Note di Matematica Vol. 16 - n. 1, 145-154 (1996)

THE VORONOVSKAYA THEOREM FOR SOME LINEAR POSITIVE OPERATORS OF FUNCTIONS OF TWO VARIABLES

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Abstract. *We give the Voronovskaya theorem for some operators $L_{m,n}^{\{i\}}$ of the Szasz-Mirakjan type defined for functions of two variables belonging to polynomial or exponential weighted spaces.*

Some approximation properties of these operators for functions of one variable are given in [1]-[5].

Key words: *linear positive operator, Voronovskaya theorem.*
A.M.S. *Subject classification: 41A26.*

1. PRELIMINARIES

1.1. Let $N:=\{1,2,\dots\}$, $N_0:=N\cup\{0\}$, $R_0:=[0,+\infty)$ and let for a fixed $p\in N_0$ and for all $x\in R_0$

$$w_0(x):=1, \qquad w_p(x):=(1+x^p)^{-1} \quad \text{if } p\geq 1. \tag{1}$$

For fixed $p_1,p_2\in N_0$ let

$$w_{p_1,p_2}(x,y):=w_{p_1}(x)w_{p_2}(y), \qquad (x,y)\in R_0^2, \tag{2}$$

where $R_0^2:=\{(x,y):x,y\in R_0\}$. Denote by $C_{1;p_1,p_2}$, $p_1,p_2\in N_0$, the space of all real-valued functions f defined on R_0^2 such that $w_{p_1,p_2}(\cdot,\cdot)f(\cdot,\cdot)$ is uniformly continuous and bounded on R_0^2 with the norm

$$\|f\|_{1;p_1,p_2}:=\sup_{(x,y)\in R_0^2} w_{p_1,p_2}(x,y)|f(x,y)|. \tag{3}$$

Moreover let $C^m_{1;p_1,p_2}$, m, $p_1,p_2\in N_0$, be the class of all functions $f\in C_{1;p_1,p_2}$, which partial derivatives of the order $\leq m$ belong also to the space $C_{1;p_1,p_2}$.

1.2. Let for a fixed $q>0$

$$v_q(x):=e^{-qx}, \qquad x\in R_0, \tag{4}$$

and, for fixed $q_1,q_2>0$, let

$$v_{q_1,q_2}(x,y):=v_{q_1}(x)v_{q_2}(y), \qquad (x,y)\in R_0^2. \tag{5}$$

Denote by $C_{2;q_1,q_2}$, $q_1,q_2>0$, the exponential weighted space of all real-valued functions f defined on R_0^2 for which $v_{q_1,q_2}(\cdot,\cdot)f(\cdot,\cdot)$ is uniformly continuous and bounded on R_0^2. The norm in $C^m_{2;q_1,q_2}$ is defined by

$$\|f\|_{2;q_1,q_2}:=\sup_{(x,y)\in R_0^2} v_{q_1,q_2}(x,y)|f(x,y)|. \tag{6}$$

Analogously as in Sec. 1.1. we define the class $C^m_{2;q_1,q_2}$ with some $m \in N_0$.

1.3. In the space $C_{1;p_1,p_2}$, $p_1, p_2 \in N_0$, and $C_{2;q_1,q_2}$, $q_1, q_2 > 0$, we introduce the following operators $L^{\{i\}}_{m,n}$, $m, n \in N$, $i = 1, 2, 3, 4$:

$$L^{\{1\}}_{m,n}(f;x,y) := \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{m,j}(x)a_{n,k}(y)f\left(\frac{2j}{m}, \frac{2k}{n}\right), \tag{7}$$

$$L^{\{2\}}_{m,n}(f;x,y) := \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{m,j}(x)a_{n,k}(y)\frac{mn}{4}\int_{\frac{2j}{m}}^{\frac{2j+2}{m}}\int_{\frac{2k}{n}}^{\frac{2k+2}{n}} f(t,z)dtdz, \tag{8}$$

$$\begin{aligned} L^{\{3\}}_{m,n}(f;x,y) \; := \; & \frac{f(0,0)}{(1+\sinh mx)(1+\sinh ny)} + \\ & + \frac{1}{1+\sinh mx}\sum_{k=0}^{\infty} b_{n,k}(y)f\left(0, \frac{2k+1}{n}\right) + \\ & + \frac{1}{1+\sinh ny}\sum_{j=0}^{\infty} b_{m,j}(x)f\left(\frac{2j+1}{m}, 0\right) + \\ & + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} b_{m,j}(x)b_{n,k}(y)f\left(\frac{2j+1}{m}, \frac{2k+1}{n}\right), \end{aligned} \tag{9}$$

$$\begin{aligned} L^{\{4\}}_{m,n}(f;x,y) \; := \; & \frac{f(0,0)}{(1+\sinh mx)(1+\sinh ny)} + \\ & + \frac{1}{1+\sinh mx}\sum_{k=0}^{\infty} b_{n,k}(y)\frac{n}{2}\int_{\frac{2k+1}{n}}^{\frac{2k+3}{n}} f(0,z)dz + \\ & + \frac{1}{1+\sinh ny}\sum_{j=0}^{\infty} b_{m,j}(x)\frac{m}{2}\int_{\frac{2j+1}{m}}^{\frac{2j+3}{m}} f(t,0)dt + \\ & + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} b_{m,j}(x)b_{n,k}(y)\frac{mn}{4}\int_{\frac{2j+1}{m}}^{\frac{2j+3}{m}}\int_{\frac{2k+1}{n}}^{\frac{2k+3}{n}} f(t,z)dtdz, \end{aligned} \tag{10}$$

$(x, y) \in R_0^2$, where

$$a_{n,k}(x) := \frac{1}{\cosh nx}\frac{(nx)^{2k}}{(2k)!}, \tag{11}$$

$$b_{n,k}(x) := \frac{1}{1+\sinh nx}\frac{(nx)^{2k+1}}{(2k+1)!}, \qquad k \in N_0 \text{ and } n \in N, \tag{12}$$

and $\sinh x$, $\cosh x$ are the elementary hyperbolic functions.

It is easily verified that $L^{\{i\}}_{m,n}$, $m, n \in N$ and $1 \le i \le 4$, are linear positive operators well-defined in every space $C_{1;p_1,p_2}$ and $C_{2;q_1,q_2}$. These operators are some analogues of the operators $L_n^{\{i\}}$, $n \in N$, $1 \leq i \leq 4$, introduced and examined for functions f of one variable in the papers [1 – 3], i.e.

$$L_n^{\{1\}}(f;x) := \sum_{k=0}^{\infty} a_{n,k}(x) f\left(\frac{2k}{n}\right), \tag{13}$$

$$L_n^{\{2\}}(f;x) := \sum_{k=0}^{\infty} a_{n,k}(x) \frac{n}{2} \int_{\frac{2k}{n}}^{\frac{2k+2}{n}} f(t)dt, \tag{14}$$

$$L_n^{\{3\}}(f;x) := \frac{f(0)}{1+\sinh mx} + \sum_{k=0}^{\infty} b_{n,k}(x) f\left(\frac{2k+1}{n}\right) \tag{15}$$

$$L_n^{\{4\}}(f;x) := \frac{f(0)}{1+\sinh nx} + \sum_{k=0}^{\infty} b_{n,k}(x) \frac{n}{2} \int_{\frac{2k+1}{n}}^{\frac{2k+3}{n}} f(t)dt, \tag{16}$$

for $x \in R_0$ and $n \in N$.

Some approximation properties of $L_{m,n}^{\{i\}}$, $i = 1, 2$, for the functions $f \in C_{1;p_1,p_2}$ are given in [1]. In particular is proved the following

Theorem A. *Suppose that $f \in C_{1;p_1,p_2}$ with some $p_1, p_2 \in N_0$. Then there exists a positive constant M_1 (p_1, p_2) such that for all $m, n \in N$, $(x, y) \in R_0^2$ and $i = 1, 2$*

$$w_{p_1,p_2}(x,y)|L_{m,n}^{\{i\}}(f;x,y) - f(x,y)| \leq M_1(p_1,p_2)\omega\left(f; \sqrt{\frac{x+1}{m}}, \sqrt{\frac{y+1}{n}}\right),$$

where $\omega(f;t,s)$ is the modulus of continuity of f.

In this paper we shall give the Voronoskaya theorem for the operators $L_{n,n}^{\{i\}}$, $1 \leq i \leq 4$, in the spaces $C_{1;p_1,p_2}$ and $C_{2;q_1,q_2}$. In Sec. 2 we shall give some auxiliary properties of the operators (7)-(10) and (13)-(16).

Below by $M_k(a,b), k = 1, 2, \ldots$, we shall denote the suitable positive constants depending only on indicated parameters a, b.

2. AUXILIARY RESULTS

From (7)-(16) we deduce that if $f \in C_{1;p_1,p_2}$ or $f \in C_{2;q_1,q_2}$ and $f(x,y) = f_1(x) f_2(y)$, then

$$L_{m,n}^{\{i\}}(f_1(t)f_2(z);x,y) = L_m^{\{i\}}(f_1(t);x) L_n^{\{i\}}(f_2(z);y), \tag{17}$$

$$L_n^{\{i\}}(1;x) = 1, \qquad L_{m,n}^{\{i\}}(1;x,y) = 1, \tag{18}$$

for all $x \in R_0$, $(x,y) \in R_0^2$, $m, n \in N$ and $1 \leq i \leq 4$.

In the papers [1 – 3] were proved the following lemmas.

Lemma 1. ([1],[3]) *For every fixed $p \in N_0$ there exists a positive constant $M_2(p)$ such that for all $n \in N$ and $1 \le i \le 4$*

$$\sup_{x \in R_0} w_p(x) L_n^{\{i\}}\left(\frac{1}{w_p(t)};x\right) \le M_2(p). \tag{19}$$

Moreover there exists a positive constant $M_3(p)$ such that

$$w_p(x) L_n^{\{i\}}\left(\frac{(t-x)^2}{w_p(t)};x\right) \le M_3(p)\frac{x+1}{n}, \tag{20}$$

for all $x \in R_0$, $n \in N$ and $1 \le i \le 4$. □

Lemma 2. ([2],[4]) *For every fixed $q > 0$ and $r > q$ there exist a positive constant $M_4(q,r)$ and a natural number $n_0, n_0 > q\,(ln(r/q))^{-1}$, such that for all $n > n_0$ and $1 \le i \le 4$*

$$\sup_{x \in R_0} v_r(x) L_n^{\{i\}}\left(\frac{1}{v_q(t)};x\right) \le M_4(q,r). \tag{21}$$

Moreover there exists a positive constant $M_5(q,r)$ such that

$$v_r(x) L_n^{\{i\}}\left(\frac{(t-x)^2}{v_q(t)};x\right) \le M_5(q,r)\frac{x+1}{n} \tag{22}$$

for all $x \in R_0$, $n > n_0$ and $1 \le i \le 4$. □

Lemma 3. ([1],[3]) *For each $n \in N$, $x \in R_0$ and $1 \le i \le 4$ we have*

$$|L_n^{\{i\}}(t-x;x)| \le M_6(i) n^{-1},$$

where $M_6(i) \le 5$ for $1 \le i \le 4$. □

Lemma 4. ([2],[4]) *For every fixed $x_0 \in R_0$ and $1 \le i \le 4$*

$$\lim_{n \to \infty} n L_n^{\{i\}}(t-x_0;x_0) = \begin{cases} 0 & \text{if} \quad i = 1,3, \\ 1 & \text{if} \quad i = 2, \end{cases}$$

$$\lim_{n \to \infty} n L_n^{\{4\}}(t-x_0;x_0) = \begin{cases} 0 & \text{if} \quad x_0 = 0, \\ 1 & \text{if} \quad x_0 > 0, \end{cases}$$

$$\lim_{n \to \infty} n L_n^{\{i\}}((t-x_0)^2;x_0) = x_0$$

□

Lemma 5. ([5]) *For every fixed $x_0 \in R_0$ there exists a positive constant $M_7(x_0)$ such that for all $n \in N$ and $1 \le i \le 4$ we have*

$$L_n^{\{i\}}((t-x_0)^4;x_0) \le M_7(x_0) n^{-2}.$$

□

Using these lemmas and (1)-(18) we immediately obtain the following properties of the operators $L_{m,n}^{\{i\}}$.

Lemma 6. *For every fixed $p_1, p_2 \in N_0$ there exists a positive constant $M_8(p_1,p_2)$ such that for all $m,n \in N$ and $1 \le i \le 4$*

$$\left\| L_{m,n}^{\{i\}}\left(\frac{1}{w_{p_1,p_2}(t,z)};\cdot,\cdot\right)\right\|_{1;p_1,p_2} \le M_8(p_1,p_2)$$

and consequently

$$\|L_{m,n}^{\{i\}}(f;\cdot,\cdot)\|_{1;p_1,p_2} \le M_8(p_1,p_2)\|f\|_{1;p_1,p_2}$$

for every $f \in C_{1;p_1,p_2}$, $m,n \in N$ and $1 \le i \le 4$. This fact and (7) − (12) show that $L_{m,n}^{\{i\}}$, $m,n \in N$, $1 \le i \le 4$, is a linear positive operator from the space $C_{1;p_1,p_2}$ into $C_{1;p_1,p_2}$, $p_1,p_2 \in N_0$. □

Lemma 7. *For every fixed $q_1, q_2 > 0$ and $r_1 > q_1, r_2 > q_2$ there exist a positive constant $M_9^* \equiv M_9\ (q_1,\, q_2,\, r_1,\, r_2)$ and natural numbers m_0 and n_0,*

$$m_0 > q_1\left(\ln\frac{r_1}{q_1}\right)^{-1}, \qquad n_0 > q_2\left(\ln\frac{r_2}{q_2}\right)^{-1}, \tag{23}$$

such that for all $m > m_0$, $n > n_0$ and $1 \le i \le 4$ we have

$$\left\| L_{m,n}^{\{i\}}\left(\frac{1}{v_{q_1,q_2}(t,z)};\cdot,\cdot\right)\right\|_{2,r_1,r_2} \le M_9^*$$

Moreover for every $f \in C_{2;q_1,q_2}$, $m > m_0$, $n > n_0$ and $1 \le i \le 4$

$$\|L_{m,n}^{\{i\}}(f;\cdot,\cdot)\|_{2;r_1,r_2} \le M_9^*\|f\|_{2;q_1,q_2}. \tag{24}$$

The inequality (24) and (7) − (12) prove that $L_{m,n}^{\{i\}}$, $1 \le i \le 4$, is a linear positive operator from the space $C_{2;q_1,q_2}$, $q_1,q_2 > 0$, into $C_{2;r_1,r_2}$ with $r_1 > q_1$, $r_2 > q_2$ provided that $m > m_0$ and $n > n_0$. □

Lemma 8. *Suppose that (x_0,y_0) is a fixed point in R_0^2 and $\varphi\,(\cdot,\cdot)$ is a given function belonging to some space $C_{1;p_1,p_2}$, $p_1,p_2 \in N_0$, and $\varphi(x_0,y_0) = 0$. Then*

$$\lim_{n\to\infty} L_{n,n}^{\{i\}}(\varphi(t,z);x_0,y_0) = 0, \qquad i = 1,2,3,4. \tag{25}$$

Proof. Let $i = 1$. By (7) we have for every $n \in N$

$$L_{n,n}^{\{1\}}(\varphi(t,z),x_0,y_0) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{n,j}(x_0)a_{n,k}(y_0)\varphi\left(\frac{2j}{n},\frac{2k}{n}\right).$$

Choose $\varepsilon > 0$. By the properties of φ there exist two positive constants $\delta = \delta(\varepsilon)$ and M_{10} such that

$$w_{p_1,p_2}(t,z)|\varphi(t,z)| \le M_{10} \quad \text{for all } (t,z) \in R_0^2, \tag{26}$$

$$w_{p_1,p_2}(t,z)|\varphi(t,z)| < \frac{\varepsilon}{4M_8} \quad \text{for } |t-x_0| < \delta \text{ and } |z-y_0| < \delta, \tag{27}$$

where $M_8 \equiv M_8(p_1,p_2)$ is positive constant given in Lemma 6. Hence we can write for every $n \in N$

$$\begin{aligned} w_{p_1,p_2}(x_0,y_0)|L_{n,n}^{\{1\}}(\varphi(t,z);x_0,y_0)| &\le w_{p_1,p_2}(x_0,y_0)\Bigg\{\sum_{\left|\frac{2j}{n}-x_0\right|<\delta}\ \sum_{\left|\frac{2k}{n}-y_0\right|<\delta} + \\ &+ \sum_{\left|\frac{2j}{n}-x_0\right|<\delta}\ \sum_{\left|\frac{2k}{n}-y_0\right|\ge\delta} + \sum_{\left|\frac{2j}{n}-x_0\right|\ge\delta}\ \sum_{\left|\frac{2k}{n}-y_0\right|<\delta} + \\ &+ \sum_{\left|\frac{2j}{n}-x_0\right|\ge\delta}\ \sum_{\left|\frac{2k}{n}-y_0\right|\ge\delta}\Bigg\} a_{n,j}(x_0)a_{n,k}(y_0)\left|\varphi\left(\frac{2j}{n},\frac{2k}{n}\right)\right| := S_1+S_2+S_3+S_4. \end{aligned} \tag{28}$$

By (27) and Lemma 6 we get for every $n \in N$

$$\begin{aligned} S_1 &< \frac{\varepsilon}{4M_8} w_{p_1,p_2}(x_0,y_0)\sum_{j=0}^{8}\sum_{k=0}^{8} a_{n,j}(x_0)a_{n,k}(y_0)\left(w_{p_1,p_2}\left(\frac{2j}{n},\frac{2k}{n}\right)\right)^{-1} \le \\ &\le \frac{\varepsilon}{4M_8}\left\|L_{n,n}^{\{1\}}\left(\frac{1}{w_{p_1,p_2}(t,z)};\cdot,\cdot\right)\right\|_{1;p_1,p_2} \le \frac{\varepsilon}{4}. \end{aligned}$$

Since $\left|\frac{2k}{n}-x_0\right| \ge \delta$ implies $\left(\frac{2k}{n}-y_0\right)^2\delta^{-2} \ge 1$, we have by (26) and (2)

$$\begin{aligned} S_2 &\le M_{10}w_{p_1,p_2}(x_0,y_0)\sum_{\left|\frac{2j}{n}-x_0\right|<\delta}\ \sum_{\left|\frac{2k}{n}-y_0\right|\ge\delta} a_{n,j}(x_0)a_{n,k}(y_0)\left(w_{p_1,p_2}\left(\frac{2j}{n},\frac{2k}{n}\right)\right)^{-1} \le \\ &\le \frac{M_{10}}{\delta^2}\left\{w_{p_1}(x_0)\sum_{j=0}^{\infty}a_{n,j}(x_0)\left(w_{p_1}\left(\frac{2j}{n}\right)\right)^{-1}\right\}\left\{w_{p_2}(y_0)\sum_{\left|\frac{2k}{n}-y_0\right|\ge\delta}^{\infty}a_{n,k}(y_0)\frac{\left(\frac{2k}{n}-y_0\right)^2}{w_{p_2}\left(\frac{2k}{n}\right)}\right\} \le \\ &\le \frac{M_{10}}{\delta^2}\left\{w_{p_1}(x_0)L_n^{\{1\}}\left(\frac{1}{w_{p_1}(t)};x_0\right)\right\}\left\{w_{p_2}(y_0)L_n^{\{1\}}\left(\frac{(z-y_0)^2}{w_{p_2}(z)};y_0\right)\right\}. \end{aligned}$$

Using (19) and (20), we get for $n \in N$

$$S_2 \le M_{10}M_2(p_1)M_3(p_2)\frac{y_0+1}{n\delta^2} \equiv M_{11}(p_1,p_2)\frac{y_0+1}{n\delta^2}.$$

Analogously we obtain

$$S_3 \leq M_{12}(p_1, p_2)\frac{x_0+1}{n\delta^2},$$
$$S_4 \leq M_{13}(p_1, p_2)\frac{(x_0+1)(y+1)}{n^2\delta^4} \quad \text{for} \quad n \in N.$$

We observe that for given positive numbers ε, δ, $M_k(p_1, p_2)$ with $11 \leq k \leq 13$ and x_0, $y_0 \in R_0$ there exist natural numbers n_1, n_2, n_3 depending only on these parameters and such that

$$M_{11}(p_1, p_2)\frac{y_0+1}{n\delta^2} < \frac{\varepsilon}{4} \quad \text{for all} \quad n > n_1,$$
$$M_{12}(p_1, p_2)\frac{x_0+1}{n\delta^2} < \frac{\varepsilon}{4} \quad \text{for all} \quad n > n_2,$$
$$M_{13}(p_1, p_2)\frac{(x_0+1)(y_0+1)}{n^2\delta^4} < \frac{\varepsilon}{4} \quad \text{for all} \quad n > n_3.$$

Hence there exists natural number $n_4 = \max\{n_1, n_2, n_3\}$, such that for all $n_4 < n \in N$ holds

$$S_k < \frac{\varepsilon}{4}, \qquad 1 \leq k \leq 4,$$

which by (28) yields

$$w_{p_1,p_2}(x_0, y_0)|L_{n,n}^{\{1\}}(\varphi(t,z);x_0,y_0)| < \varepsilon \qquad \text{for all} \quad n > n_4.$$

This proves that

$$\lim_{n\to\infty} w_{p_1,p_2}(x_0, y_0)L_{n,n}^{\{1\}}(\varphi(t,z);x_0,y_0) = 0$$

and further by (2)

$$\lim_{n\to\infty} L_{n,n}^{\{1\}}(\varphi(t,z);x_0,y_0) = 0.$$

Thus the proof of (25) for $i=1$ is completed. The proof of (25) for $i = 2, 3, 4$ is analogous.
□

Similarly we can prove the following

Lemma 9. *Suppose that (x_0, y_0) is a fixed point in R_0^2 and $\varphi(\cdot,\cdot)$ is a given function belonging to some space $C_{2;q_1,q_2}$, $q_1, q_2 > 0$, and $\varphi(x_0, y_0) = 0$. Then the statement (25) holds.*

3. THE VORONOVSKAYA THEOREM

Now using the above lemmas and denoting by $f_x' \equiv \frac{\partial f}{\partial x}, f_{xx}'' \equiv \frac{\partial^2 f}{\partial x^2}$, we shall prove the main theorems.

Theorem 1. *Assume that $f \in C^2_{1;p_1,p_2}$ with some p_1, $p_2 \in N_0$. Then for every $(x,y) \in R_0^2$ we have*

$$\lim_{n\to\infty} n\left\{L_{n,n}^{\{i\}}(f;x,y) - f(x,y)\right\} = \tag{29}$$

$$= \begin{cases} \frac{x}{2}f''_{xx}(x,y) + \frac{y}{2}f''_{yy}(x,y) & \text{if} \quad i = 1,3, \\ f'_x(x,y) + f'_y(x,y) + \frac{x}{2}f''_{xx}(x,y) + \frac{y}{2}f''_{yy}(x,y) & \text{if} \quad i = 2, \end{cases}$$

$$\lim_{n\to\infty} n\{L_{n,n}^{\{4\}}(f;x,y) - f(x,y)\} = \tag{29'}$$

$$= \begin{cases} 0 & \text{if} \quad x = y = 0, \\ f'_x(x,y) + f'_y(x,y) + \frac{x}{2}f''_{xx}(x,y) + \frac{y}{2}f''_{yy}(x,y) & \text{if} \quad x^2 + y^2 > 0. \end{cases}$$

Proof. Let (x_0, y_0) be a fixed point in R_0^2. Then by the Taylor formula for $f \in C^2_{1;p_1,p_2}$ we have for $(t,z) \in R_0^2$

$$f(t,z) = f(x_0,y_0) + f'_x(x_0,y_0)(t-x_0) + f'_y(x_0,y_0)(z-y_0) +$$

$$+\frac{1}{2}\{f''_{xx}(x_0,y_0)(t-x_0)^2 + 2f''_{xy}(x_0,y_0)(t-x_0)(z-y_0) + f''_{yy}(x_0,y_0)(z-y_0)^2\} +$$

$$+\psi(t,z;x_0,y_0)\sqrt{(t-x_0)^4 + (z-y_0)^4},$$

where $\psi(\cdot,\cdot;x_0,y_0) \in C_{1;p_1,p_2}$ and $\lim_{\substack{t\to x_0 \\ z\to y_0}} \psi(t,z;x_0,y_0) = 0$. From this and by (18) and (17) we get for every $n \in N$ and $1 \le i \le 4$

$$L_{n,n}^{\{i\}}(f(t,z);x_0,y_0) = f'_x(x_0,y_0) + f'_x(x_0,y_0)L_n^{\{i\}}(t-x_0,x_0) + \tag{30}$$

$$+f'_y(x_0,y_0)L_n^{\{i\}}(z-y_0;y_0) + \frac{1}{2}\{f''_{xx}(x_0,y_0)L_n^{\{i\}}((t-x_0)^2;x_0) +$$

$$+2f''_{xy}(x_0,y_0)L_n^{\{i\}}(t-x_0;x_0)L_n^{\{i\}}(z-y_0;y_0) + f''_{yy}(x_0,y_0)L_n^{\{i\}}((z-y_0)^2;y_0)\} +$$

$$+L_{n,n}^{\{i\}}(\psi(t,z;x_0,y_0)\sqrt{(t-x_0)^4 + (z-y_0)^4};x_0,y_0).$$

Using the Hölder inequality, we get for $n \in N$

$$|L_{n,n}^{\{i\}}(\psi(t,z;x_0,y_0)\sqrt{(t-x_0)^4 + (z-y_0)^4};x_0,y_0)| \le . \tag{31}$$

$$\le \{L_{n,n}^{\{i\}}(\psi^2(t,z;x_0,y_0);x_0,y_0)\}^{\frac{1}{2}}\{L_{n,n}^{\{i\}}((t-x)^4 + (z-y_0)^4;x_0,y_0)\}^{\frac{1}{2}} \quad \text{if} \quad i = 1,2,$$

$$|L_{n,n}^{\{i\}}(\psi(t,z;x_0,y_0)\sqrt{(t-x_0)^4 + (z-y_0)^4};x_0,y_0)| \le \tag{32}$$

$$\le 4\{L_{n,n}^{\{i\}}(\psi^2(t,z,x_0,y_0),x_0,y_0)\}^{\frac{1}{2}}\{L_{n,n}^{\{i\}}((t-x)^4 + (z-y_0)^4,x_0,y_0)\}^{\frac{1}{2}} \quad \text{if} \quad i = 3,4.$$

But by the linearity of $L_{n,n}^{\{i\}}$ and by (17), (18) and lemma 5 we see that there exists a positive constant $M_{14}(x_0,y_0)$ such that for all $n \in N$ and $1 \le i \le 4$ holds

$$L_{n,n}^{\{i\}}((t-x_0)^4 + (z-y_0)^4;x_0,y_0) = \tag{33}$$

$$= L_n^{\{i\}}((t-x_0)^4;x_0) + L_n^{\{i\}}((z-y_0)^4;y_0) \le M_{14}(x_0,y_0)n^{-2}.$$

Moreover, by the properties of $\psi(\cdot,\cdot;x_0,y_0)$ we apply Lemma 8 for $\varphi(t,z) \equiv \psi^2\,(t,z,x_0,y_0)$. Hence,

$$\lim_{n\to\infty} L_{n,n}^{\{i\}}(\psi^2(t,z;x_0,y_0);x_0,y_0) = 0, \qquad 1 \le i \le 4. \tag{34}$$

Using (33) and (34) to (31) and (32) we get

$$\lim_{n\to\infty} nL_{n,n}^{\{i\}}(\psi(t,z;x_0,y_0)\sqrt{(t-x_0)^4+(z-y_0)^4};x_0,y_0) = 0 \tag{35}$$

for $i = 1,2,3,4$. Next, using (35) and Lemma 3 and Lemma 4 to (30), we obtain

$$\lim_{n\to\infty} n\{L_{n,n}^{\{i\}}(f(t,z);x_0,y_0) - f(x_0,y_0)\} = \frac{x_0}{2}f_{xx}^{''}(x_0,y_0) + \frac{y_0}{2}f_{yy}^{''}(x_0,y_0) \quad \text{for} \quad i = 1,3$$

and

$$\lim_{n\to\infty} n\{L_{n,n}^{\{i\}}(f(t,z);x_0,y_0) - f(x_0,y_0)\} =$$

$$f_x'(x_0,y_0) + f_y'(x_0,y_0) + \frac{x_0}{2}f_{xx}^{''}(x_0,y_0) + \frac{y_0}{2}f_{yy}^{''}(x_0,y_0) \quad \text{for} \quad i = 2,4.$$

Analogously, using (35) and Lemmas 3 and 4 to (30), we obtain (29′). Since (x_0,y_0) is arbitrary fixed point we see that the proof of (29) and (29’) is completed. ■

Arguing as in the proof of Theorem 1 and using Lemmas 3-5 and Lemma 9, we can prove the following

Theorem 2. *Suppose that* $f \in C^2_{2;q_1,q_2}$ *with some* $q_1,q_2 > 0$. *Then the statements* (29) *and* (29′) *hold for every* $(x,y) \in R_0^2$ *and* $1 \le i \le 4$.

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Received September 30, 1996 and in revised form June, 1997
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