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FULLNESS AND SCALAR CURVATURE OF THE TOTALLY REAL SUBMANIFOLDS IN $S^6(1)$[1]

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Abstract. *Let M be a totally real 3-dimensional submanifold of the nearly Kähler 6-sphere $S^6(1)$. Theorems are proven on the relation between the fullness and the scalar curvature R of M. In particular, if either R is a constant different from 2, or M is compact with $R \neq 2$, then M is full in $S^6(1)$ unless M is totally geodesic. A family of examples with $R \equiv 2$, which are fully contained in some great hypersphere $S^5(1) \subset S^6(1)$, are also defined in an explicit manner.*

Key words and phrases. *Fullness Scalar curvature - Totally real submanifolds - Nearly Kähler structure - Minimality*

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1. INTRODUCTION

Denote by $S^n(c)$ the standard n-sphere with constant sectional curvature c. It is known that there is a canonical nearly Kähler structure on $S^6(1)$. So the study on complex or totally real submanifolds in $S^6(1)$ is much attracting, and much progress have been made in this direction. For example, see the references at the end.

On the other hand, one expects that the codimension of a fully immersed minimal submanifold in S^n be closely related to its curvature. Thus it was once conjectured that $S^3(c)$ can not be minimally immersed into $S^6(1)$ when $c \neq 1$[7]; while a theorem of Moore [13], which seems to support this conjecture, says that $S^3(c)$ can not be minimally immersed into $S^5(1)$ unless $c = 1$. But N. Ejiri proved in 1981 that $S^3(c)$ can be immersed into $S^6(1)$ as a totally real submanifold if and only if $c = 1$ or $\frac{1}{16}$. Since in this case total reality implies minimality [8], the conjecture in [7] thus has a counterexample. It is clear that Ejiri's example is full in $S^6(1)$, which is explicily expressed by F. Dillen etc. [6]

In this paper we study the relation between the fullness and the scalar curvature of a 3-dimensional totally real submanifold in $S^6(1)$. After some necessary preliminaries in Section 2, we give explicitly in Section 3 a family of totally real immersions of S^3 into $S^6(1)$, which are of constant scalar curvature $R = 2$ and fully contained in some $S^5(1) \subset S^6(1)$. Also they induce totally real imbeddings of the real projective 3-space RP^3. We will show that they are all G_2-congruent to each other. Therefore, up to G_2-congruence, we can view these immersions as one example and denote it by x^0. In Sections 4 and 5, we are to prove the following results, which reveal in an extent the relationship desired.

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Theorem A. Let M be a C^∞-manifold of dimension 3 and ψ: $M \to S^6(1)$ be a totally real immersion with constant scalar curvature R. If $R \neq 2$, then ψ is full in $S^6(1)$ unless it is totally geodesic, i.e. $R = 6$. Furthermore, in case $R = 2$, ψ is locally G_2-congruent to our example x^0 unless it is full in $S^6(1)$.

Theorem B. Let M be a compact C^∞-manifold of dimension 3, and ψ: $M \to S^6(1)$ be a totally real immersion with scalar curvature $R \neq 2$. Then ψ is full in $S^6(1)$ except it is totally geodesic, that is, $R \equiv 6$

In the course of proof we also obtain the following corollaries:

Corollary C. Let M be a compact and connected C^∞-manifold of dimension 3, and ψ: $M \to S^6(1)$ be a totally real immersion with scalar curvature R. If either $R \leq 2$ or $R \geq 2$, Then ψ is full in $S^6(1)$ except

1° it is totally geodesic, i.e., $R \equiv 6$; or,

2° it is covered by the immersion x^0 upto G_2-congruence.

Corollary D. Let M be a C^∞-manifold of dimension 3 and ψ: $M \to S^6(1)$ be a totally real immersion. If ψ is full in some totally geodesic $S^m(1) \subset S^6(1)$, then $m = 3, 5, 6$.

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2. PRELIMINARIES

In this section, we briefly review some basic facts on the nearly Kähler structure on $S^6(1)$ and the total reality condition. One may see, for example, [1, 6, 11] to get further information.

For convenience, we agree to the following ranges of indices and conventions:

$$1 \leq A, B, C, \ldots \leq 6, \quad 1 \leq i, j, k, \ldots \leq 3, \quad i^* = i + 3.$$

The multiplication on the Cayley numbers defines one cross-product on the set $\mathbf{R}^7$ of purely imaginary Cayley numbers by

$$x \times y = \frac{1}{2}(xy - yx). \tag{2.1}$$

The standard inner product on $\mathbf{R}^7$ can be reformulated as

$$\langle x, y \rangle = -\frac{1}{2}(xy + yx).$$

If $\{\delta_1, \delta_2, \ldots, \delta_7\}$ is the standard basis for $\mathbf{R}^7$, then we can write down the cross-product table as:

Cross-product Table of $\mathbf{R}^7$

$\times$	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	δ_7
δ_1	0	δ_3	$-\delta_2$	δ_5	$-\delta_4$	$-\delta_7$	δ_6
δ_2	$-\delta_3$	0	δ_1	δ_6	δ_7	$-\delta_4$	$-\delta_5$
δ_3	δ_2	$-\delta_1$	0	δ_7	$-\delta_6$	δ_5	$-\delta_4$
δ_4	$-\delta_5$	$-\delta_6$	$-\delta_7$	0	δ_1	δ_2	δ_3
δ_5	δ_4	$-\delta_7$	δ_6	$-\delta_1$	0	$-\delta_3$	δ_2
δ_6	δ_7	δ_4	$-\delta_5$	$-\delta_2$	δ_3	0	$-\delta_1$
δ_7	$-\delta_6$	δ_5	δ_4	$-\delta_3$	$-\delta_2$	δ_1	0

It can be shown that [11], the operation "$\times$" in (2.1) satisfies the following identity:

$$x \times (y \times z) + (x \times y) \times z = 2(x, z)y - (x, y)z - (y, z)x. \tag{2.2}$$

Since $S^6(1)$ is a hypersphere in $\mathbf{R}^7$ of radius 1, we can use the cross-product "$\times$" to introduce on $S^6(1)$ a $(1, 1)$-tensor field $\mathbf{J}$ as

$$\mathbf{J}_x(X) = x \times X, \quad x \in S^6(1), \quad X \in T_xS^6(1), \tag{2.3}$$

where $T_xS^6(1)$ denotes the tangent space of $S^6(1)$ at x. It is readily by (2.3) that $\mathbf{J}$ satisfies $\mathbf{J}^2 = -id$, and defines an almost complex structure on $S^6(1)$, which is also nearly Kählerian in the sense that $\overline{\nabla}_X Y \equiv 0$, where X, Y are vector fields on $S^6(1)$ and $\overline{\nabla}$ is the Levi-Civita connection of the standard metric of $S^6(1)$.

Define a subgroup G_2 of the orthogonal group $O(7)$ by

$$G_2 = \{g \in O(7), g(x \times y) = g(x) \times g(y), \quad \text{for all } x, y \in \mathbf{R}^7\}. \tag{2.4}$$

Then G_2 is nothing but the group of isometries on $S^6(1)$ preserving the nearly Kähler structure $\mathbf{J}$ [2,3].

Let G be the $(2, 1)$-tensor field on $S^6(1)$ defined by

$$G(X, Y) = (\overline{\nabla}_X \mathbf{J})(Y), \quad X, Y \in TS^6(1),$$

where $TS^6(1)$ denotes the tangent bundle of $S^6(1)$. Then we have

Lemma 2.1. [4,6,10] For X, Y, $Z \in T_xS^6(1)$,

$$G(X, Y) + G(Y, X) = 0, \quad G(X, \mathbf{J}Y) + \mathbf{J}G(X, Y) = 0,$$

$$\langle G(X, Y), Z\rangle + \langle G(X, Z), Y\rangle = 0, \quad G(X, Y) = X \times Y + \langle X, \mathbf{J}Y\rangle x.$$

Now let M be a C^∞-manifold of dimension 3, and ψ: $M \to S^6(1)$ be an immersion. With the induced metric by ψ, M becomes a Riemannian manifold. Let TM and NM denote respectively the tangent and normal bundles of ψ. ψ is called totally real if $\mathbf{J}TM \subset NM$; ψ is said full in $S^6(1)$ if $\psi(M)$ is not contained in some great hypersphere $S^5(1) \subset S^6(1)$. One important fact is the following

Lemma 2.2 [8] If ψ: $M \to S^6(1)$ is totally real, then it is also minimal.

Let ∇ and D be the Levi-Civita connection and normal connection on M respectively. Denote by h the vector-valued second fundamental form, and ∇h its covariant differentiation. The Weingarten map A_ξ for a normal vector ξ is related to h by

$$\langle A_\xi X, Y\rangle = \langle h(X, Y), \xi\rangle, \quad X, Y \in TS^6(1).$$

Using the total reality and Lemma 2.1, one can prove the following lemma:

Lemma 2.3. [4,6]

$$D_X(\mathbf{J}Y) = G(X, Y) + \mathbf{J}\nabla_X Y, \quad A_{\mathbf{J}X}Y = -\mathbf{J}h(X, Y), \tag{2.5}$$

$$\begin{aligned}\langle(\nabla h)(X, Y, Z), \mathbf{J}W\rangle &- \langle(\nabla h)(X, Y, W), \mathbf{J}Z\rangle \\ &= \langle h(Y, Z), G(X, W)\rangle - \langle h(Y, W), G(X, Z)\rangle,\end{aligned} \tag{2.6}$$

$$\langle h(X, Y), \mathbf{J}Z\rangle = \langle h(X, Z), \mathbf{J}Y\rangle. \tag{2.7}$$

3. EXPLICIT EXAMPLES

Let $S^3(1) = \{(x_0, x_1, x_2, x_3) \in \mathbf{R}^4, \quad x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1\}$. Define along $S^3(1)$ three vectors X_1, X_2, X_3 by

$$X_1 = (-x_1, x_0, x_3, -x_2), \ X_2 = (-x_2, -x_3, x_0, x_1), \ X_3 = (-x_3, x_2, -x_1, x_0).$$

Then X_1, X_2, X_3 are linearly independent everywhere on $S^3(1)$. Put

$$E_1 = \frac{1}{\sqrt{2}}X_1, \quad E_2 = \frac{1}{\sqrt{2}}X_2, \quad E_3 = \frac{1}{2}X_3.$$

Introduce on $S^3(1)$ a new metric g by regarding $\{E_1, E_2, E_3\}$ as one orthonomal frame field. Denote by S^3 the Riemannian manifold $(S^3(1), g)$. It is easily seen that

$$[E_1, E_2] = 2E_3, \quad [E_2, E_3] = E_1, \quad [E_3, E_1] = E_2.$$

Thus by the definition of the Levi-Civita connection of a metric [12], we have

Proposition 3.1. The Levi-Civita connection ∇ on S^3 is characterized by

$$\begin{aligned}&\nabla_{E_i}E_i = 0, \quad \nabla_{E_1}E_2 = -\nabla_{E_2}E_1 = E_3, \\ &\nabla_{E_1}E_3 = -E_2, \quad \nabla_{E_2}E_3 = E_1, \quad \nabla_{E_3}E_1 = \nabla_{E_3}E_2 = 0.\end{aligned}$$

Further calculations prove

Proposition 3.2. The components of the curvature tensor of S^3 with respect to $\{E_1, E_2, E_3\}$ are given by

$$R_{ijkl} = \begin{cases} \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}, & \text{if any of } i, j, k, l \text{ equals to } 3 \\ \delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}, & \text{otherwise.} \end{cases} \tag{3.1}$$

In particular, the sectional curvature K, the Ricci curvature Ric and the scalar curvature R satisfy the followings:

$$-1 \leq K \leq 1, \quad 0 \leq Ric \leq 2, \quad R \equiv 2.$$

For each real number θ, we define a C^∞-map x^θ of S^3 into $\mathbf{R}^7$ by

$$\begin{aligned} x^\theta(x_0, x_1, x_2, x_3) = \frac{1}{\sqrt{2}}\{&2(x_1x_2 - x_0x_3)\delta_1 + (x_0^2 + x_2^2 - x_1^2 - x_3^2)\delta_2 \\ &[2(x_0x_1 + x_2x_3)\cos\theta - 2(x_1x_3 - x_0x_2)\sin\theta]\delta_3 + (x_0^2 + x_1^2 - x_2^2 - x_3^2)\delta_5 \\ &+ 2(x_0x_3 + x_1x_2)\delta_6 + [2(x_0x_1 + x_2x_3)\sin\theta + 2(x_1x_3 - x_0x_2)\cos\theta]\delta_7\} \end{aligned} \tag{3.2}$$

It is easily verified that $\|x^\theta(x_0, x_1, x_2, x_3)\|^2 \equiv 1$, since $x_0^2 + x_1^2 + x_2^2 + x_3^2 \equiv 1$. Therefore x^θ is a map of S^3 into $S^6(1)$. Direct calculations show that

$$\begin{aligned} x_*^\theta E_1 = {} & 2(x_0x_2 + x_1x_3)\delta_1 + 2(x_2x_3 - x_0x_1)\delta_2 + (x_0^2 + x_3^2 - x_1^2 - x_2^2)\cos\theta\delta_3 \\ & +(x_0^2 + x_3^2 - x_1^2 - x_2^2)\sin\theta\delta_7, \end{aligned} \tag{3.3}$$

$$\begin{aligned} x_*^\theta E_2 = {} & -2(x_0x_2 + x_1x_3)\delta_5 - 2(x_2x_3 - x_0x_1)\delta_6 + (x_0^2 + x_3^2 - x_1^2 - x_2^2)\sin\theta\delta_3 \\ & -(x_0^2 + x_3^2 - x_1^2 - x_2^2)\cos\theta\delta_7, \end{aligned} \tag{3.4}$$

$$\begin{aligned} x_*^\theta E_3 = {} & \frac{1}{\sqrt{2}}\{-(x_0^2 + x_1^2 - x_2^2 - x_3^2)\delta_1 - 2(x_0x_3 + x_1x_2)\delta_2 \\ & - [2(x_1x_3 - x_0x_2)\cos\theta + 2(x_0x_1 + x_2x_3)\sin\theta)\delta_3 \\ & + 2(x_1x_2 - x_0x_3)\delta_5 + (x_0^2 + x_2^2 - x_1^2 - x_3^2)\delta_6 \\ & + [-2(x_1x_3 - x_0x_2)\sin\theta + 2(x_0x_1 + x_2x_3)\cos\theta]\delta_7. \end{aligned} \tag{3.5}$$

It follows that $\langle x_*^\theta E_i, x_*^\theta E_j\rangle = \delta_{ij}$, which implies that x^θ is an isometric immersion. From the definition of $\mathbf{J}$, we can find directly that

$$\begin{aligned} \mathbf{J}(x_*^\theta E_1) = {} & \tfrac{1}{\sqrt{2}}\cos\theta[(x_0^2 + x_1^2 - x_2^2 - x_3^2)\delta_1 + (x_0^2 + x_2^2 - x_1^2 - x_3^2)\delta_2 \\ & +2(x_1x_2 - x_0x_3)\delta_5 + (x_0^2 + x_2^2 - x_1^2 - x_3^2)\delta_6] \\ & +\tfrac{1}{\sqrt{2}}\sin\theta[2(x_1x_2 - x_0x_3)\delta_1 + (x_0^2 + x_2^2 - x_1^2 - x_3^2)\delta_2 \\ & -(x_0^2 + x_1^2 - x_2^2 - x_3^2)\delta_5 - 2(x_0x_3 + x_1x_2)\delta_6] \\ & +\tfrac{1}{\sqrt{2}}[2(x_1x_3 - x_0x_2)\delta_3 + 2(x_0x_1 + x_2x_3)\delta_7], \end{aligned} \tag{3.6}$$

$$\begin{aligned} \mathbf{J}(x_*^\theta E_2) = {} & \tfrac{1}{\sqrt{2}}\sin\theta[(x_0^2 + x_1^2 - x_2^2 - x_3^2)\delta_1 + (x_0^2 + x_2^2 - x_1^2 - x_3^2)\delta_2 \\ & +2(x_1x_2 - x_0x_3)\delta_5 + (x_0^2 + x_2^2 - x_1^2 - x_3^2)\delta_6] \\ & -\tfrac{1}{\sqrt{2}}\cos\theta[2(x_1x_2 - x_0x_3)\delta_1 + (x_0^2 + x_2^2 - x_1^2 - x_3^2)\delta_2 \\ & -(x_0^2 + x_1^2 - x_2^2 - x_3^2)\delta_5 - 2(x_0x_3 + x_1x_2)\delta_6] \\ & +\tfrac{1}{\sqrt{2}}[-2(x_0x_1 + x_2x_3)\delta_3 + 2(x_1x_3 - x_0x_2)\delta_7], \end{aligned} \tag{3.7}$$

$$\mathbf{J}(x_*^\theta E_3) = -\delta_4 \tag{3.8}$$

which are all orthogonal to $x_*^\theta E_1$, $x_*^\theta E_2$ and $x_*^\theta E_3$. Thus we have

Proposition 3.3. The map x^θ defined by (3.2) is a totally real and isometric immersion of S^3 into $S^6(1)$, which is contained in the great hypersphere $S^5(1)$ orthogonal to δ_4

Proposition 3.4. For each θ, there is a frame field $\{\tilde{E}_1, \tilde{E}_2, \tilde{E}_3\}$ on S^3 such that the followings hold identically:

$$\nabla_{\tilde{E}_i}\tilde{E}_i = 0, \quad \nabla_{\tilde{E}_1}\tilde{E}_2 = -\nabla_{\tilde{E}_2}\tilde{E}_1 = \tilde{E}_3, \quad \nabla_{\tilde{E}_1}\tilde{E}_3 = -\tilde{E}_2, \tag{3.9}$$

$$\nabla_{\tilde{E}_2}\tilde{E}_3 = \tilde{E}_1, \quad \nabla_{\tilde{E}_3}\tilde{E}_1 = \nabla_{\tilde{E}_3}\tilde{E}_2 = 0, \tag{3.10}$$

$$h(\tilde{E}_1, \tilde{E}_1) = -h(\tilde{E}_2, \tilde{E}_2) = \mathbf{J}(x_*^\theta \tilde{E}_1), \quad h(\tilde{E}_1, \tilde{E}_2) = h(\tilde{E}_2, \tilde{E}_2) = -\mathbf{J}(x_*^\theta \tilde{E}_2), \tag{3.11}$$

$$h(\tilde{E}_i, \tilde{E}_3) = h(\tilde{E}_3, \tilde{E}_i) = 0, \tag{3.12}$$

$$\begin{aligned} &\mathbf{J}G(x_*^\theta \tilde{E}_1, x_*^\theta \tilde{E}_2) = x_*^\theta \tilde{E}_3, \quad \mathbf{J}G(x_*^\theta \tilde{E}_2, x_*^\theta \tilde{E}_3) = x_*^\theta \tilde{E}_1, \\ &\mathbf{J}G(x_*^\theta \tilde{E}_3, x_*^\theta \tilde{E}_1) = x_*^\theta \tilde{E}_2. \end{aligned} \tag{3.13}$$

Proof. From (3.3), (3.4) and (3.5) we find easily

$$\begin{aligned} d(x_*^\theta E_1)(x_*^\theta E_1) &= -2\sqrt{2}(x_1x_2 - x_0x_3)\delta_1 - \sqrt{2}(x_0^2 + x_2^2 - x_1^2 - x_3^2)\delta_2 \\ &\quad -2\sqrt{2}(x_0x_1 + x_2x_3)\cos\theta\delta_3 - 2\sqrt{2}(x_0x_1 + x_2x_3)\sin\theta\delta_7 \\ d(x_*^\theta E_2)(x_*^\theta E_1) &= -2\sqrt{2}(x_0x_1 + x_2x_3)\sin\theta\delta_3 + 2\sqrt{2}(x_1x_2 - x_0x_3)\delta_5 \\ &\quad +\sqrt{2}(x_0^2 + x_2^2 - x_1^2 - x_3^2)\delta_6 + 2\sqrt{2}(x_0x_1 + x_2x_3)\cos\theta\delta_7, \\ d(x_*^\theta E_1)(x_*^\theta E_3) &= d(x_*^\theta E_2)(x_*^\theta E_3) = 0, \quad d(x_*^\theta E_3)(x_*^\theta E_3) = -x^\theta(x_0, x_1, x_2, x_3). \end{aligned}$$

Take the *NM*-components of the above, and compare them with (3.6) and (3.7) one sees

$$\begin{aligned} &h(E_1, E_1) = -\mathbf{J}(x_*^\theta E_1)\sin\theta + \mathbf{J}(x_*^\theta E_2)\cos\theta, \\ &h(E_1, E_2) = \mathbf{J}(x_*^\theta E_1)\cos\theta + \mathbf{J}(x_*^\theta E_2)\sin\theta, \\ &h(E_1, E_3) = h(E_2, E_3) = h(E_3, E_3) = 0. \end{aligned} \tag{3.14}$$

For some real number α we introduce a new frame field $\{\tilde{E}_1, \tilde{E}_2, \tilde{E}_3\}$ by

$$\tilde{E}_1 = E_1\cos\alpha - E_2\sin\alpha, \quad \tilde{E}_2 = E_1\sin\alpha + E_2\cos\alpha, \quad \tilde{E}_3 = E_3.$$

Then it is not hard to verify (3.9) and (3.10). From (3.14) we have also that

$$\begin{aligned} h(\tilde{E}_1, \tilde{E}_2) &= h(E_1, E_1)\sin 2\alpha + h(E_1, E_2)\cos 2\alpha \\ &= \mathbf{J}(x_*^\theta E_1)\cos(2\alpha + \theta) + \mathbf{J}(x_*^\theta E_2)\sin(2\alpha + \theta), \\ h(\tilde{E}_1, \tilde{E}_1) &= h(E_1, E_1)\cos 2\alpha - h(E_1, E_2)\sin 2\alpha \\ &= -\mathbf{J}(x_*^\theta E_1)\sin(2\alpha + \theta) + \mathbf{J}(x_*^\theta E_2)\sin(2\alpha + \theta), \\ \mathbf{J}(x_*^\theta \tilde{E}_1) &= \mathbf{J}(x_*^\theta E_1)\cos\alpha - \mathbf{J}(x_*^\theta E_2)\sin\alpha, \\ \mathbf{J}(x_*^\theta \tilde{E}_2) &= \mathbf{J}(x_*^\theta E_1)\sin\alpha + \mathbf{J}(x_*^\theta E_2)\cos\alpha. \end{aligned}$$

By choosing $\alpha = -(\frac{\pi}{2} + \frac{\theta}{3})$, we obtain (3.11) and (3.12).

To prove (3.13), we note that for each i,

$$\begin{aligned}\langle D_{E_1}\mathbf{J}x_*^\theta E_2, \mathbf{J}x_*^\theta E_i\rangle &= E_1\langle \mathbf{J}x_*^\theta E_2, \mathbf{J}x_*^\theta E_i\rangle - E_1\langle \mathbf{J}x_*^\theta E_2, D_{E_1}\mathbf{J}x_*^\theta E_i\rangle \\ &= -\langle \mathbf{J}x_*^\theta E_2, D_{E_1}\mathbf{J}x_*^\theta E_i\rangle,\end{aligned}$$

which implies $D_{E_1}\mathbf{J}x_*^\theta E_2 = 0$. Thus by (2.5),

$$\mathbf{J}G(x_*^\theta E_1, x_*^\theta E_2) = \mathbf{J}D_{E_1}(\mathbf{J}x_*^\theta E_2) - \mathbf{J}^2 x_*^\theta(\nabla_{E_1}E_2) = x_*^\theta E_3.$$

Similarly we have other two identities in (3.13).

Corollary 3.5. All the immersions x_*^θ are G_2-congruent to each other.

Proof. Let θ and θ′ be any two real numbers. Then the Corollary 3.5 follows by applying to x^θ and $x^{\theta'}$ similar arguments of Dillen etc. in [6]. See the proof of the main theorem and the remark thereafter; Also see Section 4 below for an outline of this.

Remark 3.6 let RP^3 be the real projective 3-space with the metric induced from S^3. Then it is easily seen that each x^θ induces a totally real embedding of RP^3 into $S^6(1)$.

4. PROOF OF THEOREM A

To prove Theorem A, we are able to assume that $\psi(M)$ is contained in some great hypersphere $S^5(1) \subset S^6(1)$, which is equivalent to that there is a unit vector v orthogonal identically to ψ and each of its osculating spaces. Clearly we need to prove that, if R is constant, then either $R = 6$ and ψ is totally geodesic, or $R = 2$ and ψ is locally G_2-congruent to the immersion x^0 in Section 3.

Let e_3 be the unit vector field on M such that $\psi_* e_3 = \mathbf{J}v$, and e_1, e_2 be two local unit vector fields on M such that e_1, e_2, e_3 form a local orthonormal frame field on M. By the total reality of ψ, $\{\psi_* e_i, \mathbf{J}\psi_* e_i\}$ is a local orthonormal frame field of $S^6(1)$ along ψ. For simplicity, we shall drop from now on the tangent map "ψ_*" in "$\psi_* e_i$" if causing no confusions. Put $e_{i^*} = \mathbf{J}e_i$ and write

$$G(e_A, e_B) = \sum_C G_{AB}^C e_C, \quad h(e_i, e_j) = \sum_k h_{ij}^{k^*} e_k, \quad \nabla h(e_i, e_j, e_k) = \sum_l h_{ijk}^{l^*} e_{l^*}.$$

From Lemmas 2.1 and 2.3, we have

Lemma 4.1.

$$G_{AB}^C = -G_{BA}^C = -G_{AC}^B, \quad G_{ij}^{k^*} = G_{i^*j}^k = G_{ij^*}^k, \tag{4.1}$$

$$h_{ijk}^{l^*} = h_{ikj}^{l^*}, \quad h_{ij}^{k^*} = h_{ji}^{k^*} = h_{ik}^{j^*}, \tag{4.2}$$

$$h_{ijk}^{l^*} = h_{ijl}^{k^*} + G_{li}^{m^*} h_{jk}^{m^*} - G_{ki}^{m^*} h_{jl}^{m^*}. \tag{4.3}$$

Lemma 4.2. There are suitable e_1 and e_2 such that the corresponding frame $\{e_i, e_{i^*}\}$ has the following properties:

$$\begin{aligned}&h_{11}^{1^*} = -h_{22}^{1^*} = -h_{12}^{2^*} = -h_{21}^{2^*} = a \geq 0, \text{ other } h_{ij}^{k^*} = 0,\\ &\mathbf{J}G(e_1, e_2) = e_3, \quad \mathbf{J}G(e_2, e_3) = e_1, \quad \mathbf{J}G(e_3, e_1) = e_2.\end{aligned} \tag{4.4}$$

Proof. By (4.2) and the choice of e_3 we see that, if any of i, j, k equals to 3, then $h_{ij}^{k^*} = 0$. An application of this with (4.2) and Lemma 2.2 gives

$$h_{11}^{1^*} = -h_{22}^{1^*} = -h_{12}^{2^*} = -h_{21}^{2^*}, \quad h_{11}^{2^*} = -h_{22}^{2^*} = h_{12}^{1^*} = h_{21}^{1^*}. \tag{4.5}$$

If e_1 and e_2 are changed to $\tilde{e}_1$ and $\tilde{e}_2$ by

$$\tilde{e}_1 = e_1 \cos\alpha - e_2 \sin\alpha, \quad \tilde{e}_2 = e_1 \sin\alpha + e_2 \cos\alpha, \tag{4.6}$$

in which α is a real number. Then

$$\begin{aligned} \tilde{h}_{11}^{1^*} &= \langle h(\tilde{e}_1, \tilde{e}_1), \mathbf{J}\tilde{e}_1\rangle = h_{11}^{1^*} \cos 3\alpha - h_{11}^{2^*} \sin 3\alpha, \\ \tilde{h}_{11}^{2^*} &= \langle h(\tilde{e}_1, \tilde{e}_1), \mathbf{J}\tilde{e}_2\rangle = h_{11}^{1^*} \sin 3\alpha + h_{11}^{2^*} \cos 3\alpha. \end{aligned}$$

In case that $h_{11}^{2^*} \neq 0$, one sees easily that there must be some value of α, such that $\tilde{h}_{11}^{2^*} = 0$ and $a = \tilde{h}_{11}^{1^*} \geq 0$.

To proceed, we remark that (4.4) holds if and only if any one of the last three equalities in it holds. For example, we have

$$\mathbf{J}G(e_1, e_2) = e_3 \Leftrightarrow G(e_1, e_2) = -e_{3^*} \Leftrightarrow G_{12}^{k^*} = -\delta_3^k,$$

which with (4.1) implies $G_{23}^{k^*} = -\delta_1^k$ and $G_{31}^{k^*} = -\delta_2^k$. Therefore (4.4) holds. Note also that $\mathbf{J}G(e_1, e_2) = e_3$ is invariant under the transformation (4.6) of e_1, e_2 into $\tilde{e}_1$, $\tilde{e}_2$, we need only to show that there exists one pair of e_1, e_2 satisfying $\mathbf{J}G(e_1, e_2) = e_3$. To this end, fix arbitrarily one e_2 orthogonal to e_3 and put $e_1 = \mathbf{J}G(e_2, e_3)$. Then from the above argument we know that $\mathbf{J}G(e_1, e_2) = e_3$, and so the lemma is proved.

In what follows, we shall fix the frame field $\{e_i, e_{i^*}\}$ such that (4.3) and (4.4) hold.

Lemma 4.3. Let ω_i be dual to e_i. Then, with respect to the frame field $\{e_i, e_{i^*}\}$, the connection components ω_{AB} of $S^6(1)$ restricting to ψ satisfy the following relations:

$$\omega_{1^*2^*} = \omega_{12} - \omega_3, \quad \omega_{1^*3^*} = \omega_{2^*3^*} = 0, \quad \omega_{13} = -\omega_2, \quad \omega_{23} = \omega_1, \tag{4.7}$$

$$\omega_{11^*} = a\omega_1, \quad \omega_{12^*} = -a\omega_2, \quad \omega_{22^*} = -a\omega_1, \quad \omega i3^* = \omega_{3i^*} = 0.$$

Proof. By (2.5) and (4.4),

$$G_{ij}^{k^*} e_{k^*} = G(e_i, e_j) = D_{e_i}\mathbf{J}e_j - \mathbf{J}\nabla_{e_i} e_j = \omega_{j^*k^*}(e_i)e_{k^*} - \omega_{jk}(e_i)e_{k^*}.$$

Therefore

$$\omega_{j^*k^*} = G_{ij}^{k^*} \omega_i + \omega_{jk}. \tag{4.8}$$

On the other hand, by the choice of e_3, $\omega_{1^*3^*} = \omega_{2^*3^*} = 0$. This together with (4.4) and (4.8) gives (4.7). Other parts of the lemma is direct.

From the equation of Gauss and (4.3), we can write out the components of the curvature tensor of ψ as follows:

$$R_{ijkl} = \begin{cases} \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}, & \text{if some of } i,\ j,\ k,\ l \text{ equals to } 3, \\ (1 - 2a^2)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), & \text{otherwise.} \end{cases} \tag{4.9}$$

Lemma 4.4. The components of ∇h satisfy the following equations:

$$h_{223}^{2^*} = -h_{123}^{1^*} = -h_{113}^{2^*} = a, \quad h_{ijk}^{3^*} = h_{33i}^{j^*} = 0, \quad h_{123}^{2^*} = h_{113}^{1^*} = h_{223}^{1^*} = 0, \tag{4.10}$$

$$\begin{aligned} 3a\omega_{12} &= h_{111}^{2^*}\omega_1 + h_{112}^{2^*}\omega_2 = h_{112}^{1^*}\omega_1 + h_{122}^{1^*}\omega_2, \\ da &= h_{111}^{1^*}\omega_1 + h_{112}^{2^*}\omega_2 = -h_{112}^{2^*}\omega_1 - h_{122}^{2^*}\omega_2. \end{aligned} \tag{4.11}$$

Proof. By the definition of ∇h, we have

$$h_{ijk}^{l^*}\omega_k = dh_{ij}^{l^*} + h_{kj}^{l^*}\omega_{ki} + h_{ik}^{l^*}\omega_{kj} + h_{ij}^{k^*}\omega_{k^*l^*}.$$

Then Lemma 4.4 follows from Lemmas 4. 1, 4. 2 and 4. 3 by direct calculations.

Now we are in a position to complete the proof of Theorem A.

By the equation of Gauss, $R = 6 - 4a^2$. Thus the condition that R =const is equivalent to a =const. By (4.11), we have either $a = 0$ or $\omega_{12} \equiv 0$.

If $a = 0$, then $R = 6$ and ψ is totally geodesic;

If $a \neq 0$, then $\omega_{12} \equiv 0$. Take exterior differentiation of this identity, we get by Lemma 4.3 and (4.9),

$$0 \equiv d\omega_{12} = \omega_{13} \wedge \omega_{32} - \frac{1}{2}R_{12ij}\omega_i \wedge \omega_j = 2(2a^2 - 1)\omega_1 \wedge \omega_2 \tag{4.12}$$

Thus we obtain that $a^2 = 1$, namely $R = 2$. In this case, the connection forms of $S^6(1)$ restricting to ψ are completely determined by Lemma 4.3 as follows:

$$\begin{aligned} &\omega_{1^*2^*} = -\omega_3, \quad \omega_{1^*3^*} = \omega_{2^*3^*} = 0, \quad \omega_{13} = -\omega_2, \quad \omega_{23} = \omega_1, \\ &\omega_{11^*} = \omega_1, \quad \omega_{12^*} = -\omega_2, \quad \omega_{22^*} = -\omega_1, \quad \omega_{23^*} = \omega_{3i^*} = 0. \end{aligned}$$

From these equalities one gets easily that

$$\begin{aligned} &\nabla_{e_i}e_i = 0, \quad \nabla_{e_1}e_2 = -\nabla_{e_2}e_1 = e_3, \quad \nabla_{e_1}e_3 = -e_2, \\ &\nabla_{e_2}e_3 = e_1, \quad \nabla_{e_3}e_1 = \nabla_{e_3}e_2 = 0, \\ &h(e_1, e_1) = -h(e_2, e_2) = e_{1^*}, \quad h(e_1, e_2) = -e_{2^*}, \quad h(e_i, e_3) = 0, \end{aligned}$$

all having the same constant coefficients as in Proposition 3.4. Using Lemma 5.6 in [6], M is homogeneous and locally isometric to S^3. Let $\{E_i\}$ be a frame on S^3 satisfying Proposition 3.4 for the fixed immersion x^0. Then there is a local isometry φ of an open $U \subset M$ into S^3, such that $\varphi_* e_i = E_i$. Let ϕ be the map between the normal bundles of U and $\varphi(U)$ in $S^6(1)$ defined by $\phi e_{i^*} = E_{i^*}$. Then ϕ preserves the bundle metric, the second fundamental form and the normal connection. By the rigidity theorem of submanifolds, U and $\varphi(U)$ are congruent, that is, there is a motion $\sigma \in SO(7)$ such that $\sigma \circ x^0 \circ \varphi = \psi$ on U.

On the other hand, By Lemma 2.1 and the total reality, we know that $G(e_i, e_j) = e_i \times e_j$, $G(E_i, E_j) = E_i \times E_j$. Here we have omitted the tangent maps x_*^0 and ψ_*. An application of this fact with (3.13) and (4.4) gives that

$$\begin{aligned} &e_{1^*} = \psi \times e_1 = e_3 \times e_2, \quad e_{2^*} = \psi \times e_2 = e_1 \times e_3, \quad e_{3^*} = \psi \times e_3 = e_2 \times e_1, \\ &E_{1^*} = x^0 \times E_1 = E_3 \times E_2, \quad E_{2^*} = x^0 \times E_2 = E_1 \times E_3, \\ &E_{3^*} = x^0 \times E_3 = E_2 \times E_1. \end{aligned}$$

Since $\sigma e_i = E_i$ and $\sigma e_{i^*} = E_{i^*}$, it is clearly that σ preserves the cross product and so belongs to G_2. This completes the proof of Theorem A.

Remark 4.5 The argument of G_2-congruence is similar to that in [6]. If M is complete, connected and simply connected, then M is globally isometric to S^3, and the argument above shows that ψ is G_2-congruent to x^0.

5. PROOFS OF THEOREM B AND THE COROLLARIES

1°. *Proof of Theorem B.*

Not loss of generality, we still assume $\psi(M) \subset S^5(1) \subset S^6(1)$. Then from (4.8) and the minimality we get an equality as

$$3a\omega_{12} = a_{,2}\omega_1 - a_{,1}\omega_2, \text{ where } da = a_{,1}\omega_1 + a_{,2}\omega_2,$$

which can be written into

$$6a^2\omega_{12} = a^2_{,2}\omega_1 - a^2_{,1}\omega_2.$$

Thus at points where $a^2 \neq 0$,

$$\omega_{12} = \frac{1}{6}[(\log a^2)_{,2}\omega_1 - (\log a^2)_{,1}\omega_2].$$

Since (see (4.13)) $d\omega_{12} = 2(a^2 - 1)\omega_1 \wedge \omega_2$, we obtain the following identities:

$$\Delta \log a^2 = 12(1 - a^2), \text{ at points where } a^2 \neq 0, \tag{5.1}$$

$$\Delta a^2 = 12a^2(1 - a^2) + 16|\nabla a|^2, \text{on} M. \tag{5.2}$$

We can also assume M to be connected, otherwise we need only to consider each of its connected components. Thus by the equation of Gauss, either $a^2 > 1$ or $a^2 < 1$. Since M is compact, the former case is impossible and hence $a^2 < 1$. Now (5.2) and the compactness imply that $a \equiv 0$, that is ψ is totally geodesic.

2°. *Proof of Corollary C.*

If $R \geq 2$, namely, $a^2 \leq 1$, then by (5.2) and the compactness, we have either $a^2 \equiv 0$ or $a^2 \equiv 1$;

If $R \leq 2$, namely, $a^2 \geq 1$, then by (5.1) and the compactness, we have $a^2 \equiv 1$, that is $R \equiv 2$.

Thus by Remark 4.5, it is not hard to see that either ψ is totally geodesic or, the universal covering of it is G_2-congruent to x^0.

3°. *Proof of Corollary D.*

We need only to show that $\psi(M)$ can not be contained in a $S^4(1) \in S^6(1)$ unless it is totally geogesic and this is rather clear. In fact, if we are able to choose one more constant unit normal vector field e_{2^*} at the begining, then $h(e_i, e_2)$ will also be zero identically, which with (4.13) implies $a \equiv 0$.

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