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GENERALIZED CARDINAL INTERPOLATION BY REFINABLE FUNCTIONS: SOME NUMERICAL RESULTS

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Abstract. *We present a particular class of refinable functions, that are solutions of a refinement equation when the mask satisfies suitable conditions. Some general properties of these functions, such as symmetry and monotonicity, are proved. We extend an interpolation problem, already considered in the context of the cardinal splines, to the refinable functions. Some numerical results and graphs are displayed.*

1. INTRODUCTION

A *refinement equation* (RE) is a functional equation of the type

$$\varphi(x) = \sum_{j\in\mathbb{Z}} a_j\, \varphi(2x - j) \tag{1.1}$$

where the "mask" $\mathbf{a} = \{a_j\}_{j\in\mathbb{Z}}, a_j \in \mathbb{R}$, is a given sequence satisfying suitable conditions. We shall refer to φ as a *refinable function.*

A well known example of such functions is provided by the set of B-splines of a given order.

The refinable functions play a crucial role in the study of stationary subdivision schemes [1,3,6,7] as well in the construction of basis of orthonormal wavelets [2,4,10].

The interest in analysing how the behaviour of a refinable function depends on the mask is suggested not only by the involvement of refinable functions in the mentioned subjects, but also by some recent researches [5], where it is just examined how an alteration even slight on the mask affects the corresponding refinable function.

In this paper we shall consider the case of finitely supported refinable functions whose mask **a** satisfies the following conditions:

$$a_k > 0, \quad k = 0, \dots, n+1 \qquad a_k = 0, \quad k < 0, \quad k > n+1 \tag{1.2}$$

$$\sum_{j\in\mathbb{Z}} a_{k+2j} = 1 \quad \forall k \in \mathbb{Z}. \tag{1.3}$$

These conditions ensure [12] that there exists a unique continuous function φ satisfying (1.1) and such that

$$\varphi(x) = 0, \quad x \notin (0, n+1) \tag{1.4}$$

$$\varphi(x) > 0, \quad x \in (0, n+1) \tag{1.4'}$$

and such that

$$\sum_{j\in\mathbb{Z}} \varphi(x - j) = 1, \quad \forall x \in \mathbb{R}. \tag{1.5}$$

Moreover we suppose that the associated polynomial

$$p(z) = \sum_{j=0}^{n+1} a_j z^j \tag{1.6}$$

is left-plane stable, that is a Hurwitz polynomial, so that the refinable function φ is also a ripplet [8], that is it satisfies:

$$\det_{l,j=1,\dots,r} \varphi(x_l - i_j) \geq 0 \qquad \forall x_1 < \ldots < x_r,\ i_1 < \ldots < i_r,\ x_l \in \mathbb{R},\ i_j \in \mathbb{Z} \tag{1.7}$$

which, in particular, joined to (1.5), implies that

$$0 \leq \varphi(x) \leq 1. \tag{1.8}$$

Moreover, the functions $\varphi(\cdot - k), k \in \mathbb{Z}$, are linearly independent so they can be used in interpolation problems. In Section 2 we shall introduce the generalized Cardinal Interpolation Problem and we shall give some procedure to construct the interpolating function belonging to $span\{\varphi(\cdot - k)\}$. In Section 3 we shall study the behaviour of a particular class of refinable functions and their performances with respect to the interpolation problem; this will be done mostly from a computational perspective, providing some graphs and numerical tables.

2. CARDINAL INTERPOLATION BY REFINABLE FUNCTIONS

In relation with the cardinal splines, $M_n(x)$, of degree n, Schoenberg proposed [13,14] the Cardinal Interpolation Problem (CIP) consisting in the search of a function $f \in span\{M_n(\cdot - k\}$:

$$f(x) = \sum_{k \in \mathbb{Z}} b_k M_n(x-k) \tag{2.1}$$

satisfying the interpolation conditions

$$f(j+\alpha) = y_j, \quad j \in \mathbb{Z}$$

where $\mathbf{y} = \{y_j\}_{j\in\mathbb{Z}}, y_j \in \mathbb{R}$, is a given sequence and $0 \leq \alpha < 1$.

Micchelli treated the same problem in the wider context of the cardinal $\mathcal{L}$-splines [11], which are determined by operators of the type:

$$\mathcal{L}(D)y = \sum_{k=0}^{n+1} a_k D^k y.$$

This problem was solved under various conditions on $\mathbf{y}$; in particular in both cases it was proved that, if $\mathbf{y} \in l^1(\mathbb{Z})$, the CIP has a unique solution for each $\alpha \in [0, 1)$ different from an exceptional value α_0. The value of α_0 depends on the particular class of $\mathcal{L}$-splines.

On the other hand the cardinal splines satisfy a refinement equation.

Thus, a generalized CIP can be set up as follows.

Given a sequence $\mathbf{y} = \{y_j\} \in l^1(\mathbb{Z})$ of data $y_j \in \mathbb{R}$, and a refinable function φ whose mask $\mathbf{a}$ satisfies the conditions specified in Section 1, one finds a function $F \in span\{\varphi(\cdot - k\}$, i.e.

$$F(x) = \sum_{k \in \mathbb{Z}} c_k \varphi(x - k), \tag{2.2}$$

such that the conditions

$$F(\alpha + j) = y_j, \quad j \in \mathbb{Z} \tag{2.3}$$

are fulfilled, with α fixed in $[0, 1)$.

Following a line of reasoning basically developed by Schoenberg [15], it is possible to show that the solution to this problem exists and is unique providing that the Euler-Frobenius polynomial

$$\Pi(\alpha; z) = \sum_{j=0}^{n} \varphi(\alpha + j)\, z^j \tag{2.4}$$

does not vanish on the unit circle $|z| = 1$.

Since under the present assumptions φ is a ripplet [see (1.7)], the sequence of the coefficients in (2.4) is totally positive. Therefore, for any fixed $\alpha > 0$, the polynomial (2.4) has only real negative zeros, and it has only real non positive zeros for $\alpha = 0$ [9]; thus we are interested to avoid only the values of α such that

$$\Pi(\alpha; -1) = 0. \tag{2.5}$$

Now let us assume that at most a value α exists such that (2.5) does not hold, so that, for any set of data $\mathbf{y}$, the conditions (2.3) identify the unique interpolating function (2.2). It is worth noting that such a value of α depends only on the refinable function φ.

In the next Section we shall consider a case in which F exists for any $\alpha \in [0, 1)$ different from an exceptional value α_0; now we get a procedure for obtaining the interpolating function F.

Let us denote by $\mathcal{L}_\varphi(x) \in span\{\varphi(\cdot - k)\}$ the function which solves the generalized CIP when $\mathbf{y} = \{\delta_j\}$, that is

$$\mathcal{L}_\varphi(\alpha + j) = \delta_j = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{if } j \neq 0 \end{cases} \tag{2.6}$$

where α is such that $\Pi(\alpha; -1) \neq 0$.

One can write

$$F(x) = \sum_{\nu \in \mathbb{Z}} y_\nu \mathcal{L}_\varphi(x - \nu). \tag{2.7}$$

It follows immediately that $F \in span\{\varphi(\cdot - k)\}$ and satisfies (2.3), due to (2.6).

Let us consider the Laurent series of the reciprocal of the Euler-Frobenius polynomial (2.4):

$$\frac{1}{\Pi(\alpha; z)} = \sum_{\nu \in \mathbb{Z}} \omega_\nu z^\nu \tag{2.8}$$

which converges in an open set $r < |z| < \frac{1}{r}, r > 0$, including the unit circle $|z| = 1$.

From (2.4) and (2.8), multiplying the two sums, one gets

$$1 \equiv \sum_{j\in\mathbb{Z}} z^j \sum_{\nu=j}^{j-n} \omega_\nu \varphi(\alpha + j - \nu)$$

and, recalling that φ is finitely supported, this is equivalent to

$$\sum_{\nu\in\mathbb{Z}} \omega_\nu \varphi(\alpha + j - \nu) = \delta_j. \tag{2.9}$$

Thus one obtains

$$\mathcal{L}_\varphi(x) = \sum_{\nu\in\mathbb{Z}} \omega_\nu \varphi(x - \nu). \tag{2.10}$$

Remark. The finite support of φ implies that the sum in (2.10) contains only a finite number of terms for fixed x, just $n+1$. Moreover if also $\mathbf{y}$ reduces to a finite number of data different from zero, then the interpolant F in (2.7) is expressed by a finite sum.

The problem to express the interpolating function F is so reduced to find the coefficients in the Laurent series (2.8).

3. A CASE STUDY

Some inverse problems related to the construction of interpolating functions belonging to the space span$\{\varphi(\cdot - k)\}$ leads to consider the finitely supported mask

$$a_0^{(h)} = a_3^{(h)} = \frac{2^{h+1}}{(2^h+1)^2}, \quad a_1^{(h)} = a_2^{(h)} = \frac{2^{2h}+1}{(2^h+1)^2}, \quad h \in \mathbb{N} \cup \{0\}. \tag{3.1}$$

For any h this mask has the properties (1.2), (1.3) with $n = 2$ and the associated polynomial

$$p(z) = \sum_{k=0}^{3} a_k^{(h)} z^k = \frac{1}{(2^h+1)^2}(z+1)(2^{h+1}z^2 + (2^h-1)^2 z + 2^{h+1})$$

is a Hurwitz polynomial.

So it is possible to ensure [8, 12] that, for any fixed h, there exists a unique function $\varphi_h \in C^0(-\infty,\infty)$, supported on $(0,3)$, satisfiying the RE

$$\varphi_h(x) = \sum_{k=0}^{3} a_k^{(h)} \varphi_h(2x - k), \quad \forall x \in \mathbb{R}, \tag{3.2}$$

and (1.4), (1.4'), (1.5).

Because of the symmetry of $a_i^{(h)}$, also φ_h is symmetric with respect to $\frac{3}{2}$ [12], that is

$$\varphi_h(3 - x) = \varphi_h(x), \quad \forall x \in \mathbb{R}. \tag{3.3}$$

Theorem 3.1. *The refinable function $\varphi_h, h > 0$, defined in (3.2) is increasing in* $[0, \frac{3}{2}]$.

Proof. Given a function $f \in C^0[\alpha, \beta]$, we consider the transformation

$$Tf(x) = \sum_{j \in J} k(x,j) f(j), \qquad J = \mathbb{Z}$$

where $k(x,j) = \varphi_h(2x - j)$.

The properties (1.5), (1.7) assure [9, §3.6] that T maps increasing functions in increasing functions.

Then assuming $f(t) = a_0^{(h)} + (a_1^{(h)} - a_0^{(h)})t$, which is increasing for $t \in [0, 1]$ and $h > 0$, it follows that

$$Tf(x) = a_0^{(h)} \varphi_h(2x) + a_1^{(h)} \varphi_h(2x - 1)$$

is increasing on $\mathbb{R}$.

In particular, when $x \in [0, 1]$, by (3.1) it follows

$$Tf(x) = \varphi_h(x)$$

which implies the monotonicity of φ_h in the same interval.

Assuming now $x \in [1, \frac{3}{2}]$, the expression (3.2) reduces to

$$\varphi_h(x) = a_0^{(h)} \varphi_h(2x) + a_1^{(h)} \varphi_h(2x - 1) + a_1^{(h)} \varphi_h(2x - 2) = Tf(x) + a_1^{(h)} \varphi_h(2x - 2).$$

By the proof above, $\varphi_h(2x - 2)$ is increasing when $x \in [1, \frac{3}{2}]$, then the thesis follows. □

By the symmetry of φ_h it follows immediatly

Corollary. *The refinable function $\varphi_h, h > 0$, is decreasing in* $[\frac{3}{2}, 3]$.

In the following we shall consider only the case $h > 0$ and, for short notation, we shall suppress the index h.

Moreover, it is useful to recall that the set of the dyadic rationals of level r is defined to be

$$D_r = \left\{ \frac{k}{2^r}, k \in \mathbb{Z} \right\}, r \in \mathbb{Z}$$

and the set of dyadic rationals is defined by

$$D = \bigcup_{r \in \mathbb{Z}} D_r.$$

From (3.1), (3.2) one obtains the values of φ on D_0

$$\varphi(1) = \varphi(2) = \frac{1}{2}. \tag{3.4}$$

Then the values of φ on a set D_r produce the values of φ on the next set D_{r+1} by means of the RE (3.2).

Taking into account the symmetry property (3.3), the computation can be confined to the interval $[0, 3/2]$.

Then by means of (3.1), (3.2) one gets

$$\varphi\left(\frac{1}{2}\right) = a_0\varphi(1) = \frac{a_0}{2}, \quad \varphi\left(\frac{3}{2}\right) = 2a_1\varphi(1) = a_1 \tag{3.5}$$

and, by induction,

$$\varphi\left(\frac{1}{2^r}\right) = a_0^r\varphi(1), \quad r \geq 1 \tag{3.6}$$

$$\varphi\left(\frac{3}{2^r}\right) = 3a_0^{r-1}a_1\varphi(1), \quad r > 1. \tag{3.7}$$

Several numerical cases have been checked corresponding to different values of h, which correspond to different choises of the mask, and some examples are quoted in the following Fig. 1. Here graphs corresponding to $h = 0, 1, 2, 3, 5, 9$ are shown and the cardinal B-spline of order 3, N_3, is also given.

Indeed the mask $\mathbf{a} = \{\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{4}\}$ of the cardinal B-spline is obtained from (3.1) assumig $h = \log(3 + 2\sqrt(2)) / \log 2 \simeq 2.5431$: we can observe that when h varies in $[0, +\infty)$, the shape of the refinable functions under consideration is quite similar to the shape of N_3 when $h = 2, 3$, and tends to the graph of the non continuous function

$$\varphi_\infty(x) = \begin{cases} 1 & 1 < x < 2 \\ \frac{1}{2} & x = 1, x = 2 \\ 0 & \text{otherwise} \end{cases}$$

when $h \to +\infty$.

The refinable functions (3.2) were also considered in the generalized cardinal interpolation problem.

The Euler-Frobenius polynomial

$$\Pi(z; \alpha) = \varphi(\alpha + 2)z^2 + \varphi(\alpha + 1)z + \varphi(\alpha), \qquad \alpha \in [0, 1) \tag{3.8}$$

has only non negative zeros (see Section 2).

When $\alpha = 0$, the zeros of (3.8) are

$$\lambda_1 = 0, \quad \lambda_2 = \frac{-\varphi(1)}{\varphi(2)}$$

and so the polynomial $\Pi(z; 0)$ vanishes on the unit circle iff $\varphi(2) = \varphi(1)$. The following proposition is stated.

Theorem 3.2. *In the generalized CIP the value $\alpha = 0$ is an exceptional value for any refinable function φ, supported on $(0, 3)$ iff*

$$\varphi(2) = \varphi(1). \tag{3.9}$$

Theorem 3.3. *If the refinable functions (3.2) are strictly monotone, then the unique exceptional value with respect to CIP is $\alpha = 0$.*

Proof. The refinable functions (3.2) satisfy (3.9) due to the symmetry; thus $\alpha = 0$ is exceptional according to the previous theorem.

Suppose now that an exceptional value $\bar{\alpha} \neq 0$ exists. Then, from (3.8) and (2.5), it follows that

$$R(\alpha) = \varphi(\bar{\alpha}) + \varphi(\bar{\alpha}+2) - \varphi(\bar{\alpha}+1) = 0, \qquad R(\alpha) \text{ given by } \Pi(z; \bar{\alpha}) = (z+1)Q(\alpha; z) + R(\alpha).$$

Taking (1.5) into account, this would imply

$$\varphi(\bar{\alpha} + 1) = \frac{1}{2},$$

which is impossible because, from (3.4) and the hypothesis, it follows that

$$\varphi(x) > \frac{1}{2} \qquad \forall x \in (1, 2). \quad \Box$$

Remark. The hypothesis that the functions φ are strictly monotone is probably redundant, as numerical tests seem to imply that this is always the case.

Experiments have been carried out with finitely supported data sets, namely

$$y_j = 0, \quad j < 0, j > m,$$

and using the values $\alpha = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ that are not exceptional values, as can be checked directly using (3.8).

In Figs. 2, 3, 4 the graphs of the interpolant (2.7) are quoted when the data set $\{y_j\}$ is obtained by the functions:

$$f_1(x) = \begin{cases} x & 0 \leq x \leq 4.5 \\ 0 & \text{otherwise} \end{cases}$$

$$f_2(x) = \begin{cases} 1 & -4. \leq x \leq 4. \\ 0 & \text{otherwise} \end{cases}$$

corresponding to equidistributed nodes with step h. Of course the cardinal interpolation condition (2.3) is obtained homothetically.

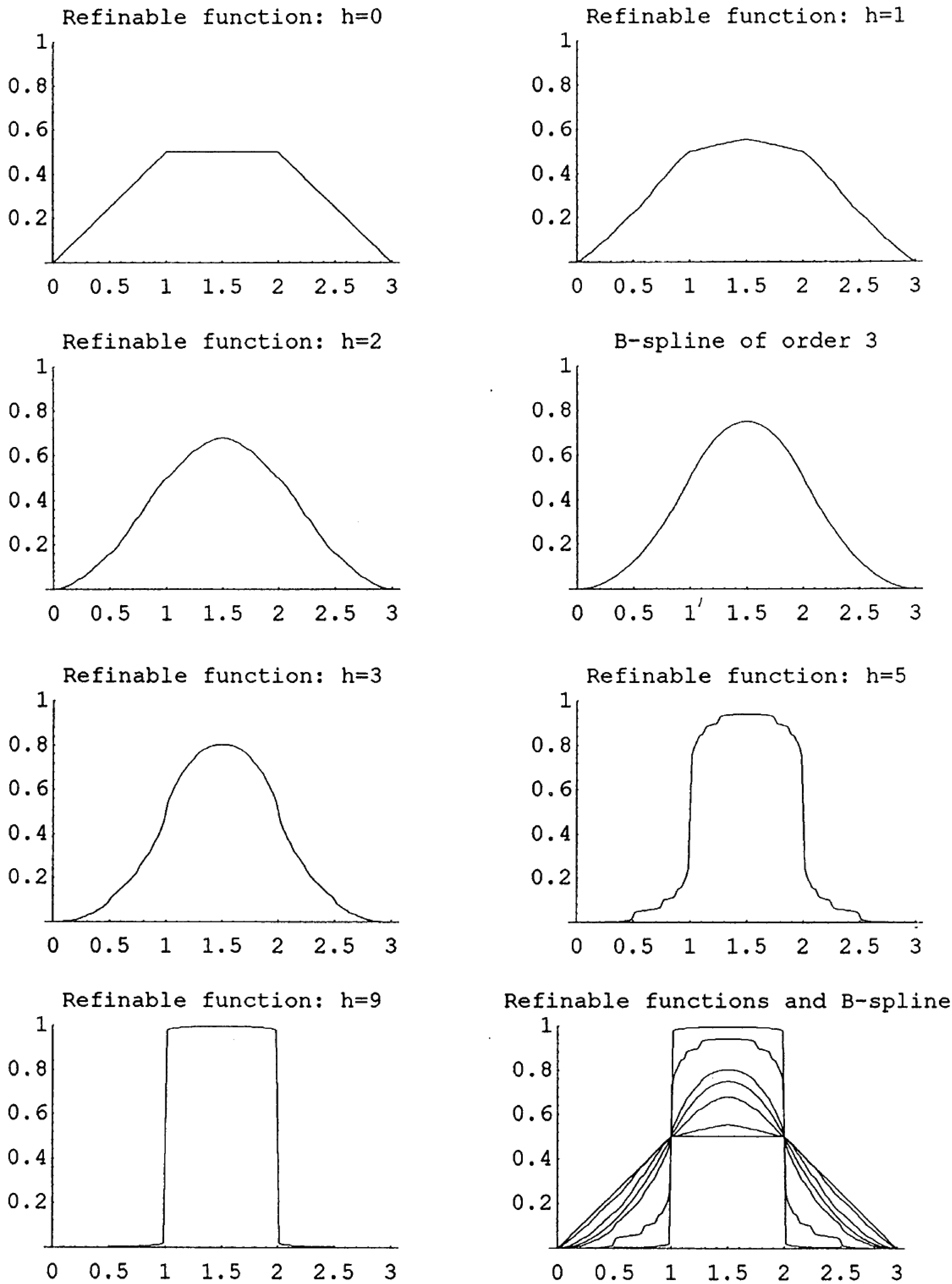

FIG. 1. Graphs of the refinable functions for different values of h.

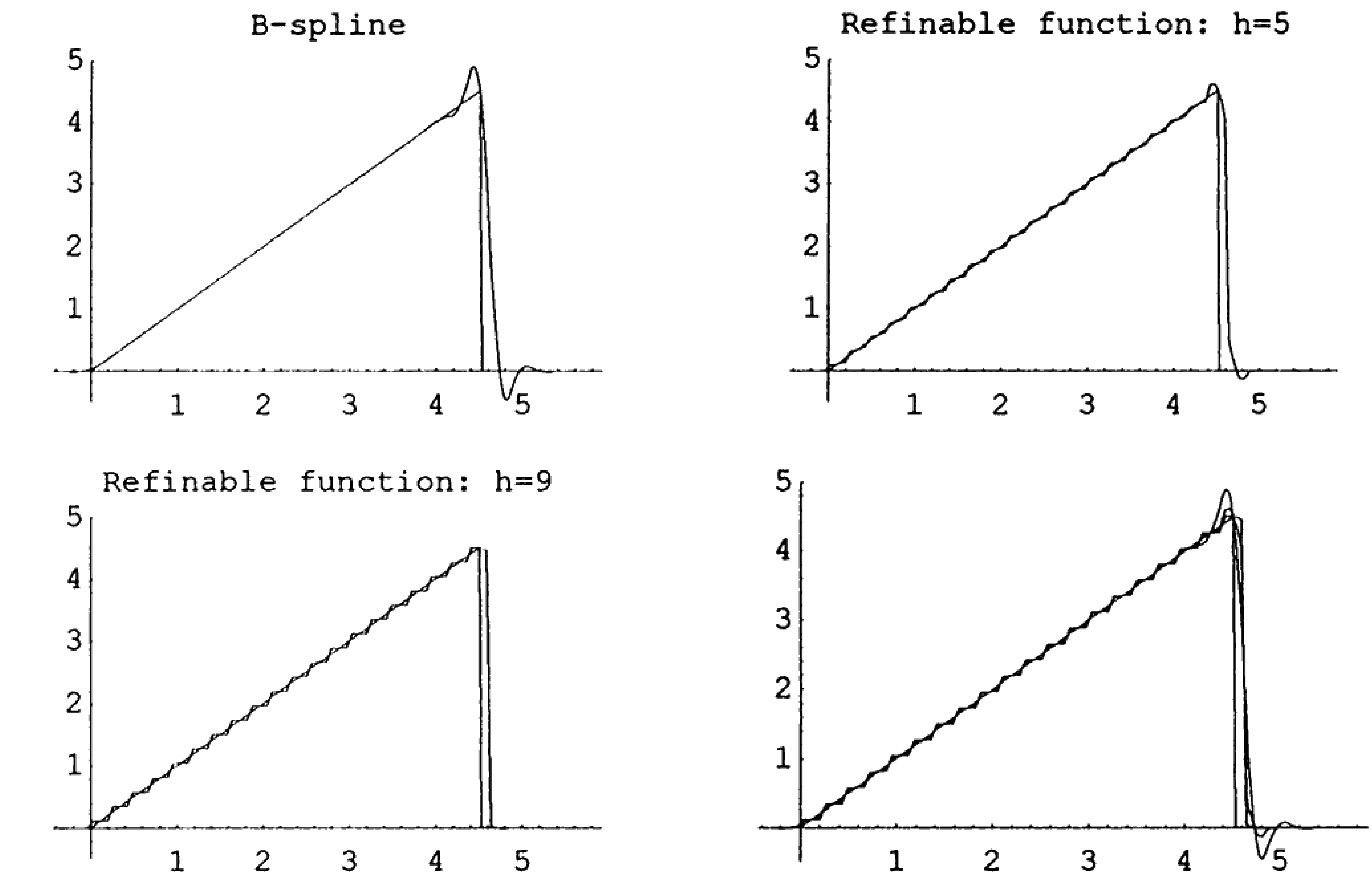

FIG. 2. Graphs of the interpolants (2.7) to f_1 for $\alpha = \frac{1}{2}$ and $m = 20$

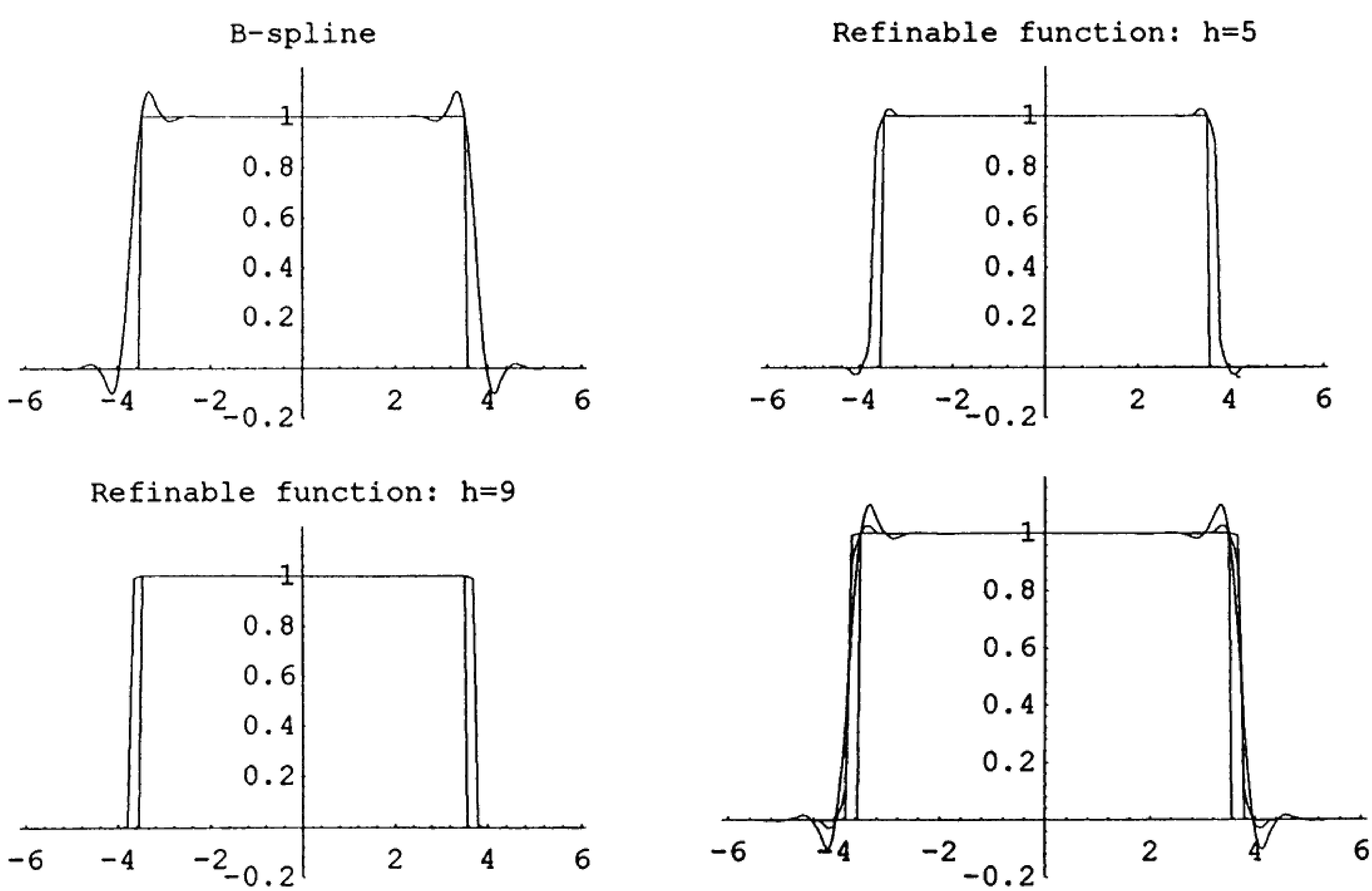

FIG. 3. Graphs of the interpolants (2.7) to f_2 for $\alpha = \frac{1}{2}$ and $m = 15$

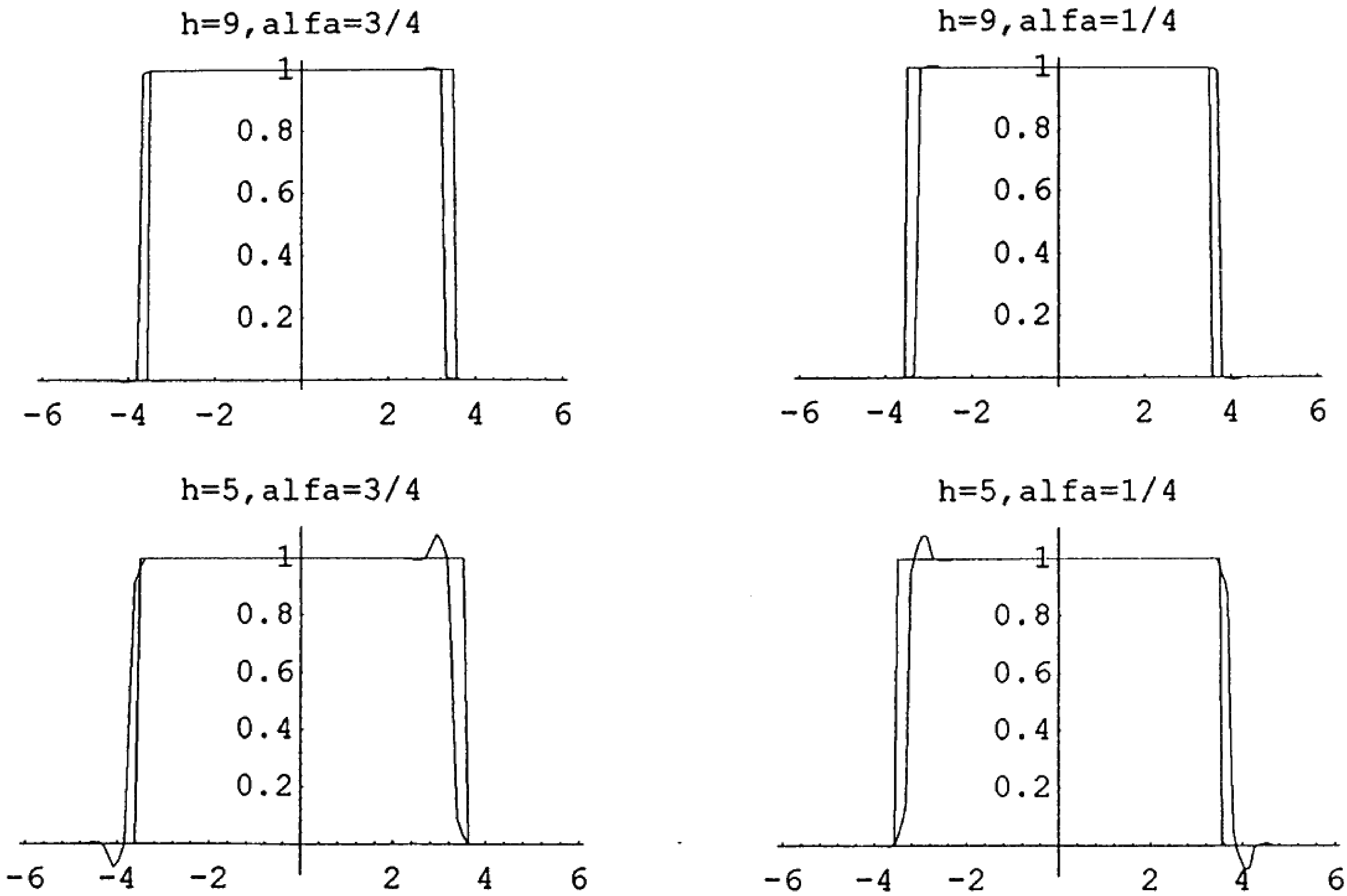

FIG. 4. Graphs of the interpolants (2.7) to f_2 for $\alpha = \frac{1}{4}, \frac{3}{4}$ and $m = 15$

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