|\varphi_0\|.$$

The contradiction abtained shows that

$$|x(t; t_0, \varphi_0)| \leq M\|\varphi_0\| \text{ for } \|\varphi_0\| < \delta \text{ and } t > t_0$$

Theorem 2. *Let the following conditions hold:*

1. Conditions H1-H16 hold.
2. For $(t,x,\tilde{x}) \in (t_0,\infty) \times S(\rho)S(\rho)$ the inequality

$$[x,f(t,x\tilde{x}]_+ \leq g(t,|x|)$$

is valid, where

$$[x,y]_+ = \lim_{\sigma\to 0+} \sup \frac{1}{\sigma}[|x+\sigma y| - |x|], x,y \in R^n.$$

Then the zero solution of problem (1) is uniformly Lipschitz stable (globally uniformly Lipschitz stable).

Proof. Let $\rho^* = \min(\rho,\rho_0)$. From condition H16 it follows that there exist constants $M > 0$ and $\delta > 0$ ($M\delta < \rho^*$) such that for $0 \leq u_0 < \delta$ and $t > t_0$ the following inequality is valid

$$u^+(t,t_0,u_0) \leq Mu_0.$$

We shall prove that $|(t,t_0,\varphi_0)| \leq M\|\varphi_0\|$ for $\|\varphi_0\| < \delta$ and $t > t_0$.

Suppose that this is not true. Then, as in the proof of Theorem 1 we find a solution $x(t) = x(t,t_0,\varphi_0)$, $\|\varphi_0\| < \delta$ of problem (1) and $t^0 \in (\tau_k,\tau_{k+1}]$, $k \in N$ such that $M\|\varphi_0\| < |x(t^0)| < \rho$ and $|x(t)| < \rho$, $t_0 < t \leq t^0$.

Set $m(t) = |x(t)|$ and $u_0 = \|\varphi_0\|$. Use condition 2 of Theorem 2 for $t \in (t_0,t^0]$, $t \neq \tau_j$, $j = 1,2,\dots,k$ and obtain the inequalities

$$D^+m(t) = \lim_{\sigma\to 0+} \sup \frac{1}{\sigma}[m(t+\sigma) - m(t)] =$$

$$= \lim_{\sigma\to 0+} \sup \frac{1}{\sigma}[|x(t+\sigma)| - |x(t)|] \leq$$

$$\leq \lim_{\sigma\to 0+} \sup \frac{1}{\sigma}[|x(t+\sigma) - x(t)] - f(t,x(t),x(t-h))|+$$

$$+ \lim_{\sigma\to 0+} \sup \frac{1}{\sigma}[|x(t) + \sigma f(t,x(t),x(t-h))| - |x(t)|] =$$

$$= [x(t),f(t,x(t),x(t-h))]_+ \leq g(t,|x(t)|) = g(t,m(t)).$$

Further on the proof of Theorem 2 is completed as the proof of Theorem 1.

Introduce the following conditions:

H17. The function $\rho : R_+ \to R_+$ is continuous; nondecreasing in R_+; positive in $(0,\infty)$ and submultiplicative, i.e.

$$\rho(\lambda u) \leq \rho(\lambda)\rho(u) \quad \text{for} \quad \lambda > 0, u > 0.$$

H18. For $(t,x\tilde{x}) \in (t_0,\infty) \times R^n \times R^n$ the inequality

$$|f(t,x,\tilde{x})| \leq m(t)\rho(|x|)$$

is valid, where $m(t)$ is continuous and nonnegative in (t_0,∞).

H19. For $x \in R^n$ and any $k = 1, 2, \ldots$ the inequalities

$$|I_k(x)| \leq \beta_k |x|$$

are valid, where $\beta_k = \text{const} \geq 0, k = 1, 2, \ldots$

Theorem 3. *Let the following conditions hold:*

1. *Conditions H1-H8, H17-H19 are met*
2. $\rho(\lambda u) \geq \mu(\lambda)\rho(u)$ *for* $\lambda > 0, u > 0$, *where* $\mu(\lambda) > 0$ *for* $\lambda > 0$.
3. $G(u) = \int_a^u \frac{ds}{\rho(s)}, u \geq a > 0$.
4. $G(\infty) = \infty$.
5. $G^{-1}\left[G\left(\prod_{\sigma < \tau_k < \infty}(1 + \beta_k)\right) + \frac{\rho(\|\varphi_0\|)}{\|\varphi_0\|}\int_\sigma^\infty \frac{1+\beta_k}{\mu(1+\beta_k)} m(s)ds\right] < \infty$

for any $\varphi_0 \in C_0$ *and any* $\sigma \geq t_0$.

Then the zero solution of problem (1) is globally uniformly Lipschitz stable.

Proof. Since in the interval $(\tau_k, \tau_{k+1}], k = 1, 2, \ldots, x(t) = x(t, t_0, \varphi_0)$ coincides with the solution of problem (1), (3), we conclude that for $\tau_k < t \leq \tau_{k+1}$ the function $x(t, t_0, \varphi_0)$ satisfies the integral equation

$$x(t, t_0, \varphi_0) = x(\tau_k) + I_k(x(\tau_k)) + \int_{\tau_k}^t f(s, x(s), x(s-h))ds.$$

From this we obtain inductively

$$x(t, t_0, \varphi_0) = x(t_0) + \sum_{t_0 < \tau_k < t} I_k(x(\tau_k)) + \int_{t_0}^t f(s, x(s), x(s-h))ds.$$

From conditions H17-H19 and condition 2 of Theorem 3 and the above equality we get to the inequalities

$$\begin{aligned} |x(t, t_0, \varphi_0)| &\leq |x(t_0)| + \sum_{t_0 < \tau_k < t} |I_k(x(\tau_k))| + \\ &+ \int_{t_0}^t |f(s, x(s), x(s-h))|ds \leq \\ &\leq \|\varphi_0\| + \sum_{t_0 < \tau_k < t} \beta_k |x(\tau_k, t_0, \varphi_0)| + \\ &+ \int_{t_0}^t m(s)\rho(|x(s, t_0, \varphi_0)|)ds, \end{aligned}$$

from which we obtain the estimates

$$\frac{|x(t, t_0, \varphi_0)|}{\|\varphi_0\|} \leq 1 + \sum_{t_0 < \tau_k < t} \beta_k \frac{|x(\tau_k, t_0, \varphi_0)|}{\|\varphi_0\|} +$$

$$+ \int_{t_0}^t \frac{m(s)}{\|\varphi_0\|}\rho\left(\|\varphi_0\|\frac{|x(s, t_0, \varphi_0)|}{\|\varphi_0|}\right) ds \leq 1 + \sum_{t_0 < \tau_k < t} \beta_k \frac{|x(\tau_k, t_0, \varphi_0)|}{\|\varphi_0\|} +$$

$$+\int_{t_0}^{t}\frac{\rho(\|\varphi_0\|)}{\|\varphi_0\|}m(s)\left(\frac{|x(s,t_0,\varphi_0)|}{\|\varphi_0\|}\right)ds.$$

To the last inequality we apply Lemma 2 for $u(t)=\frac{|x(t)|}{\|\varphi_0\|}$, $k(s)=\frac{\rho(\|\varphi_0\|)}{\|\varphi_0\|}m(s)$ and are led to the inequality

$$|x(t,t_0,\varphi_0)|\le\|\varphi_0\|G^{-1}[G\left(\prod_{t_0<\tau_k<t}(1+\beta_k)\right)+$$

$$+\frac{\rho(\|\varphi_0\|)}{\|\varphi_0\|}\int_{t_0}^{t}\left(\prod_{s<\tau_k<t}\frac{1+\beta_k}{\mu(1+\beta_k)}\right)m(s)ds].$$

From condition 5 of Theorem 3 it follows that

$$|x(t,t_0,\varphi_0)|\le M\|\varphi_0\|$$

for any $\varphi_0\in C_0$ and $t>t_0$.

Theorem 4. *Let the following conditions hold:*

1. Conditions H1-H8, H17-H19 are met.

2. $G_k=\int_{C_k}^{u}\frac{ds}{\rho(s)}, C_k=(1+\beta_k)G_{k-1}^{-1}\left(\int_{\tau_{k-1}}^{\tau_k}m(s)ds\right), k=1,2,\ldots,$ *and* $G_0=\int_{C}^{u}\frac{ds}{\rho(s)}, u\ge c>0.$

3. $G_k(\infty)=\infty, k=0,1,2,\ldots$

4. For any $k=1,2,\ldots$, $t\in(\tau_k,\tau_{k+1}]$ *and any* $\varphi_0\in C_0$ *the following condition is valid*

$$G_k^{-1}\left(\frac{\rho(\|\varphi_0\|)}{\|\varphi_0\|}\int_{\tau_k}^{t}m(s)ds\right)\le M,$$

$M=const>0$.

Then the zero solution of problem (1) is globally uniformly Lipschitz stable.

Proof. As in the proof of Theorem 3 we obtain the inequality

$$\frac{|x(t,t_0,\varphi_0)|}{\|\varphi_0\|}\le 1+\sum_{t_0<\tau_k<t}\beta_k\frac{|x(\tau_k,t_0,\varphi_0)|}{\|\varphi_0\|}+$$

$$+\int_{t_0}^{t}\frac{\rho(\|\varphi_0\|)}{\|\varphi_0\|}m(s)\rho\left(\frac{|x(s,t_0,\varphi_0)|}{\|\varphi_0\|}\right)ds, t>t_0,$$

to which we apply Lemma 3 an obtain the inequality

$$|x(t,t_0,\varphi_0)|\le\|\varphi_0\|G_k^{-1}\left(\frac{\rho(\|\varphi_0\|)}{\|\varphi_0\|}\int_{t_0}^{t}m(s)ds\right),$$

$$t\in(\tau_k,\tau_{k+1}], k=1,2,\ldots$$

From the above inequality and condition 4 of Theorem 4 it follows that

$$|x(t,t_0,\varphi_0)|\le M\|\varphi_0\| \text{ for } \varphi_0\in C_0 \text{ and } t>t_0.$$

4. EXAMPLES

4.1. Consider the linear impulsive system of differential-difference equations

$$\begin{aligned}\{\dot{x}(t) &= Ax(t) + Bx(t-h), t > t_0, t \neq \tau_k,\\ x(t) &= \varphi_0(t), t \in [t_0 - h, t_0],\\ \Delta x(\tau_k) &= C_k x(\tau_k), k = 1, 2, \ldots, \tau_k > t_0,\end{aligned} \tag{10}$$

where A, B and $C_k, k = 1, 2, \ldots$ are constant matrices of type $(n \times n)$, $\varphi_0 \in C_0$.

If conditions H1, H2 are fulfilled as well as the conditions:

1) $\|E + C_k\|_1 \leq d_k, k = 1, 2, \ldots, d_k = \text{const} > 0$.
2) $\prod_{k=1}^{\infty} d_k < \infty$.
3) $\mu(A, B) \leq 0$

then the zero solution of the problem

$$\begin{aligned}\dot{u} &= \mu(A, B)u, t > t_0, t \neq \tau_k,\\ u(t_0 + 0) &= u_0, u_0 \in R_+,\\ \Delta u(\tau_k) &= (d_k - 1)u(\tau_k)\end{aligned}$$

is globally uniformly Lipschitz stable (Bainov and Simeonov, 1989).

Then, if $[x(t), Ax(t) + Bx(t-h)]_+ \leq \mu(A, B)|x(t)|$, then by Theorem 2 the zero solution of problem (10) is globally uniformly Lipschitz stable.

4.2. Consider the problem

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)x(t-h), t > t_0, t \neq \tau_k,\\ x(t) &= \varphi_1(t), t \in [t_0 - h, t_0],\\ \Delta x(\tau_k) &= C_k x(\tau_k), k = 1, 2, \ldots, \tau_k > t_0,\end{aligned} \tag{11}$$

where $A, B : (t_0, \infty) \to R^n$ are $(n \times n)$-matrices; $C_k, k = 1, 2, \ldots$ are contants $(n \times n)$-matrices; $\varphi_1 \in C_0$.

If conditions H1, H2 are met as well as the conditions

1) The matrices $A(t)$ and $B(t)$ are piecewise continuous in (t_0, ∞) with points of discontinuity of the first kind $\tau_k, k = 1, 2, \ldots$ at which they are continuous from the left.
2) $\lim_{t \to \infty} \sup(\int_{t_0}^{t} \mu(A(s), B(s))ds) < \infty$.
3) $\|E + C_k\|_1 \leq d_k, k = 1, 2, \ldots$
4) $\bigcap_{k=1}^{\infty} d_k < \infty$,

then the zero solution of the problem

$$\begin{aligned}\dot{u} &= \mu(A(t), B(t))u, t > t_0, t \neq \tau_k,\\ u(t_0 + 0) &= u_0, u_0 \in R_+,\\ \Delta u(\tau_k) &= (d_k - 1)u(\tau_k)\end{aligned}$$

is globally uniformly Lipschitz stable (Bainov and Simeonov, 1989).

Then by Theorem 2 if $[x(t), A(t)x(t) + B(t)x(t-h)]_+ \leq \mu(A,B)|x(t)|$, then the zero solution of problem (11) is globally uniformly Lipschitz stable.

4.3. Consider problem (1). If conditions H1-H16 are met as well as the conditions

1) $[x, f(t,x,\tilde{x})]_+ \leq \rho(t)F(|x|)$ for $(t,x,\tilde{x}) \in (t_0,\infty) \times S(\rho) \times S(\rho)$, $\rho \in C[(t_0,\infty), R_+]$, $F \in \mathcal{K}$.

2) $|x + I_k(x)| \leq G_k(|x|), x \in S(\rho), k = 1,2,\ldots$, where $G_k : [0,\rho_0) \to [0,\rho)$ and $G_k \in \mathcal{K}$, $k = 1,2,\ldots$

3) For any $\sigma \in (0,\rho_0)$ we have

$$\int_{\tau_k}^{\tau_{k-1}} \rho(s)ds + \int_{\sigma}^{G_k(\sigma)} \frac{ds}{F(s)} \leq 0, k = 1,2,\ldots,$$

then the zero solution of the problem

$$\begin{aligned}
&\dot{u} = \rho(t)F(u), t > t_0, t \neq \tau_k,\\
&u(t_0+0) = u_0, u_0 \in R_+,\\
&\Delta u(\tau_k) = G_k(u(\tau_k)) - u(\tau_k)
\end{aligned}$$

is uniformly Lipschitz stable (Bainov and Simeonov, 1989).

From Theorem 2 it follows that the zero solution of problem (1) is uniformly Lipschitz stable.

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