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ON GLOBAL STABILITY OF SETS FOR LINEAR IMPULSIVE DIFFERENTIAL-DIFFERENCE EQUATIONS

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Abstract. *Some results for stability of sets of general type for linear impulsive differential-difference equations with variable impulsive perturbations are obtained. The main results are proved with the aid of piecewise continuous functions, which are analogues of the Lyapunov functions.*

1. INTRODUCTION

The impulsive differential-difference equations are adequate mathematical models of numerous real processes and phenomena studied at physics, biology, population dynamics, bio-technologies, control theory, industrial robotics, etc.

In spite of the great possibilities of applications, the theory of these equations is developing rather slowly.

The difficulties arising when one studies impulsive differential-difference equations with variable impulsive perturbations due to the presence of phenomena as "beating" of the solutions, bifurcation, loss of property of autonomy, etc. On the other hand, the presence of delay in the argument needs to introduce new methods, as well as to modify the standard methods in the investigation of these equations.

At the present work sufficient conditions are found for global stability of set with respect to linear system of impulsive differential-difference equations with variable impulsive perturbations. The main results were carried out with the aid of piecewise continuous auxiliary functions [2].

2. PRELIMINARY NOTES AND DEFINITIONS

Let $\mathbb{R}_+ = [0,\infty)$; $\mathbb{R}^n$ be the n-dimensional Euclidean space with elements $x = \mathrm{col}\,(x_1,\ldots,x_n)$, the norm $|\cdot|$ and the distance $d(\cdot,\cdot)$.

Let $h > 0$ and $t_0 \in \mathbb{R}$.

We consider the linear system of impulsive differential-difference equations

$$\dot{x}(t) = \begin{cases} A(t)\,x(t) + B(t)x(t-h), \; x(t) > 0, t \neq \tau_k(x(t)), t > t_0, \\ 0, \; x(t) \leq 0, t \neq \tau_k(x(t)), t > t_0, \end{cases}$$

$$\Delta x(t)|_{t=\tau_k(x(t))} = \begin{cases} C_k x(t), \; x(t) > 0, t > t_0, \\ 0, \; x(t) < 0, t > t_0, \end{cases} \tag{1}$$

where $x : (t_0, \infty) \to \mathbb{R}^n, A(t)$ and $B(t)$ are $n \times n$-matrix valued functions; C_k, $k = 1, 2, \ldots$, are $n \times n$-matrices; $\tau_k : \mathbb{R}^n \to (t_0, \infty)$ and $\Delta x(t) = x(t+0) - x(t-0)$.

Let $\tau_0(x) \equiv t_0$ for $x \in \mathbb{R}^n$.

We introduce the following conditions:

H1. $\tau_k \in C[\mathbb{R}^n, (t_0, \infty)], k = 1, 2, \ldots$

H2. $t_0 < \tau_1(x) < \tau_2(x) < \ldots, x \in \mathbb{R}^n$.

H3. $\tau_k(x) \to \infty$ as $k \to \infty$ uniformly with respect to $x \in \mathbb{R}^n$.

Under the assumptions that conditions H1, H2, and H3 are fulfilled, we introduce the following notations:

$$G_k = \{(t, x) \in [t_0, \infty) \times \mathbb{R}^n : \tau_{k-1}(x) < t < \tau_k(x)\}, \quad k = 1, 2, \ldots,$$

$$\sigma_k = \{(t, x) \in [t_0, \infty) \times \mathbb{R}^n : t = \tau_k(x)\},$$

i.e., $\sigma_k, k = 1, 2, \ldots$ are hypersurfaces with the equations $t = \tau_k(x(t))$.

Let $\varphi_0 \in C[[t_0 - h, t_0], \mathbb{R}^n] \equiv C_0$.

We denote by $x(t) = x(t;\, t_0, \varphi_0)$ the solution of the system (1) which satisfies the initial condition

$$x(t) = \varphi_0(t), \quad t \in [t_0 - h, t_0], \tag{2}$$

and let $J^+(t_0, \varphi_0)$ denotes the maximal interval of the type (t_0, β), at which the solution $x(t, t_0, \varphi_0)$ is defined; $\|\varphi_0\| = \max_{t \in [t_0 - h, t_0]} |\varphi_0(t)|$ is the norm of the function $\varphi_0 \in C_0$.

We shall describe the solution $x(t) = x(t;\, t_0, \varphi_0)$ of the initial problem (1), (2):

1. For $t_0 - h \le t \le t_0$ the solution $x(t)$ coincides with the initial function $\varphi_0 \in C_0$.

2. The function $x(t)$ is piecewise continuous on $J^+(t_0, \varphi_0)$, it is continuous from the left and $x(t+0) = x(t) + C_k x(t)$ for $t \in J^+(t_0, \varphi_0)$, $t \neq \tau_k(x(t))$, $t \neq \beta$, $k = 1, 2, \ldots$.

3. For $t \in J^+(t_0, \varphi_0), t \neq \tau_k(x(t)), k = 1, 2, \ldots$, the function $x(t)$ is differentiable and

$$\dot{x}(t) = A(t)x(t) + B(t)x(t-h).$$

Let $M \subset [t_0 - h, \infty) \times \mathbb{R}^n$.

We introduce the notations:

$$M(t) = \{x \in \mathbb{R}^n : (t, x) \in M, t \in (t_0, \infty)\},$$

$$M_0(t) = \{x \in \mathbb{R}^n : (t, x) \in M, t \in [t_0 - h, t_0]\},$$

$d(x, M(t)) = \inf_{y \in M(t)} |x - y|$ is the distance between $x \in \mathbb{R}^n$ and $M(t)$;

$M(t, \varepsilon) = \{x \in \mathbb{R}^n : d(x, M(t)) < \varepsilon\}$ is an ε-neighbourhood of $M(t)$;

$d_0(\varphi, M_0(t)) = \max_{t \in [t_0 - h, t_0]} d(\varphi(t), M_0(t))$ is the distance between $\varphi \in C_0$ and $M_0(t)$;

$M_0(t, \varepsilon) = \{\varphi \in C_0 : d_0(\varphi, M_0(t)) < \varepsilon\}$ is an ε-neighbourhood of $M_0(t)$;

$\overline{S}_\alpha = \{x \in \mathbb{R}^n : |x| \le \alpha\}, (\alpha > 0)$;

$\overline{S}_\alpha(C_0) = \{\varphi \in C_0 : \|\varphi\| \le \alpha\}$.

We introduce the following assumptions:

H4. The matrix-valued functions $A(t)$ and $B(t)$ are continuous for $t \in (t_0, \infty)$.

H5. The elements of the matrices $C_k, k = 1, 2, \ldots$ are nonnegative.

H6. The integral curves of the system (1) meet successively each one of the hypersurfaces $\sigma_1, \sigma_2, \ldots$ exactly once.

Let $t_1, t_2, \ldots (t_0 < t_1 < t_2 < \ldots)$ are the moments at which the integral curve $(t, x(t;\, t_0, \varphi_0))$ of the problem (1), (2) meets the hypersurfaces $\sigma_k, k = 1, 2, \ldots$.

We shall note that if conditions H1-H4 and H6 are fulfilled, then $t_k \to \infty$ as $k \to \infty$ and $J^+(t_0, \varphi_0) = (t_0, \infty)$.

Definition 1. The solutions of the system (1) are said to be *uniformly M-bounded* if:

$$(\forall \eta > 0)(\exists \beta = \beta(\eta) > 0)(\forall t_0 \in \mathbb{R})(\forall \alpha > 0)$$

$$(\forall \varphi_0 \in \overline{S}_\alpha(C_0) \cap \overline{M_0(t, \eta)})(\forall t > t_0):$$

$$x(t; t_0, \varphi_0) \in M(t, \beta).$$

Definition 2. The set M is said to be:

a) stable with respect to the system (1), if

$$(\forall t_0 \in \mathbb{R})(\forall \alpha > 0)(\forall \varepsilon > 0)(\exists \delta = \delta(t_0, \alpha, \varepsilon) > 0)$$

$$(\forall \varphi_0 \in S_\alpha(C_0) \cap M_0(t, \delta))(\forall t > t_0):$$

$$x(t; t_0, \varphi_0) \in M(t, \varepsilon).$$

b) uniformly stable with respect to the system (1), if the number δ in *a)* depends on ε only.

c) uniformly globally attractive with respect to the system (1), if

$$(\forall \eta > 0)(\forall \varepsilon > 0)(\exists \sigma = \sigma(\eta, \varepsilon) > 0)$$

$$(\forall t_0 \in \mathbb{R})(\forall \alpha > 0)(\forall \varphi_0 \in S_\alpha(C_0) \cap \overline{M_0(t, \eta)})$$

$$(\forall t \in [t_0 + \sigma, \infty) \cap J^+(t_0, \varphi_0)):$$

$$x(t; t_0, \varphi_0) \in M(t, \varepsilon).$$

d) uniformly globally asymptotically stable with respect to the system (1), if it is uniformly stable, uniformly globally attractive with respect to the system (1), and the solutions of the system (1) are uniformly M-bounded.

In the following investigations the class $\mathcal{V}_0$ of the piecewise continuous functions $V : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ will be used. These functions are analogues of the classical Lyapunov's functions [2].

Definition 3. We will say that the function $V : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ belongs to the class $\mathcal{V}_0$, if:

1. The function V is continuous in $\bigcup_{k=1}^{\infty} G_k$ and it is locally Lipschitzian with respect to its second argument on each of the sets $G_k, k = 1, 2, \ldots$.
2. $V(t, x) = 0$ for $(t, x) \in M, t \geq t_0$ and $V(t, x) > 0$ for $(t, x) \in \{[t_0, \infty) \times \mathbb{R}^n\} \setminus M$.
3. For each $k = 1, 2, \ldots$ and $(t_0^*, x_0^*) \in \sigma_k$ there exist the finite limits

$$V(t_0^* - 0, x_0^*) = \lim_{\substack{(t,x) \to (t_0^*, x_0^*) \\ (t,x) \in G_k}} V(t, x), \quad V(t_0^* + 0, x_0^*) = \lim_{\substack{(t,x) \to (t_0^*, x_0^*) \\ (t,x) \in G_{k+1}}} V(t, x).$$

4. The equality $V(t_0^* - 0, x_0^*) = V(t_0^*, x_0^*)$ holds true.

We introduce the following functional classes:

$PC[[t_0,\infty),\mathbb{R}^n] = \{x : [t_0,\infty) \to \mathbb{R}^n : x(t)$ is piecewise continuous function with points of discontinuity of the first kind (i.e., the left and right limits exist there, and they are bounded) belonging to the interval (t_0,∞), at which the function is continuous from the left;

$$\Omega = \{x \in PC[[t_0,\infty),\mathbb{R}^n] : V(s,x(s)) \le V(t,x(t)), t-h \le s \le t, t \ge t_0, V \in \mathcal{V}_0\}.$$

Let $V \in \mathcal{V}_0$ and $x \in PC[[t_0,\infty),\mathbb{R}^n]$.

Let $t \ne \tau_k(x(t)), k = 1,2,\ldots$.

Introduce the function

$$D_-V(t,x(t)) = \liminf_{\sigma\to 0^-} \frac{1}{\sigma}\left[V(t+\sigma, x(t) + \sigma(A(t)x(t) + B(t)x(t-h))) - V(t,x(t))\right].$$

Definition 4. Let $\lambda : (t_0,\infty) \to \mathbb{R}_+$ be a measurable function. Then we say that $\lambda(t)$ is integral positive if $\int_J \lambda(t)dt = \infty$, whenever $J = \bigcup_{k=1}^{\infty}[\alpha_k,\beta_k]$, $\alpha_k < \beta_k < \alpha_{k+1}$ and $\beta_k - \alpha_k \ge \theta > 0, k = 1,2,\ldots$.

In proving the main results of the paper we will use the following statements:

Theorem 1. *Let the following assumptions hold:*

1. Conditions H1-H4 *and* H6 *are met.*

2. $g \in PC[[t_0,\infty)\times\mathbb{R}_+,\mathbb{R}_+]$ *and* $g(t,0) = 0$ *for* $t \in [t_0,\infty)$.

3. $B_k \in C[\mathbb{R}_+,\mathbb{R}_+], B_k(0) = 0$ *and the functions* $\psi_k : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi_k(u) = u + B_k(u)$ *are nondecreasing with respect to* $u, k = 1,2,\ldots$.

4. The maximal solution $r(t;\, t_0,u_0)$ *of the problem*

$$\begin{gathered} \dot{u} = g(t,u), \quad t \ne t_k, \quad k = 1,2,\ldots, \\ u(t_0+0) = u_0 \ge 0, \\ \Delta u(t_k) = B_k(u(t_k)), \quad k = 1,2,\ldots \end{gathered} \tag{3}$$

is defined on the interval (t_0,∞).

5. The function $V \in \mathcal{V}_0$ *is such that*

$$V(t_0,\varphi_0(t_0)) \le u_0$$

and the inequalities

$$\begin{gathered} D_-V(t,x(t)) \le g(t,V(t,x(t))), \quad t \ne \tau_k(x(t)), \quad k = 1,2,\ldots \\ V(t+0, x(t) + C_kx(t)) \le \psi_k(V(t,x(t))), t = \tau_k(x(t)), k = 1,2,\ldots \end{gathered} \tag{4}$$

are satisfied for $t \ge t_0$ *and* $x \in \Omega$.

Then

$$V(t,x(t;t_0,\varphi_0)) \le r(t;t_0,u_0), \quad t \in (t_0,\infty). \tag{5}$$

Proof. The maximal solution $r(t;\, t_0, u_0)$ of the problem (3) for $t \in (t_0, \infty)$ is defined by the equality

$$r(t;t_0,u_0) = \begin{cases} r_0(t;t_0,u_0^+), & t_1 < t \le t_1, \\ r_1(t;t_1,u_1^+), & t_1 < t \le t_2, \\ \dots\dots\dots\dots\dots\dots \\ r_k(t;t_k,u_k^+), & t_k < t \le t_{k+1}, \\ \dots\dots\dots\dots\dots\dots, \end{cases}$$

where $r_k(t;t_k,u_k^+)$ is the maximal solution of the equation $\dot{u} = g(t,u)$ without impulses in the interval (t_k, t_{t+1}), $k = 0,1,2,\dots$, for which $u_k^+ = \psi_k(r_{k-1}(t_k;t_{t-1},\, u_{k-1}^+)), k = 1,2,\dots$ and $u_0^+ = u_0$.

Let $t \in (t_0, t_1]$. Then, it follows from the corresponding comparison lemma in the continuous case [1], that

$$V(t, x(t;t_0, \varphi_0)) \le r(t;t_0,u_0),$$

i.e., the inequality (5) is satisfied for $t \in (t_0, t_1]$.

Now, suppose that (5) holds for $t \in (t_{k-1}, t_k], k > 1$. Then, using (4) and the fact that the function ψ_k is nondecreasing, we obtain

$$V(t_k+0, x(t_k+0;t_0,\varphi_0)) \le \psi_k(V(t_k, x(t_k;t_0,\varphi_0))) \le$$

$$\le \psi_k(r(t_k;t_0,u_0)) = \psi_k(r_{k-1}(t_k;t_{k-1},u_{k-1}^+)) = u_k^+.$$

Applying again the comparison lemma from [1] for $t \in (t_k, t_{k+1}]$, we obtain

$$V(t, x(t;t_0,\varphi_0)) \le r_k(t;t_k,u_k^+) = r(t;t_0,u_0),$$

i.e., the inequality (5) remains true for $t \in (t_k, t_{k+1}]$. The proof is completed by induction. ■

Corollary 1. *Let the following assumptions hold:*

1. Conditions H1-H4 *and* H6 *are met.*

2. The function $V \in \mathcal{V}_0$ is such that for $t \ge t_0$ and $x \in \Omega$ the following inequalities are valid:

$$D_V(t,x(t)) \le 0, t \ne \tau_k(x(t)), k = 1,2,\dots,$$
$$V(t+0, x(t)+C_k x(t)) \le V(t,x(t)), t = \tau_k(x(t)), k = 1,2,\dots.$$

Then

$$V(t, x(t;t_0,\varphi_0)) \le V(t_0, \varphi_0(t_0)), \quad t \in [t_0, \infty).$$

3. MAIN RESULTS

Theorem 2. *Let the following conditions hold:*

1. Conditions H1-H6 *are fulfilled.*

2. There exists continuous, real $n \times n$-matrix-valued function $D(t), t \in (t_0, \infty)$, which is symmetric, positively definite, differentiable for $t \neq \tau_k(x(t))$, $k = 1, 2, \ldots$ and it is such that

$$x^T \left[A^T(t)D(t) + D(t)A(t) + \dot{D}(t)\right] x \leq -a(t)|x|^2, \tag{6}$$

$$x \in \mathbb{R}^n, \quad t \neq \tau_k(x(t)),$$

$$x^T \left[C_k^T D(t) + D(t)C_k + C_k^T D(t)C_k\right] x \leq 0, \tag{7}$$

$$t = \tau_k(x(t)),$$

where $a(t) > 0$ is a continuous function.

3. There exists an integrally positive function $\lambda(t)$ such that

$$b(t) = a(t) - \max\{\alpha(t)\lambda(t), \beta(t)\lambda(t)\} \geq 0, \tag{8}$$

$$\frac{2\beta^{1/2}(t)}{\alpha^{1/2}(t-h)}|D(t)B(t)| \leq b(t), \tag{9}$$

where $\alpha(t)$ and $\beta(t)$ are respectively the smallest and the largest eigenvalues of $D(t)$.

Then the set $M = [-h, \infty) \times \{x \in \mathbb{R}^n : x \leq 0\}$ is uniformly globally asymptotically stable with respect to the system (1).

Proof. First of all, we shall prove that the set M is uniformly stable set with respect to the system (1).

For arbitrary $\varepsilon > 0$ we choose the positive number $\delta = \delta(\varepsilon) > 0$ such that $\beta(t_0)\delta^2 < \alpha(t)\varepsilon^2$, $t > t_0$.

Let $\alpha > 0$, $\varphi_0 \in S_\alpha(C_0) \cap M_0(t, \delta)$ and let $x(t) = x(t;\, t_0, \varphi_0)$ be the solution of the initial problem (1), (2).

We define the function

$$V(t,x) = \begin{cases} x^T D(t)x & \text{for} \quad x > 0, \\ 0 \quad \text{for} & x \leq 0. \end{cases}$$

The fact that $D(t)$ is a symmetric, real and positively definite matrix ensures that for $x \in \mathbb{R}^n$,

$$\alpha(t)|x|^2 \leq x^T D(t)x \leq \beta(t)|x|^2. \tag{10}$$

For the function $V(t,x)$, the set Ω is defined by the equality

$$\Omega = \{x \in PC[[t_0, \infty), \mathbb{R}^n] : x^T(s)D(s)x(s) \leq x^T(t)D(t)x(t), t - h \leq s \leq t, t \geq t_0\}.$$

For $t \in (t_0, \infty)$ and $x \in \Omega$ the following inequalities are valid:

$$\alpha(t-h)|x(t-h)|^2 \leq x^T(t-h)D(t-h)x(t-h) \leq$$

$$x^T(t)D(t)x(t) \leq \beta(t)|x(t)|^2.$$

The above inequalities lead to the estimate

$$|x(t-h)| \leq \frac{\beta^{1/2}(t)}{\alpha^{1/2}(t-h)}|x(t)|. \tag{11}$$

We will estimate $D_{-}V(t,x(t))$ for $t \in (t_0,\infty), t \neq \tau_k(x(t))$ and $x \in \Omega$. It follows from (6), (8), (9) and (11) that

$$D_{V}(t,x(t)) \leq \begin{cases} -a(t)|x(t)|^2 + 2|D(t)B(t)\|x(t)\|x(t-h)|, & x(t)>0, \\ 0, \quad x(t) \leq 0 \end{cases}$$

$$\leq \begin{cases} -[a(t)-b(t)]|x(t)|^2, & x(t)>0, \\ 0, \quad x(t)\leq 0 \end{cases} \leq -\lambda(t)V(t,x(t)). \tag{12}$$

Let $t = \tau_k(x(t))$. Using (7) we obtain

$$V(t+0, x(t)+C_k x(t)) =$$

$$= \begin{cases} (x^T(t)+x^T(t)C_k^T)D(t)(x(t)+C_k x(t)), & x(t)>0, \\ 0, \quad x(t)\leq 0 \end{cases}$$

$$= \begin{cases} x^T(t)D(t)x(t) + x^T(t)[C_k^T D(t) + D(t)C_k + C_k^T D(t)C_k], & x(t)>0, \\ 0, \quad x(t)\leq 0 \end{cases}$$

$$\leq V(t,x(t)). \tag{13}$$

Hence, the conditions of Corollary 1 are fulfilled, and therefore

$$V(t,x(t;t_0,\varphi_0)) \leq V(t_0,\varphi_0(t_0)), \quad t \in (t_0,\infty). \tag{14}$$

The last inequality implies the inequalities

$$\alpha(t)|x(t)|^2 \leq x^T(t)D(t)x(t) \leq \varphi_0^T(t_0)D(t_0)\varphi_0(t_0) \leq$$

$$\leq \beta(t_0)|\varphi_0(t)|^2 \leq \beta(t_0)\|\varphi_0\|^2 \leq \beta(t_0)\delta^2 < \alpha(t)\varepsilon^2, \quad t>t_0.$$

Since $d(x,M(t)) = |x|$ for $t>t_0$ and $x>0$, then $x(t)\in M(t,\varepsilon)$ for $t>t_0$. Thus it is proved that the set M is uniformly stable with respect to the system (1).

Now we shall prove that the set M is uniformly globally attractive with respect to the system (1).

Let $\varepsilon>0$ and $\eta>0$ are arbitrary chosen. We choose the number $\delta = \delta(\varepsilon)>0$ such that $\beta(t^*)\delta^2 < \alpha(t)\varepsilon^2, t \geq t^* \geq t_0$.

We will prove that there exists $\sigma = \sigma(\varepsilon,\eta)>0$ such that for each solution $x(t) = x(t;t_0,\varphi_0)$ of the problem (1), (2), such that $t_0 \in \mathbb{R}$, $\varphi_0 \in S_\alpha(C_0) \cap \overline{M_0(t,\eta)}$ ($\alpha>0$ is arbitrary), and for each $t^* \in [t_0, t_0+\sigma]$, the inequality

$$d(x(t^*),M(t^*)) < \delta(\varepsilon) \tag{15}$$

is valid.

We suppose the opposite. Then for each $\sigma > 0$ there exists a solution $x(t) = x(t;\, t_0, \varphi_0)$ of the problem (1), (2) for what $t_0 \in \mathbb{R}$, $\varphi_0 \in S_\alpha(C_0) \cap \overline{M_0(t,\eta)}$, $\alpha > 0$, and such that

$$d(x(t), M(t)) < \delta(\varepsilon) \tag{16}$$

for $t \in [t_0, t_0 + \sigma]$.

It follows from the inequalities (12) and (13) that

$$V(t,x(t)) - V(t_0, \varphi_0(t_0)) \leq \int_{t_0}^{t} D_-V(s,x(s))ds \leq$$

$$\leq -\int_{t_0}^{t} \lambda(s)V(s,x(s))ds \leq -\int_{t_0}^{t} \lambda(s)\alpha(s)|x(s)|^2 ds, \tag{17}$$

for $t \geq t_0$.

The properties of the function $V(t,x) = x^T D(t)x \in \mathcal{V}_0$ in the interval (t_0, ∞) imply that there exists the finite limit

$$\lim_{t\to\infty} V(t,x(t)) = v_0 \geq 0. \tag{18}$$

Then (10), (16)-(18) yield

$$\int_{t_0}^{\infty} \lambda(t)\alpha(t)|x(t)|^2 dt \leq \beta(t_0)\eta^2 - v_0.$$

By virtue of the integral positivity of the function $\lambda(t)$ and the fact that $D(t)$ is positively definite matrix it follows, that the number σ can be chosen so that

$$\int_{t_0}^{t_0+\sigma} \lambda(t)\alpha(t)dt > \frac{\beta(t_0)\eta^2 - v_0 + 1}{\delta(\varepsilon)^2}.$$

Thus

$$\beta(t_0)\eta^2 - v_0 \geq \int_{t_0}^{\infty} \lambda(t)\alpha(t)|x(t)|^2 dt \geq \int_{t_0}^{t_0+\sigma} \lambda(t)\alpha(t)|x(t)|^2 dt \geq$$

$$\geq \delta^2(\varepsilon) \int_{t_0}^{t_0+\sigma} \lambda(t)\alpha(t)dt > \beta(t_0)\eta^2 - v_0 + 1.$$

The contradiction obtained shows that there exists a positive constant $\sigma = \sigma(\varepsilon, \eta)$ such that for each solution $x(t) = x(t;\, t_0, \varphi_0)$ of the problem (1), (2) for which $t_0 \in \mathbb{R}$, $\varphi_0 \in S_\alpha(C_0)$ $\cap \overline{M_0(t,\eta)}$, $\alpha > 0$, there exists $t^* \in [t_0, t_0 + \sigma]$ such that the inequality (15) holds true.

Then for $t \geq t^*$ (and thus for $t \geq t_0 + \sigma$ also) it follows from Corollary 1 the validity of the inequalities

$$\begin{aligned} \alpha(t)|x(t)|^2 &\leq x^T(t)D(t)x(t) \leq x^T(t^*)D(t^*)x(t^*) \leq \beta(t^*)|x(t^*)|^2 \leq \\ &\leq \beta(t^*)\delta^2 < \alpha(t)\varepsilon^2, \end{aligned}$$

which show that the set M is uniformly globally attractive with respect to the system (1).

Finally, we shall prove that the solutions of the system (1) are uniformly M-bounded.

Let $\eta > 0$ and let $\beta = \beta(\eta) > 0$ be such a number that

$$\beta(t_0)\eta^2 < \alpha(t)\beta^2, \quad t > t_0.$$

We choose arbitrary $\alpha > 0$, $\varphi_0 \in S_\alpha(C_0) \cap \overline{M_0(t,\eta)}$ and let $x(t) = x(t;\, t_0, \varphi_0)$. Then for $t > t_0$ the following inequalities hold:

$$\begin{aligned} \alpha(t)|x(t)|^2 &\leq x^T(t)D(t)x(t) \leq \varphi_0^T(t_0)D(t_0)\varphi_0(t_0) \leq \beta(t_0)|\varphi_0(t_0)|^2 \leq \\ &\leq \beta(t_0)\|\varphi_0(t)\|^2 \leq \beta(t_0)\eta^2 < \alpha(t)\beta^2, \quad t > t_0. \blacksquare \end{aligned}$$

Therefore, $x(t) \in M(t, \beta)$ for $t > t_0$.

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REFERENCES

[1] V. LAKSHMIKANTHAM, S. LEELA, A.A. MARTYNYUK, *Stability Analysis of Nonlinear Systems*,Marcel Dekker Inc., New York, 1989.
[2] P.S. SIMEONOV, D.D. BAINOV, Stability with respect to part of the variables in systems with impulse effect, *J. Math. Anal. Appl.*, 117 (1986), 247-263.

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