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THE SIMPLICIAL COMPLEX OF GRAPHS OF GROUPS

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Abstract. *Graphs of groups with length functions are defined to parametrize simplicial G-trees. Collapses of graphs of groups define a partial order and a simplicial structure on the set of simplicial G-trees. There is a bijective continuous map from this simplicial complex to the space of non-abelian simplicial projective translation length functions of G. Any two small splittings of* F_n *are connected by a sequence of blow ups and collapses.*

0. INTRODUCTION

Let G be a group with actions on trees. The G-trees can be parametrized by the space of the translation length functions. The outer automorphism group Out(G) acts on it. Culler and Vogtmann [5] defined the outer space, which is the space of free F_n-actions on simplicial trees. From the Bass-Serre theory, a simplicial G-tree is encoded by a quotient graph of groups. We construct a simplicial complex $C(G)$. Two quotient graphs of groups are equivalent if they have the same translation length function up to a scalar factor. The vertices of $C(G)$ are equivalence classes of quotient graphs of groups of G-trees. An elementary collapse of a graph of groups $\mathfrak{A}$ is a graph of groups obtained from $\mathfrak{A}$ by squeezing an edge to a vertex with the amalgamated free product or the HNN-extension as the new vertex group. The composition of elementary collapses is called a collapse. This defines a partial order and a simplicial structure on $C(G)$. Two vertices are adjacent if one is a collapse of the other. The group Out(G) acts simplicially on $C(G)$. The stabilizers of vertices are explicitly described in Theorem 8.1 of [2]. If G is a finitely presented group, the subcomplex of reduced and small graphs of groups is finite dimensional. We prove that there is a continuous bijective map between equivalence classes of minimal non-abelian graphs of groups and non-abelian projective translation length functions. Finally, we study the most important special case $G = F_n$, the free group of rank n. We prove that any pair of small splittings of F_n are connected by a sequence of blow ups and collapses.

1. THE SIMPLICIAL COMPLEX $C(G)$

1.1. A graph of groups with a length function. Fix a group G. A G-tree is encoded by a quotient graph of groups. In this paper all G-trees and all graphs of groups are minimal and non-trivial. To define equivalent graphs of groups, we need to specify their translation length functions. In order to specify the translation length functions, we need to specify the isomorphism between G and the fundamental group of the quotient graph of groups. This leads to the following definition. By a *graph of groups with a length function* $\mathfrak{A}(L, T, g)$ we

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understand:

(i) a graph of groups $\mathfrak{A} = (A, \mathcal{A})$,
(ii) a map $L : E(A) \longrightarrow \mathbf{R}^+$,
(iii) a maximal subtree T of A,
(iv) a map $g : E(A) \longrightarrow G$.

The above data are assumed to satisfy the following conditions.

(a) The vertex groups $\mathcal{A}_a$ ($a \in V(A)$) and the edge groups $\mathcal{A}_e$ ($e \in E(A)$) are subgroups of G.
(b) A is a simplicial graph with the edge-length function L.
(c) $g(e) = 1$ for $e \in E(T)$.
(d) If $e \in E(A) - E(T)$, exactly one of $g(e)$ and $g(\overline{e})$ is non-trivial.
(e) $\alpha_e = \mathrm{ad}(g(e))$.
(f) The homomorphism $\psi^{\mathfrak{A}} : \pi(\mathfrak{A}) \longrightarrow G$ defined by

$$\psi^{\mathfrak{A}}(s) = s \text{ for } s \in \mathcal{A}_a,$$
$$\psi^{\mathfrak{A}}(e) = g(e)g(\overline{e})^{-1} \text{ for } e \in E(A),$$

restricts to an isomorphism $\psi_a^{\mathfrak{A}} : \pi_1(\mathfrak{A}, a) \longrightarrow G$.

In this paper, all graphs of groups are graphs of groups with length functions. If there is no ambiguity, we will simply call a graph of groups with a length function a graph of groups. For a graph of groups $\mathfrak{A}(L, T, g)$, let $l_{\mathfrak{A}}$ be the translation length function of the $\pi_1(\mathfrak{A}, a)$-action on the universal cover tree $\widetilde{(\mathfrak{A}, a)}$ (cf. (1.16) of [1]). We then call $l_{\mathfrak{A}} \circ (\psi_a^{\mathfrak{A}})^{-1}$ the *translation length function* of $\mathfrak{A}(L, T, g)$.

1.2. Equivalence of graphs of groups. Let $\Phi = (\phi, \{\gamma\}) : \mathfrak{A} \longrightarrow \mathfrak{B}$ be an isomorphism of graphs of groups (cf. Definition 2.1 of [1]). Let L_A and L_B be positive edge-length functions of A and B, respectively. If $\phi : A \longrightarrow B$ is an isometry with respect to the shortest-path metrics, then $\Phi : \mathfrak{A}(L_A, T_A, g_A) \longrightarrow \mathfrak{B}(L_B, T_B, g_B)$ is called *an isomorphism of graphs of groups with length functions*.

Multiplying the edge-length function L by a constant c yields another edge-length function cL. Graphs of groups $\mathfrak{A}(L, T, g)$ and $\mathfrak{A}'(L', T', g')$ are *equivalent* if there is an isomorphism $\Phi : \mathfrak{A}(cL, T, g) \longrightarrow \mathfrak{A}'(L', T', g')$ for some $c > 0$, such that the following diagram commutes

$$\begin{array}{ccc} G & = & G \\ {\scriptstyle \psi_a^{\mathfrak{A}}}\uparrow & & \uparrow{\scriptstyle \psi_{a'}^{\mathfrak{A}'}} \\ \pi_1(\mathfrak{A}, a) & \xrightarrow[\Phi_a]{} & \pi_1(\mathfrak{A}', a') \end{array}$$

where $\psi_a^{\mathfrak{A}}$ and $\psi_{a'}^{\mathfrak{A}'}$ are defined by (1.1)(f). The isomorphism Φ is called *a natural isomorphism*. By 2.9 of [1],

$$\Phi_a = \mathrm{ad}(\gamma) \circ \delta\Phi_a, \text{ where } \gamma \in \pi_1(\mathfrak{A}', a') \text{ and } \delta\Phi = (\phi, \{\phi_a\}, \{\phi_e\}, \{\delta_e\}). \qquad (1.2.1)$$

For the simplicity of notations, we assume that $\Phi = \delta\Phi$ if there is no ambiguity. The following theorem explains the above definition.

1.3. Theorem (Theorem 1.6 of [6]). Let X and Y be minimal non-abelian simplicial G-trees with translation length functions l_X and l_Y, respectively. Form graphs of groups

$$G\backslash\backslash X = \mathfrak{A}(L_A, T_A, g_A),$$
$$G\backslash\backslash Y = \mathfrak{B}(L_B, T_B, g_B).$$

Let $\varphi \in \mathrm{Aut}(G)$. The following conditions are equivalent.

(a) $l_Y(\varphi(g)) = l_X(g)$ for all $g \in G$.

(b) There exists an isomorphism $\Phi : \mathfrak{A}(L_A, T_A, g_A) \longrightarrow \mathfrak{B}(L_B, T_B, g_B)$ such that the following diagram commutes

$$\begin{array}{ccc} G & \xrightarrow{\ \varphi\ } & G \\ {\scriptstyle \psi_a^{\mathfrak{A}}}\uparrow & & \uparrow{\scriptstyle \psi_{\phi(a)}^{\mathfrak{B}}} \\ \pi_1(\mathfrak{A}, a) & \xrightarrow[\Phi_a]{} & \pi_1(\mathfrak{B}, \phi(a)) \end{array}$$

where $\psi_a^{\mathfrak{A}}$ and $\psi_{\phi(a)}^{\mathfrak{B}}$ are isomorphisms defined in (1.1) (f). ∎

1.4. Corollary. Non-abelian graphs of groups $\mathfrak{A}(L, T, g)$ and $\mathfrak{A}'(L', T', g')$ are equivalent if and only if their translation length functions are the same up to a scalar factor, i.e. $l_{\mathfrak{A}} \circ (\psi_a^{\mathfrak{A}})^{-1} = c l_{\mathfrak{A}'} \circ (\psi_{a'}^{\mathfrak{A}'})^{-1}$ for some $c > 0$. ∎

1.5. Collapses of graphs of groups. Let $\mathfrak{A}(L_A, T_A, g_A)$ be a graph of groups. Let $e_1 \in E(A)$. An *elementary collapse* of $\mathfrak{A}(L_A, T_A, g_A)$ is a non-trivial graph of groups $\mathfrak{B}(L_B, T_B, g_B)$ obtained from $\mathfrak{A}(L_A, T_A, g_A)$ by squeezing e_1 to a vertex $\tilde{b}$. Let B' be a graph consisting of one geometric edge, e_1. Then B' is a subgraph of A. Let $\mathfrak{B}'$ be graph of groups defined as follows

$$\mathcal{B}'_x = \mathcal{A}_x \text{ for } x = \partial_0 e_1, \partial_1 e_1, e_1, \text{ and } \overline{e}_1, \text{ and } \mathcal{B}'_x = \alpha_x \text{ for } x = e_1, \text{ and } \overline{e}_1.$$

Thus, $\mathfrak{B}' = \mathfrak{A}|_{B'}$ is the subgraph of groups of $\mathfrak{A}$ (Corollary 1.14 and (2.15) of [1]). Note that

$$\pi_1(\mathfrak{B}', \underset{0}{\partial} e_1) = \begin{cases} \mathcal{A}_{\partial_0 e_1} *_{\mathcal{A}_{e_1}} \mathcal{A}_{\partial_1 e_1} & \text{if } \partial_0 e_1 \neq \partial_1 e_1, \\ \langle \mathcal{A}_{\partial_0 e_1}, e_1 | e_1 \alpha_{\overline{e}_1}(s) \overline{e}_1 = \alpha_{e_1}(s), s \in \mathcal{A}_{e_1} \rangle & \text{if } \partial_0 e_1 = \partial_1 e_1. \end{cases} \tag{1.5.1}$$

It follows from (2.15) of [1] that $i : \pi_1(\mathfrak{B}', \partial_0 e_1) \longrightarrow \pi_1(\mathfrak{A}, \partial_0 e_1)$ is injective. Let $h_{\partial_0 e_1}$ be the shortest edge path in T_A from a_0 to $\partial_0 e_1$, then the following composition is a injective homomorphism.

$$\pi_1(\mathfrak{B}', \partial_0 e_1) \xrightarrow{\ i\ } \pi_1(\mathfrak{A}, \partial_0 e_1) \xrightarrow{\ \mathrm{ad}(h_{\partial_0 e_1})\ } \pi_1(\mathfrak{A}, a_0) \xrightarrow{\ \psi_{a_0}^{\mathfrak{A}}\ } G.$$

Let the vertex groups $\mathcal{B}_a$ ($a \in V(B)$) be defined by

$$\mathcal{B}_a = \begin{cases} \mathcal{A}_a & \text{if } a \neq \tilde{b}, \\ (\psi_{a_0}^{\mathfrak{A}} \circ \mathrm{ad}(h_{\partial_0 e_1}) \circ i)(\pi_1(\mathfrak{B}', \partial_0 e_1)) & \text{if } a = \tilde{b}. \end{cases} \tag{1.5.2}$$

The edge length function is given by $L_B = L_A|_{E(A)-\{e_1,\bar{e}_1\}}$. The maximal subtree T_B is either $T_A - \{e_1,\bar{e}_1\}$ if $e_1 \in E(T_A)$ or T_A if $e_1 \notin E(T_A)$. Finally, $g_B = g_A|_{E(B)}$. By Proposition 2.8 of [6], $\mathfrak{B}(L_B, T_B, g_B)$ is a graph of groups. While $\mathfrak{B}$ is called an elementary collapse of $\mathfrak{A}$, $\mathfrak{A}$ is called an *elementary blow up* of $\mathfrak{B}$. The composition of elementary collapses is called a *collapse*.

1.6. Simplicial complex $C(G)$**.** Collapses define a partial order and a simplicial structure on the set of equivalence classes of graphs of groups. Namely, $\overline{\mathfrak{A}} < \overline{\mathfrak{B}}$ if $\mathfrak{A}$ is a collapse of $\mathfrak{B}$, and $(\overline{\mathfrak{A}_0}, \ldots, \overline{\mathfrak{A}_n})$ is an n-simplex if $\overline{\mathfrak{A}_0} < \ldots < \overline{\mathfrak{A}_n}$, where $\overline{\mathfrak{A}_i}$ is the equivalence class of $\mathfrak{A}_i$. Let $C(G)$ be the simplicial complex.

1.7. $\mathrm{Aut}(G)$**-action on** $C(G)$**.** Let $\mathfrak{A}(L,T,g)$ be a graph of groups, and let $\varphi \in \mathrm{Aut}(G)$. Define a graph of groups $\varphi(\mathfrak{A}(L,T,g)) = \mathfrak{A}_\varphi(L_\varphi, T_\varphi, g_\varphi)$ as follows:

(i) $A_\varphi = A$ and $T_\varphi = T$,
(ii) $(\mathcal{A}_\varphi)_a = \varphi(\mathcal{A}_a)$ and $(\mathcal{A}_\varphi)_e = \varphi(\mathcal{A}_e)$ for $a \in V(A)$ and $e \in E(A)$,
(iii) $g_\varphi(e) = \varphi(g(e))$ for $e \in E(A)$, and $L_\varphi = L$.

By (1.2), inner automorphisms of G fix every vertex of $C(G)$. So $\mathrm{Out}(G)$ acts on $C(G)$.

1.8. Proposition Let $\mathfrak{A}(L,T,g)$ and $\mathfrak{A}'(L',T',g')$ be graphs of groups.

(a) $\varphi(\mathfrak{A}(L,T,g))$ is a graph of groups.
(b) $\overline{\mathfrak{A}(L,T,g)} = \overline{\mathfrak{A}'(L',T',g')}$ if and only if $\overline{\varphi(\mathfrak{A}(L,T,g))} = \overline{\varphi(\mathfrak{A}'(L',T',g'))}$.
(c) $\overline{\mathfrak{A}(L,T,g)} < \overline{\mathfrak{A}'(L',T',g')}$ if and only if $\overline{\varphi(\mathfrak{A}(L,T,g))} < \overline{\varphi(\mathfrak{A}'(L',T',g'))}$.

In particular, $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$ act simplicially on $C(G)$.

Proof. Let $\Phi_\varphi = (\phi, \{\phi_a\}, \{\phi_e\}, \{\delta_e\}) : \mathfrak{A}(L,T,g) \longrightarrow \varphi(\mathfrak{A}(L,T,g))$ be defined as follows: $\phi = \mathrm{Id} : A \longrightarrow A$, $\phi_a = \varphi : \mathcal{A}_a \longrightarrow (\mathcal{A}_\varphi)_a = \varphi(\mathcal{A}_a)$, $\phi_e = \varphi : \mathcal{A}_e \longrightarrow (\mathcal{A}_\varphi)_e = \varphi(\mathcal{A}_e)$, and $\delta_e = 1$. Since $(\alpha_\varphi)_e \circ \varphi = \varphi \circ \alpha_e$, Φ_φ is an isomorphism, and the following diagram commutes.

$$\begin{array}{ccc} G & \xrightarrow{\ \varphi\ } & G \\ {\scriptstyle \psi_a^{\mathfrak{A}}}\uparrow & & \uparrow{\scriptstyle \psi_a^{\varphi(\mathfrak{A})}} \\ \pi_1(\mathfrak{A},a) & \xrightarrow[\ \Phi_\varphi\ ]{} & \pi_1(\varphi(\mathfrak{A}),a) \end{array} \tag{1.8.1}$$

Since $\psi_a^{\mathfrak{A}}$ is an isomorphism, $\psi_a^{\varphi(\mathfrak{A})}$ is also an isomorphism. It is easy to check that $\varphi(\mathfrak{A}(L,T,g))$ satisfies (1.1)(a)–(f), so $\varphi(\mathfrak{A}(L,T,g))$ is a graph of groups, which is (a).

Suppose that $\mathfrak{A}(L,T,g)$ and $\mathfrak{A}'(L',T',g')$ are equivalent. There exists a natural isomorphism $\Phi = (\phi, \{\phi_a\}, \{\phi_e\}, \{\delta_e\}) : \mathfrak{A}(L,T,g) \longrightarrow \mathfrak{A}'(L',T',g')$. Define the morphism

$$\Phi' = (\phi', \{\phi'_a\}, \{\phi'_e\}, \{\delta'_e\}) : \varphi(\mathfrak{A}(L,T,g)) \longrightarrow \varphi(\mathfrak{A}'(L',T',g'))$$

as follows:

(i) $\phi' = \phi : A \longrightarrow A'$,
(ii) $\phi'_a = \varphi \circ \phi_a \circ \varphi^{-1} : \varphi(\mathcal{A}_a) \longrightarrow \varphi(\mathcal{A}'_{\phi(a)})$,
(iii) $\phi'_e = \varphi \circ \phi'_a \circ \varphi^{-1} : \varphi(\mathcal{A}_e) \longrightarrow \varphi(\mathcal{A}'_{\phi(e)})$,
(iv) $\delta'_e = \varphi(\delta_e)$.

Note that

$$(\alpha_\varphi)_e(\varphi(s)) = \varphi(g(e))^{-1}\varphi(s)\varphi(g(e)) = \varphi(\alpha_e(s)) \text{ and } \varphi \circ \mathrm{ad}(\delta_e) = \mathrm{ad}(\delta'_e) \circ \varphi.$$

Thus,

$$\begin{aligned}\phi'_a \circ (\alpha_\varphi)_e &= \varphi \circ \phi_a \circ \varphi^{-1} \circ (\alpha_\varphi)_e \\ &= \varphi \circ \phi_a \circ \alpha_e \circ \varphi^{-1} \\ &= \varphi \circ \mathrm{ad}(\delta_e) \circ \alpha'_{\phi(e)} \circ \phi_e \circ \varphi^{-1} \\ &= \mathrm{ad}(\delta'_e) \circ (\alpha'_\varphi)_{\phi(e)} \circ \phi'_e\end{aligned}$$

and Φ' is an isomorphism. It is easy to see that

$$(\Phi_\varphi)_{a'} \circ \Phi_a = \Phi'_a \circ (\Phi_\varphi)_a. \tag{1.8.2}$$

Note that

$$\begin{aligned}&(1.8.3) & \psi^{\mathfrak{A}'}_{a'} \circ \Phi_a &= \psi^{\mathfrak{A}}_a \quad (\Phi_a \text{ is a natural isomorphism}),\\ && \varphi \circ \psi^{\mathfrak{A}}_a &= \psi^{\varphi(\mathfrak{A})}_a \circ (\Phi_\varphi)_a \quad (\text{cf. } 1.8.1),\\ && \varphi \circ \psi^{\mathfrak{A}'}_{a'} &= \psi^{\varphi(\mathfrak{A}')}_{a'} \circ (\Phi_\varphi)_{a'} \quad (\text{cf. } 1.8.1),\end{aligned}$$

By (1.8.2) and (1.8.3),

$$\begin{aligned}\psi^{\varphi(\mathfrak{A}')}_{a'} \circ \Phi'_{a'} &= \varphi \circ \psi^{\mathfrak{A}'}_{a'} \circ (\Phi_\varphi)^{-1}_{a'} \circ (\Phi_\varphi)_{a'} \circ \Phi_a \circ (\Phi_\varphi)^{-1}_a \\ &= \varphi \circ \psi^{\mathfrak{A}'}_{a'} \circ \Phi_a \circ (\Phi_\varphi)^{-1}_a \\ &= \varphi \circ \psi^{\mathfrak{A}}_a \circ (\Phi_\varphi)^{-1}_a = \psi^{\varphi(\mathfrak{A})}_a\end{aligned}$$

and $\Phi'_{a'}$ is a natural isomorphism. So $\varphi(\mathfrak{A}(L, T, g))$ and $\varphi(\mathfrak{A}'(L', T', g'))$ are equivalent, whence (b) holds.

It suffices to show (c) for an elementary collapse. Suppose that $\mathfrak{B}(L_B, T_B, g_B)$ is an elementary collapse of $\mathfrak{A}(L_A, T_A, g_A)$ by squeezing e_1. Note that the following diagram commutes.

$$\begin{array}{ccccccc}\pi_1(\mathfrak{B}', \partial_0 e_1) & \xrightarrow{i} & \pi_1(\mathfrak{A}, \partial_0 e_1) & \xrightarrow{\mathrm{ad}(h_{\partial_0 e_1})} & \pi_1(\mathfrak{A}, a_0) & \xrightarrow{\psi^{\mathfrak{A}}_{a_0}} & G \\ {\scriptstyle \Phi_\varphi}\downarrow & & \downarrow{\scriptstyle \Phi_\varphi} & & \downarrow{\scriptstyle \Phi_\varphi} & & \downarrow{\scriptstyle \varphi} \\ \pi_1(\varphi(\mathfrak{B}'), \partial_0 e_1) & \xrightarrow{i} & \pi_1(\varphi(\mathfrak{A}), \partial_0 e_1) & \xrightarrow{\mathrm{ad}(h_{\partial_0 e_1})} & \pi_1(\varphi(\mathfrak{A}), a_0) & \xrightarrow{\psi^{\varphi(\mathfrak{A})}_{a_0}} & G\end{array}$$

By (1.5.1) and (1.5.2), $\varphi(\mathfrak{B}(L_B, T_B, g_B))$ is an elementary collapse of $\varphi(\mathfrak{A}(L_A, T_A, g_A))$ by squeezing e_1. Thus, (c) holds. ∎

1.9. Vertex stabilizers. By Theorem 1.3, an automorphism fixes a translation length function if and only if, up to conjugacy, the automorphism is induced by an automorphism of the corresponding quotient graph of groups. A filtration of the stabilizer is given in Theorem 8.1 of [1], and successive quotients are described explicitly. To apply these results to the Aut(G)-action on $C(G)$, we need to show that the stabilizer of a translation length function is the stabilizer of the graph of groups.

1.10. Lemma Let $\varphi \in \mathrm{Aut}(G)$, and let $\mathfrak{A}(L, T, g)$ be a non-abelian graph of groups. Then $\varphi(\mathfrak{A}(L, T, g))$ and $\mathfrak{A}(L, T, g)$ are equivalent if and only if $l \circ \varphi = cl$ for some $c > 0$, where $l = l_{\mathfrak{A}} \circ (\psi_a^{\mathfrak{A}})^{-1}$.

Proof For $g \in G$, suppose that $(\psi_a^{\mathfrak{A}})^{-1}(g) = h(g_0 e_1 \dots e_n g_n)h^{-1} \in \pi_1(\mathfrak{A}, a)$, where $e_1, \dots, e_n$ are the edges on an edge loop at a, $\partial_0 e_i = a_{i-1} = \partial_1 e_{i-1}$, $g_i \in \mathcal{A}_{a_i}$, $h \in \pi_1(\mathfrak{A}, a)$. Furthermore, we can assume that $g_0 e_1 \dots e_n g_n$ is cyclically reduced. By Proposition 8.3 of [4],

$$l(g) = l_{\mathfrak{A}} \circ (\psi_a^{\mathfrak{A}})^{-1}(g) = l_{\mathfrak{A}}(h(g_0 e_1 \dots e_n g_n)h^{-1}) = \sum_{i=1}^{n} L(e_i). \tag{1.10.1}$$

Suppose that $\varphi(\mathfrak{A}(L, T, g))$ and $\mathfrak{A}(L, T, g)$ are equivalent. Let $\Phi : \mathfrak{A}(cL, T, g) \longrightarrow \varphi(\mathfrak{A}(L, T, g))$ be the natural isomorphism. Then the following diagram commutes.

$$\begin{array}{ccccc}
G & \xrightarrow{\varphi} & G & = & G \\
\uparrow{\scriptstyle \psi_a^{\mathfrak{A}}} & & \uparrow{\scriptstyle \psi_a^{\varphi(\mathfrak{A})}} & & \uparrow{\scriptstyle \psi_a^{\mathfrak{A}}} \\
\pi_1(\mathfrak{A}, a) & \xrightarrow[(\Phi_\varphi)_a]{} & \pi_1(\varphi(\mathfrak{A}), a) & \xleftarrow[\Phi_a]{} & \pi_1(\mathfrak{A}, a)
\end{array}$$

Since $\Phi^{-1} \circ \Phi_\varphi : \mathfrak{A}(cL, T, g) \longrightarrow \mathfrak{A}(L, T, g)$ is an isomorphism, it follows from Theorem 1.3 that $l \circ \varphi = cl$.

Suppose that $l \circ \varphi = cl$. It follows from Theorem 1.3 and (1.8.1) that the following diagram commutes.

$$\begin{array}{ccccc}
G & \xleftarrow{\varphi} & G & \xrightarrow{\varphi} & G \\
\uparrow{\scriptstyle \psi_a^{\mathfrak{A}}} & & \uparrow{\scriptstyle \psi_a^{\mathfrak{A}}} & & \uparrow{\scriptstyle \psi_a^{\varphi(\mathfrak{A})}} \\
\pi_1(\mathfrak{A}, a) & \xleftarrow[\Phi_a]{} & \pi_1(\mathfrak{A}, a) & \xrightarrow[\Phi_\varphi]{} & \pi_1(\varphi(\mathfrak{A}), a)
\end{array}$$

Then $\Phi_\varphi \circ \Phi^{-1} : \mathfrak{A}(cL, T, g) \longrightarrow \varphi(\mathfrak{A}(L, T, g))$ is a natural isomorphism. By Corollary 1.4 $\mathfrak{A}(L, T, g)$ and $\varphi(\mathfrak{A}(L, T, g))$ are equivalent. ∎

By Theorem 8.1 of [2], the vertex stabilizers of the Aut(G)-action and the Out(G)-action on $C(G)$ can be described explicitly.

Bestvina and Feighn [3] proved that if G is a finitely presented group and T is a reduced G-tree with small edge stabilizers, then there is a uniform upper bound for the number of vertices and edges of the quotient graph $T\,/\,G$. A minimal simplicial G-tree is *reduced* if for every vertex of valence two, the vertex group properly contains both edge groups incident to it, provided the two edges are distinct. A group is *small* if it does not contain a free group of rank two. A G-tree and a graph of groups are *small* if all the edge groups are small. A graph of groups is *reduced* if the corresponding G-tree is reduced.

1.11. Corollary If G is a finitely presented group, the subcomplex of reduced and small graphs of groups is finite dimensional. The groups Aut(G) and Out(G) act simplicially on it.

Proof. A collapse of a reduced and small graph of groups is also reduced and small. So the equivalence classes of reduced and small graphs of groups form a subcomplex of $C(G)$. If the reduced and small graphs of groups have at most m edges, then the largest linear ordered subset of reduced and small graphs of groups contains m elements. Therefore the subcomplex is $(m-1)$-dimensional. If $\mathfrak{A}(L, T, g)$ is reduced and small, so is $\varphi(\mathfrak{A}(L, T, g))$. Therefore Aut($G$) and Out($G$) act on the subcomplex.

2. THE RELATION BETWEEN $C(G)$ AND PL(G)

2.1. Let PL(G) be the projective space of translation length functions of non-abelian G-actions on real trees. Let $(\overline{\mathfrak{A}_0}, \ldots, \overline{\mathfrak{A}_n})$ be an n-simplex of $C(G)$. Suppose that $\mathfrak{A}_i(L_i, T_i, g_i)$ is representative of $\overline{\mathfrak{A}_i}$. Let $\sum_{i=0}^{n} \lambda_i \overline{\mathfrak{A}_i}$ be a point in the n-simplex, where $\sum_{i=0}^{n} \lambda_i = 1$, $0 \leq \lambda_i \leq 1$. Suppose that L_i is normalized for all i such that the sum of the lengths of all geometric edges is one. Since $\mathfrak{A}_{i-1}$ is a collapse of $\mathfrak{A}_i$, $E(A_{i-1}) \subset E(A_i)$ for all i. Thus, $E(A_0) \subset \ldots \subset E(A_n)$. Then we extend L_i to $E(A_n)$ by

$$L_i(e) = \begin{cases} L_i(e) & \text{if } e \in E(A_i) \\ 0 & \text{if } e \in E(A_n) - E(A_i). \end{cases}$$

Suppose that $\lambda_k \neq 0, \lambda_{k+1} = \ldots = \lambda_n = 0$. Let

$$L(e) = \sum_{i=0}^{n} \lambda_i L_i(e), \quad \mathfrak{A}(L, T, g) = \mathfrak{A}_k(L, T_k, g_k), \text{ and}$$

$$f(\sum_{i=0}^{n} \lambda_i \overline{\mathfrak{A}_i}) = \bar{l}, \text{ where } l = l_{\mathfrak{A}} \circ (\psi_a^{\mathfrak{A}})^{-1}.$$

Let $E_+(A_n)$ be the set of geometric edges of A_n. Note that

$$\sum_{e \in E_+(A_n)} L(e) = \sum_{e \in E_+(A_n)} \sum_{i=0}^{n} \lambda_i L_i(e) = \sum_{i=0}^{n} \lambda_i \sum_{e \in E_+(A_n)} L_i(e) = \sum_{i=0}^{n} \lambda_i = 1.$$

2.2. Proposition $f : C(G) \longrightarrow \text{PL}(G)$ is a continuous map.

Proof Let $s = (\overline{\mathfrak{A}_0}, \ldots, \overline{\mathfrak{A}_n})$ be an n-simplex of $C(G)$. To show that f is continuous, it suffices to show that the restriction of f on s is continuous. For $\bar{l} \in f(s) \subset \text{PL}(G)$, let $\mathfrak{A}(L, T, g)$ be a normalized graph of groups representing $\bar{l} = f(\sum_{i=0}^{n} a_i \overline{\mathfrak{A}_i})$. Let U be an open neighborhood of $\bar{l}$. Suppose that $U = U_1 \ldots U_n$, where $U_i = \{\bar{l'} \in \text{PL}(G) \| l(g_i) - l'(g_i)| < \epsilon\}$ for some $g_i \in G$. Without loss of generality, we can assume that $U = U_1$. Suppose that $g_1 = \psi_a^{\mathfrak{A}}(h(h_0e_1 \ldots e_mh_m)h^{-1})$, where $h_0e_1 \ldots e_mh_m \in \pi_1(\mathfrak{A}, a)$ is cyclically reduced. Suppose that $\sum_{i=0}^{n} a_i'\overline{\mathfrak{A}_i} \in s$, $\sum_{i=0}^{n} |a_i - a_i'| < \varepsilon / m$, and $f(\sum_{i=0}^{n} a_i'\overline{\mathfrak{A}_i}) = \overline{l}'$. Then

$$|L(e) - L'(e)| = |\sum_{i=0}^{n} a_iL_i(e) - \sum_{i=0}^{n} a_i'L_i(e)| \le \sum_{i=0}^{n} |a_i - a_i'|L_i(e) < \varepsilon / m.$$

By (1.10.1), $|l(g_1) - l'(g_1)| < \varepsilon$. Thus, $\overline{l}' \in U$ and $f|_s$ is continuous, so f is continuous. ∎

Let l be a translation length function of a minimal G-tree. Let $\mathfrak{A}(L, T, g)$ be the normalized graph of groups of the G-tree. If G is finitely generated, then the quotient graph A is finite. Suppose $E(A) = \{e_0, \overline{e}_0, \ldots, e_n, \overline{e}_n\}$. Let s be an n-simplex in $\mathbf{R}^{n+1}$ with vetices $(1, \ldots, 0), \ldots, (0, \ldots, 1)$, and let $s(\mathfrak{A})$ be the barycentric subdivision of s. There are $(n+1)!$ n-dimensional subsimplexes. For example, if $n = 2$, the vertices of $s(\mathfrak{A})$ are $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1/2, 1/2, 0)$, $(1/2, 0, 1/2)$, $(0, 1/2, 1/2)$, and $(1/3, 1/3, 1/3)$. We now define $f_1 : s(\mathfrak{A}) \longrightarrow C(G)$ on the vertices of $s(\mathfrak{A})$. For a vertex $\mathbf{a} = (a_0, \ldots, a_n)$ of $s(\mathfrak{A})$, let $L_{\mathbf{a}}(e_i) = a_i$, $0 \le i \le n$, and let

$$f_1(\mathbf{a}) = \begin{cases} \mathfrak{A}(L_{\mathbf{a}}, T, g) & \text{if } 0 < a_i < 1 \text{ for all } i, \\ \mathfrak{A}'(L_{\mathbf{a}}, T', g') & \text{if } a_i = 0 \text{ for some } i, \end{cases}$$

where $\mathfrak{A}'$ is obtained from $\mathfrak{A}$ by collapsing $\{e_i \in E(A) | a_i = 0\}$. For example, $f_1(1, 0, \ldots, 0)$ is obtained from $\mathfrak{A}$ by collapsing all edges except e_0 and $\overline{e}_0$. Next we define $f_1(v)$ for an arbitrary point $v \in s(\mathfrak{A})$ as follows

$$f_1(v) = \sum_{i=0}^{n} \lambda_i f_1(v_i)$$

where $v = \sum_{i=0}^{n} \lambda_i v_i$, and $(v_0, \ldots, v_n) \subset s$ is an n-simplex.

2.3. Lemma $f_1 : s(\mathfrak{A}) \longrightarrow C(G)$ is an embedding.

Proof. Let $\mathbf{a} = (a_0, \ldots, a_n)$ be a vertex of $s(\mathfrak{A})$. First we show that $f_1(\mathbf{a})$ is a graph of groups of a G-action on a $\mathbf{Z}$-tree. It is the same as showing that $a_i = 0$ or $1/p$, where p is the number of nonzero components of $\mathbf{a}$. Note that $\mathbf{a}$ may not be a vertex of s. Suppose that $\mathbf{a}$ is on a $(p-1)$-simplex of s and not on a $(p-2)$-simplex of s. Since $\mathbf{a}$ is a vertex of the barycentric subdivision of s, $\mathbf{a}$ is the average of the p vertices of the $(p-1)$-simplex. Thus, $a_i = 0$ or $1/p$, which shows that $f_1(\mathbf{a})$ is a vertex of $C(G)$. Therefore f_1 is well defined.

Next we show that f_1 is an embedding. Let $L'(e_i) = 1/(n+1)$ for all i. Replace the edge length function L of $\mathfrak{A}(L, T, g)$ by L'. Then $\mathfrak{A}(L', T, g)$ is a vertex of $C(G)$. Note that

$$f_1(s(\mathfrak{A})) = \{(\mathfrak{A}_0, \ldots, \mathfrak{A}_n) \subset C(G) | \mathfrak{A}_n = \mathfrak{A}(L', T, g)\}.$$

In fact f_1 is a bijection between the vertices of $s(\mathfrak{A})$ and the vertices of $\{(\mathfrak{A}_0, \ldots, \mathfrak{A}_n) \subset C(G) | \mathfrak{A}_n = \mathfrak{A}(L', T, g)\}$. We define a partial order in the set of vetices of $s(\mathfrak{A})$. If $y_i = 0$ implies $x_i = 0$ for all i, then let $(x_0, \ldots, x_n) \le (y_0, \ldots, y_n)$. For example, $(0, 0, 1) < (1/2, 0, 1/2) < (1/3, 1/3, 1/3)$. By induction on n, it is easy to see that the set of vertices of an n-simplex of $s(\mathfrak{A})$ is a linear ordered set of $n+1$ vertices. It follows from the definition of f_1 that $f_1(\mathbf{x})$ is a collapse of $f_1(\mathbf{y})$ if and only if $\mathbf{x} < \mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in V(s(\mathfrak{A}))$. Therefore f_1 is an embedding. ∎

2.4. Theorem Suppose G is a finitely generated group, and f is defined as in (2.1). Then $f : C(G) \longrightarrow \mathrm{PL}(G)$ is a bijective continuous map.

Proof First we show that f is surjective. Let $\bar{l}$ be a simplicial projective translation length function. Suppose that $l = l_{\mathfrak{A}} \circ (\psi_a^{\mathfrak{A}})^{-1}$, where $\mathfrak{A}(L, T, g)$ is the graph of groups of the simplicial G-tree. Note that $0 < L(e_i) < 1$ and $v = (L(e_0), \ldots, L(e_n))$ is a point in $s(\mathfrak{A})$. Thus, v belongs to an n-dimensional subsimplex $s' \subset s(\mathfrak{A})$. Let $v_0, \ldots, v_n$ be the vertices of s'. Suppose that $v = \sum_{i=0}^n a_i v_i$, and $f_1(v_i) = \overline{\mathfrak{A}}_i(L_i, T_i, g_i) \in C(G)$. Note that $v_i = (L_i(e_0), \ldots, L_i(e_n))$ for $i = 0, \ldots, n$. Then

$$(L(e_0), \ldots, L(e_n)) = v = \sum_{i=0}^n a_i v_i = (\sum_{i=0}^n a_i L_i(e_0), \ldots, \sum_{i=0}^n a_i L_i(e_n)).$$

Thus,

$$\sum_{i=0}^n a_i L_i(e) = L(e), \quad f(\sum_{i=0}^n a_i \overline{\mathfrak{A}_i}) = \bar{l},$$

and f is surjective.

To show that f is injective, suppose that

$$\overline{l_{\mathfrak{A}} \circ (\psi_a^{\mathfrak{A}})^{-1}} = f(\sum_{i=0}^n a_i \overline{\mathfrak{A}}_i(L_i, T_i, g_i)) = f(\sum_{i=0}^m a_i' \overline{\mathfrak{A}}_i'(L_i', T_i', g_i')) = \overline{l_{\mathfrak{A}'} \circ (\psi_{a'}^{\mathfrak{A}'})^{-1}}.$$

Without loss of generality, we can assume that $a_n \neq 0$ and $a_m' \neq 0$. Thus,

$$\mathfrak{A} = \mathfrak{A}_n(\sum_{i=0}^n a_i L_i, T_n, g_n) \text{ and } \mathfrak{A}' = \mathfrak{A}_m'(\sum_{i=0}^m a_i' L_i', T_m', g_m').$$

Since they are non-abelian and have the same translation length functions, up to a scalar factor, by Corollary 1.4, they are equivalent. We can assume the scalar factor is 1. Let $\Phi : \mathfrak{A} \longrightarrow \mathfrak{A}'$ be the natural isomorphism. In particular, $\sum_{i=0}^n a_i L_i(e) = \sum_{j=0}^m a_j' L_j'(\phi(e))$. Since $\phi : A \longrightarrow A'$ is an isometry, we can assume that $A = A'$. Note that $L_j'(e) = L_j'(\phi(e))$. Then

$$L(e) = \sum_{i=0}^n a_i L_i(e) = \sum_{j=0}^m a_j' L_j'(e) = L'(e). \tag{2.4.1}$$

Let $E_+(A) = \{e_0, \ldots, e_t\}$. Note that both $\sum_{i=0}^n a_i \overline{\mathfrak{A}}_i(L_i, T_i, g_i)$ and $\sum_{i=0}^m a_i' \overline{\mathfrak{A}}_i'(L_i', T_i', g_i')$ belong to $f_1(s(\mathfrak{A}))$. Let

$$v_i = (L_i(e_0), \ldots, L_i(e_t)), \quad v_i' = (L_i'(e_0), \ldots, L_i'(e_t)).$$

Since $\overline{\mathfrak{A}}_i$'s are vertices of $f_1(s(\mathfrak{A}))$ and $\mathfrak{A}_i$ is a collapse of $\mathfrak{A}_{i+1}$, $(v_0, \ldots, v_n)$ is an n-simplex s_1 of s. Similarly, $(v'_0, \ldots, v'_n)$ is also an n-simplex s_2 of s. Note that

$$f_1(\sum_{i=0}^{n} a_i v_i) = \sum_{i=0}^{n} a_i \overline{\mathfrak{A}}_i(L_i, T_i, g_i) \text{ and } f_1(\sum_{i=0}^{n} a'_i v'_i) = \sum_{i=0}^{m} a'_i \overline{\mathfrak{A}}'_i(L'_i, T'_i, g'_i).$$

By (2.4.1),

$$\sum_{i=0}^{n} a_i v_i = (L(e_0), \ldots, L(e_t)) = (L'(e_0), \ldots, L'(e_t)) = \sum_{i=0}^{n} a'_i v'_i.$$

By Lemma 2.3,

$$\sum_{i=0}^{n} a_i \overline{\mathfrak{A}}_i(L_i, T_i, g_i) = \sum_{i=0}^{m} a'_i \overline{\mathfrak{A}}'_i(L'_i, T'_i, g'_i).$$

Thus, f is injective. ∎

3. THE COMPLEX OF SMALL F_n-TREES

In this section, we study $CS(F_n)$, which is the subcomplex of small simplicial F_n-trees. A graph of groups is *free* if all vertex groups are trivial. We shall prove

3.1. Theorem. For any $\overline{\mathfrak{A}}, \overline{\mathfrak{A}}' \in CS(F_n)$ $(n > 1)$, there exist $\overline{\mathfrak{A}}_0, \ldots, \overline{\mathfrak{A}}_m \in CS(F_n)$ such that $\overline{\mathfrak{A}}_0 = \overline{\mathfrak{A}}$, $\overline{\mathfrak{A}}_m = \overline{\mathfrak{A}}'$, and either $\mathfrak{A}_i$ is a collapse of $\mathfrak{A}_{i-1}$ or $\mathfrak{A}_{i-1}$ is a collapse of $\mathfrak{A}_i$ for $1 \leq i \leq m$. Namely, $CS(F_n)$ is path connected.

3.2. Lemma. Let $\overline{\mathfrak{A}}, \overline{\mathfrak{A}}' \in CS(F_n)$ $(n > 1)$. Suppose that $\mathfrak{A}$ and $\mathfrak{A}'$ are free. Then there is an edge path in $CS(F_n)$ joining $\overline{\mathfrak{A}}$ and $\overline{\mathfrak{A}}'$.

Proof. After collapsing the maximum subtrees of A and A', we can assume that A and A' are n-leaf roses. Each leaf corresponds to a generator of F_n. Let $E_+(A) = \{e_1, \ldots, e_n\}$ and $E_+(A') = \{e'_1, \ldots, e'_n\}$. Let $g_i = \psi_a^{\mathfrak{A}}(e_i)$, $g'_i = \psi_{a'}^{\mathfrak{A}'}(e'_i)$, and $\alpha(g_i) = g'_i$ for $1 \leq i \leq n$. Then $\alpha \in \mathrm{Aut}(F_n)$. Without loss of generality, we can assume that α is an elementary Nielsen transformation. Suppose $g'_1 = \alpha(g_1) = g_1^{-1}$ and $g'_i = \alpha(g_i) = g_i$ for $2 \leq i \leq n$. Let $\Phi = (\phi, \{\phi_a\}, \{\phi_e\}, \{\delta_e\}) : \mathfrak{A} \longrightarrow \mathfrak{A}'$ be defined as follows:

$$\phi(e_i) = \begin{cases} {e'}_1^{-1} & \text{if } i = 1 \\ e'_i & \text{if } i > 1, \end{cases} \quad \phi_a = \phi_e = \mathrm{Id}, \ \delta_e = 1 \text{ for } e \in E(A).$$

Then

$$\psi_a^{\mathfrak{A}}(e_1) = g_1 = {g'}_1^{-1} = \psi_a^{\mathfrak{A}'}({e'}_1^{-1}) = \psi_a^{\mathfrak{A}'} \circ \Phi_a(e_1),$$

so Φ is a natural isomorphism. By Corollary 1.4, $\overline{\mathfrak{A}} = \overline{\mathfrak{A}}'$.

Now suppose $g'_1 = \alpha(g_1) = g_1 g_2$ and $g'_i = \alpha(g_i) = g_i$ for $2 \leq i \leq n$. Let $\mathfrak{B}$ be obtained from $\mathfrak{A}$ by collapsing the loop e_2. Let $\mathfrak{B}'$ be obtained from $\mathfrak{A}'$ by collapsing the loop e'_2.

Since $n > 1$, $\mathfrak{B}$ and $\mathfrak{B}'$ are non-trivial. Let $\Phi = (\phi, \{\phi_a\}, \{\phi_e\}, \{\delta_e\}) : \mathfrak{B} \longrightarrow \mathfrak{B}'$ be defined as follows:

$$\phi(e_i) = e_i' \text{ if } i \neq 2, \ \phi_a = \text{Id}, \ \phi_e = \text{Id}, \ \delta_e = \begin{cases} 1 & \text{if } e \neq \overline{e}_1 \\ g_2 & \text{if } e = \overline{e}_1. \end{cases}$$

Then

$$\psi^{\mathfrak{B}'} \circ \Phi(e_1) = \psi^{\mathfrak{B}'}(\delta_{e_1} e_1' \delta_{\overline{e}_1}^{-1}) = \psi^{\mathfrak{B}'}(e_1' g_2'^{-1}) = g_1' g_2'^{-1} = g_1 g_2 g_2^{-1} = \psi^{\mathfrak{B}}(e_1),$$

so Φ is a natural isomorphism, and $\overline{\mathfrak{B}} = \overline{\mathfrak{B}'}$. Since $\mathfrak{B}$ is a collapse of $\mathfrak{A}$ and $\mathfrak{B}'$ is a collapse of $\mathfrak{A}'$, $(\overline{\mathfrak{B}}, \overline{\mathfrak{A}})$ and $(\overline{\mathfrak{B}}', \overline{\mathfrak{A}}')$ are edges of $CS(F_n)$. So there is an edge path in $CS(F_n)$ joining $\overline{\mathfrak{A}}$ and $\overline{\mathfrak{A}}'$. ∎

Let $\overline{\mathfrak{A}} \in CS(F_n)$. We will find a path in $CS(F_n)$ from $\overline{\mathfrak{A}}$ to $\overline{\mathfrak{B}}$, where $\overline{\mathfrak{B}}$ is free. Then Theorem 3.1 follows from Lemma 3.2. Since we only consider small F_n-actions, the edge stabilizers are either trivial or they are infinitely cyclic.

Case 1 $\mathcal{A}_e = \{1\}$ for some $e \in E(A)$.

If $A - \{e, \overline{e}\}$ is not connected, we then collapse two components of $A - \{e, \overline{e}\}$ to two vertices of e, and obtain a new graph of groups, which is a free product, because the edge group is trivial. If $A - \{e, \overline{e}\}$ is connected, we collapse $A - \{e, \overline{e}\}$ to a vertex, and obtain an HNN-extension.

Case 1.1 Free product.

In this case, $F_n = F_p * F_q$, and $p + q = n$. We then blow up two vertices to roses of p-leaf and q-leaf, respectively. The resulting graph of groups $\mathfrak{B}$ is free. This gives us a path in $CS(F_n)$ from $\overline{\mathfrak{A}}$ to $\overline{\mathfrak{B}}$.

Case 1.2. ***HNN*-extension.**

In this case, since the edge group is trivial, $F_n = \langle F_{n-1}, e \rangle = F_{n-1} * \langle e \rangle$. We then blow up the vertex to a rose of $(n-1)$-leaf. The resulting graph of groups is free. This concludes the proof of Case 1.

Case 2. $\mathcal{A}_e \cong \mathbf{Z}$ **for all** $e \in E(A)$**.**

We choose an edge $e \in E(A)$, and collapse $A - \{e, \overline{e}\}$ to a vertex if $A - \{e, \overline{e}\}$ is connected. Otherwise we will collapse two components of $A - \{e, \overline{e}\}$ to two vertices of e. The resulting graph of groups is an *HNN*-extension in the formal case, and an amalgamated free product in the latter case.

Case 2.1. Amalgamated free product.

In this case, $F_n = G_0 *_{\mathbf{Z}} G_1$, where $\mathbf{Z} = \langle t \rangle$. Let γ_i be the image of t in G_i. Note that G_i is a free group. We need the following theorem of Morgan.

3.3. Theorem [8] Let G_0 and G_1 be finitely presented groups and let $G = G_0 *_{\mathbf{Z}} G_1$ be a free product with amalgamation. Let γ_i denote the image in G_i of a generator of the amalgamating subgroup. Suppose that G acts freely on an $\mathbf{R}$-tree. Then one of the following occurs:

(1) γ_0 is a generator of a free factor in G_0;
(2) γ_1 is a generator of a free factor in G_1; or
(3) for $i = 0, 1$ there are free product decompositions $G_i = H_i * F_i$ where F_i is a free group containing γ_i and γ_i is a quadratic word with respect to some free basis for F_i.

Note that F_n acts freely on an **R**-tree, so the theorem applies. In case (1) of Theorem 3.3, since $\overline{\mathfrak{A}}$ is non-trivial, the amalgamated free product is non-trivial. Thus, $\langle\gamma_0\rangle$ is a proper free factor of G_0. Note that G_0 is a free group. We blow up the vertex of G_0 to an edge, and the new splitting is

$$F' * \langle\gamma_0\rangle *_{\mathbf{Z}} G_1.$$

Now we collapse $\langle\gamma_0\rangle *_{\mathbf{Z}} G_1$ to a vertex group, the result is a free product, which is covered by Case 1.1. Case (2) of Theorem 3.3 can be handled similarly. In case (3) of Theorem 3.3, if H_i is non-trivial, then we can reduce it to the previous case. If H_i's are trivial, let $\{g_0, \ldots, g_p\}$ be a basis for G_0, and $\{h_0, \ldots, h_q\}$ be a basis for G_1. Suppose that g_0 appears only once in γ_0, then let $\{\gamma_0, g_1, \ldots, g_p\}$ be a new basis for G_0, which is in case (1) of Theorem 3.3. Suppose that each g_i appears exactly twice in γ_0, and each h_i appears exactly twice in γ_1. Then F_n will be a surface group, a contradiction. This settles Case 2.1.

Case 2.2. ***HNN*-extension.**

We need the following theorem of Morgan and Skora:

3.4. Theorem (Theorem 5.2 of [9]) Let the finitely presented group G act freely on an **R**-tree. If G has an *HNN*-decomposition $G = H*_{\langle s\rangle}$, where $i_0(s) = s_0$, $i_1(s) = s_1$, then either

(1) H splits as a free product $H = H' * \mathbf{Z}$, where one of s_0, s_1 is a generator of the $\mathbf{Z}$-factor, and the other conjugated by some $h \in H$ has image in H'; or

(2) H splits as free product $H = H' * F$, where F is a free group and where the elements s_0 and hs_1h^{-1}, for some $h \in H$, have image in F. Furthermore, s_0, hs_1h^{-1} form a purely quadratic system with respect to some basis of F.

Suppose $F_n = \langle H, t | ts_0t^{-1} = s_1\rangle$. Consider case (1) of Theorem 3.4, suppose that $hs_1h^{-1} \in H'$ and $s_0 \in \mathbf{Z}$. Let $\mathfrak{A}(L, T, g)$ be defined as follows:

$$\begin{aligned} &A \text{ is a loop with a vertex } a,\\ &\mathcal{A}_a = H,\ \mathcal{A}_e = \langle s_0\rangle,\ \alpha_e(s_0) = s_0,\ \alpha_{\bar{e}}(s) = s_1,\\ &L(e) = 1,\ g(e) = 1,\ g(\bar{e}) = t. \end{aligned}$$

Let $\mathfrak{A}'(L', T', g')$ be defined as follows:

$$\begin{aligned} &A' = A,\ \mathcal{A}'_a = \mathcal{A}_a,\ \mathcal{A}'_e = \mathcal{A}_e,\ \alpha'_e = \alpha_e,\ \alpha'_{\bar{e}} = \mathrm{ad}(h) \circ \alpha_{\bar{e}},\\ &L'(e) = 1,\ g'(e) = 1,\ g'(\bar{e}) = ht. \end{aligned}$$

Let $\Phi : \mathfrak{A}(L, T, g) \longrightarrow \mathfrak{A}'(L', T', g')$ be defined as follows:

$$\Phi = (\mathrm{Id}_A, \{\phi_a = \mathrm{Id}\}, \{\phi_e = \mathrm{Id}\}, \{\delta_e = 1, \delta_{\bar{e}} = h^{-1}\}).$$

We are going to prove that $\Phi : \mathfrak{A} \longrightarrow \mathfrak{A}'$ is a natural isomorphism. Since

$$\mathrm{ad}(\delta_{\bar{e}}) \circ \alpha'_{\bar{e}} \circ \phi_{\bar{e}} = \mathrm{ad}(h^{-1}) \circ \mathrm{ad}(h) \circ \alpha_{\bar{e}} \circ \phi_{\bar{e}} = \alpha_{\bar{e}} \circ \phi_{\bar{e}} = \phi_a \circ \alpha_{\bar{e}},$$

Φ is an isomorphism. It is easy to see that $\psi_a^{\mathfrak{A}}(x) = \psi_a^{\mathfrak{A}'} \circ \Phi_a(x)$ for $x \in H$. Note that

$$\begin{aligned} &\Phi(e) = \delta_e \phi(e) \delta_{\bar{e}}^{-1} = eh,\\ &\psi_a^{\mathfrak{A}'}(e) = g_{A'}(e) g_{A'}(\bar{e})^{-1} = g_A(e) g_A(\bar{e}) h^{-1}. \end{aligned}$$

Thus,

$$\psi_a^{\mathfrak{A}'} \circ \Phi_a(e) = \psi_a^{\mathfrak{A}'}(eh) = \psi_a^{\mathfrak{A}'}(e)h$$
$$= g_A(e)g_A(\overline{e})h^{-1}h = \psi_a^{\mathfrak{A}}(e),$$

so $\psi_a^{\mathfrak{A}} = \psi_a^{\mathfrak{A}'} \circ \Phi_a$, and Φ_a is natural. By Corollary 1.4, $\mathfrak{A}$ and $\mathfrak{A}'$ are equivalent, so we can take $\mathfrak{A}'$ as the reperesentative of the equivalence class. Let $s_1' = hs_1h^{-1} \in H'$. If case (1) of Theorem 3.4 holds, then $G = \langle H', s_0, e | es_0e^{-1} = s_1' \rangle$. We then blow up the vertex to an edge e' with trivial edge group. The result is the splitting $H = H' * \langle s_0 \rangle$. We now collapse the edge e. The new splitting is $\langle H', e' \rangle$, which is in Case 1.2.

If case (2) of Theorem 3.4 holds, then we blow up the vertex to an edge e'. The resulting splitting is $G = H' * F_{\langle s \rangle}$. If H' is non-trivial, we collapse e to a vertex, the new splitting is $G = H' * F'$, which is in Case 1.1. If H' is trivial, since s_0, hs_1h^{-1} form a purely quadratic system with respect to some basis of F, $G = F_{\langle s \rangle}$ must be a surface group, a contradiction. This concludes the proof of Theorem 3.1. ∎

Lustig [7] proved that every automorphism of F_n fixes a point of the compactified outer space. The following result follows from Theorem 2.12 of [6] and Theorem 3.1.

3.5. Corollary Let $\overline{\mathfrak{A}}$ be the base point of $SC(F_n)$. For $\varphi \in \mathrm{Aut}(F_n)$, suppose that φ fixes a point $\overline{\mathfrak{A}}' \in SC(F_n)$, i.e.,

$$l_{\mathfrak{A}'} \circ (\psi_a^{\mathfrak{A}'})^{-1}(\varphi(g)) = l_{\mathfrak{A}'} \circ (\psi_a^{\mathfrak{A}'})^{-1}(g)$$

for all $g \in F_n$. Then there exist $\overline{\mathfrak{A}}_i \in SC(F_n)$ $(0 \leq i \leq m)$, such that $\mathfrak{A} = \mathfrak{A}_0$, $\mathfrak{A}' = \mathfrak{A}_m$, and either $\mathfrak{A}_{i-1}$ is a collapse of $\mathfrak{A}_i$ or vice versa. Furthermore, the following diagram commutes

$$\begin{array}{ccccccccccc} G & = & \dots & = & G & \xrightarrow{\varphi} & G & = & \dots & = & G \\ \uparrow \psi_a^{\mathfrak{A}} & & \uparrow & & \uparrow \psi_{a'}^{\mathfrak{A}'} & & \uparrow \psi_{a'}^{\mathfrak{A}'} & & \uparrow & & \uparrow \psi_a^{\mathfrak{A}} \\ \pi_1(\mathfrak{A},a) & \xrightarrow[f_1]{} & \dots & \xrightarrow[f_m]{} & \pi_1(\mathfrak{A}',a') & \xrightarrow[\Phi_{a'}]{} & \pi_1(\mathfrak{A}',a') & \xleftarrow[f_m]{} & \dots & \xleftarrow[f_1]{} & \pi_1(\mathfrak{A},a) \end{array}$$

where $f_i : \pi_1(\mathfrak{A}_{i-1}, a_{i-1}) \longrightarrow \pi_1(\mathfrak{A}_i, a_i)$ is the isomorphism induced by the collapse or the blow up, and $\Phi_{a'}$ is induced by an automorphism of $\mathfrak{A}'$. In particular,

$$\varphi = \psi_a^{\mathfrak{A}} \circ (f_m \circ \dots \circ f_1)^{-1} \circ \Phi_{a'} \circ (f_m \circ \dots \circ f_1) \circ (\psi_a^{\mathfrak{A}})^{-1}. \quad ∎$$

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