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ON THE CLASS OF CONTACT METRIC MANIFOLDS WITH A 3-τ-STRUCTURE

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1. INTRODUCTION

In [7] Gouli-Andreou and Xenos introduced the notion of a contact metric structure being a 3-τ-structure and developed some of its basic properties. Known examples however are contact metric manifolds satisfying the stronger condition that their Ricci operator commute with the fundamental collineation ϕ. In this paper we show that contact metric manifolds with a 3-τ-structure indeed form a larger class and the example we give is also of interest in terms of special directions introduced in [3] on contact metric manifolds with negative sectional curvature for plane sections containing the characteristic vector field ξ.

2. CONTACT METRIC MANIFOLDS WITH A 3-τ-STRUCTURE

By a *real contact manifold* we mean a C^∞ manifold M^{2n+1} together with a 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. It is well known that given η there exists a unique vector field ξ such that $d\eta(\xi, X) = 0$ and $\eta(\xi) = 1$ called the *characteristic vector field* or *Reeb vector field* of the contact structure η. A classical theorem of Darboux states that on a contact manifold there exist local coordinates with respect to which $\eta = dz - \sum_{i=1}^{n} y^i dx^i$. We denote the *contact subbundle* or *contact distribution* defined by the subspaces $\{X \in T_m M : \eta(X) = 0\}$ by $\mathcal{D}$. Roughly speaking the meaning of the contact condition, $\eta \wedge (d\eta)^n \neq 0$, is that the contact subbundle is as far from being integrable as possible. In fact for a contact manifold the maximum dimension of an integral submanifold of $\mathcal{D}$ is only n; whereas a subbundle defined by a 1-form η is integrable if and only if $\eta \wedge d\eta \equiv 0$.

A Riemannian metric g is an *associated metric* for a contact form η if there exists a tensor field ϕ of type (1,1) such

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(X) = g(X, \xi), \quad d\eta(X, Y) = g(X, \phi Y).$$

We refer to (η, g) or (ϕ, ξ, η, g) as a *contact metric structure*. All associated metrics have the same volume element, viz., $\frac{(-1)^n}{2^n n!}\eta \wedge (d\eta)^n$. Since $d\eta(\xi, X) = 0$ and $\eta(\xi) = 1$, computing Lie derivatives, we have $\mathcal{L}_\xi \eta = 0$ and $\mathcal{L}_\xi d\eta = 0$. Thus the flow generated by ξ is volume preserving with respect to any associated metric.

In the theory of contact metric manifolds there is another tensor field that plays a fundamental role, viz. $h = \frac{1}{2}\mathcal{L}_\xi \phi$. h is a symmetric operator which anti-commutes with ϕ, $h\xi = 0$ and h vanishes if and only if ξ is Killing. We denote by ∇ the Levi-Civita connection of g and by R its curvature tensor. On a contact metric manifold we have the following important relation involving h,

$$\nabla_X \xi = -\phi X - \phi h X. \qquad (*)$$

Since $h\phi + \phi h = 0$, if λ is an eigenvalue of h with eigenvector X, then $-\lambda$ is also an eigenvalue with eigenvector ϕX. Thus, since $h\xi = 0$, in dimension 3 we have only one eigenfunction λ on the manifold to be concerned with.

The sectional curvature of a plane section containing ξ is called a *ξ-sectional curvature*. In this paper, except for the result from [4] described in the next paragraph, we do not need the notion of a Sasakian manifold, though it may be worth pointing out that the ξ-sectional curvature of a Sasakian manifold is $+1$. For a general reference to the ideas so far in this section see [2].

In [4] it was shown that a 3-dimensional contact metric manifold M^3 whose Ricci operator Q commutes with the tensor field ϕ is either Sasakian, flat or locally isometric to a left-invariant metric on the Lie group $SU(2)$ or $SL(2,\mathbb{R})$. In the latter cases M^3 has constant ξ-sectional curvature $k = 1 - \lambda^2 < 1$ and the sectional curvature of a plane section orthogonal to ξ is $-k$ (see also [5]), and the stucture occurs on these Lie groups with $k > 0$ for $SU(2)$ and $k < 0$ for $SL(2,\mathbb{R})$. It was also shown in [5] (see Lemma 3.1) that on a 3-dimensional contact metric manifold satisfying $Q\phi = \phi Q$, the eigenfunction λ is a constant.

On 3-dimensional contact metric manifolds the condition $Q\phi = \phi Q$ is equivalent to other important curvature conditions (see e.g. [5]). It is equivalent to the contact metric manifold being η-Einstein and it is equivalent to the characteristic vector field ξ belonging to the k-nullity distribution, i.e. $R_{XY}\xi = k(\eta(Y)X - \eta(X)Y)$.

In [7] Gouli-Andreou and Xenos introduced the notion of a *3-τ-manifold*, namely a 3-dimensional contact metric manifold on which

$$\nabla_\xi h = 0.$$

The name comes from the equivalent condition $\nabla_\xi \tau = 0$ where $\tau = \mathcal{L}_\xi g$; in particular τ and h are related by $\tau(X,Y) = 2g(h\phi X, Y)$. Known examples, however, are contact metric manifolds satisfying the stronger condition that their Ricci operator Q commutes with ϕ and the two conditions are not unrelated. The following proposition is proved in [3] but for completeness we give the proof here as well.

Proposition. A 3-dimensional contact metric manifold on which $Q\phi = \phi Q$ is a 3-τ-manifold. A 3-τ-manifold on which $Q\xi$ is collinear with ξ satisfies $Q\phi = \phi Q$.

Proof. If $Q\phi = \phi Q$, then $\phi\xi = 0$ gives $\phi Q\xi = 0$ and hence that $Q\xi$ is collinear with ξ. In [8] (Proposition 3.1) Perrone proved that on a 3-dimensional contact metric manifold

$$(\nabla_\xi \tau)(X,Y) = g(Q\phi X, \phi Y) - g(QX,Y) + \eta(X)g(Q\xi, Y) + \eta(Y)g(Q\xi, X)$$

$$-\eta(X)\eta(Y)g(Q\xi,\xi).$$

Thus if $Q\phi = \phi Q$, $\nabla_\xi \tau = 0$ giving the first statement.

If $(\nabla_\xi \tau)(X,Y) = 0$ and $Q\xi = f\xi$, Perrone's formula yields $g(Q\phi X, \phi Y) - g(QX,Y) + f\eta(X)\eta(Y) = 0$ or

$$-\phi Q\phi X - QX + f\eta(X)\xi = 0.$$

Applying ϕ and noting that $\eta(Q\phi X) = g(\xi, Q\phi X) = g(Q\xi, \phi X) = 0$, we have $Q\phi = \phi Q$ as desired. □

We may regard equation $(*)$ as indicating how ξ or, by orthogonality, the contact subbundle, rotates as one moves around on the manifold. For example when $h = 0$, as we move in a direction X orthogonal to ξ, ξ is always "turning" or "falling" toward $-\phi X$. If $hX = \lambda X$, then $\nabla_X \xi = -(1+\lambda)\phi X$ and again ξ is turning toward $-\phi X$ if $\lambda > -1$ or toward ϕX if $\lambda < -1$. Recall that we noted above that if λ is an eigenvalue of h with eigenvector X, then $-\lambda$ is also an eigenvalue with eigenvector ϕX. However one can ask if there can ever be directions, say Y orthogonal to ξ, along which ξ "falls" forward or backward in the direction of Y itself. In [3] the author proved the following result.

Theorem. Let M^{2n+1} be a contact metric manifold. If the tensor field h admits an eigenvalue $\lambda > 1$ at a point P, then there exists a vector Y orthogonal to ξ at P such that $\nabla_Y \xi$ is collinear with Y. In particular if M^{2n+1} has negative ξ-sectional curvature such directions Y exist.

Note that when there exists a direction Y along which $\nabla_Y \xi$ is collinear with Y, say $\nabla_Y \xi = \alpha Y$, $\alpha = -\sqrt{\lambda^2 - 1}$, and there is also a second such direction Z. For Z we have $\nabla_Z \xi = -\alpha Z$; thus we think of ξ as falling backward as we move in the direction Y and falling forward as we move in the direction Z. We also note that

$$g(Y, Z) = \frac{-1}{\lambda}$$

and hence that such directions Y and Z are never orthogonal.

3. ANOSOV FLOWS

Classically an Anosov flow is defined as follows [1, pp. 6-7]. Let M be a compact differentiable manifold, ξ a non-vanishing vector field and $\{\psi_t\}$ its 1-parameter group of diffeomorphisms. $\{\psi_t\}$ is said to be an *Anosov flow* (or ξ to be *Anosov*) if there exist subbundles E^s and E^u which are invariant along the flow and such that $TM = E^s \oplus E^u \oplus \{\xi\}$ and there exists a Riemannian metric such that

$$|\psi_{t*}Y| \le a e^{-ct}|Y| \text{ for } t \ge 0 \text{ and } Y \in E^s_p,$$

$$|\psi_{t*}Y| \le a e^{ct}|Y| \text{ for } t \le 0 \text{ and } Y \in E^u_p$$

where a and c are positive constants independent of $p \in M$ and Y in E^s_p or E^u_p. E^s and E^u are called the *stable* and *unstable* subbundles or the *contracting* and *expanding* subbundles.

When M is compact the notion is independent of the Riemannian metric. If M is not compact the notion is metric dependent. In our example of a contact metric manifold with a 3-τ-structure below, we will give a metric on $\mathbb{R}^3$ with respect to which the coordinate field $\frac{\partial}{\partial z} = \frac{1}{2}\xi$ is Anosov, even though $\frac{\partial}{\partial z}$ is clearly not Anosov with respect to the Euclidean metric on $\mathbb{R}^3$.

Now let M be a 3-dimensional contact metric manifold with negative ξ-sectional curvature. It was shown in [3] that if the characteristic vector field ξ generates an Anosov flow and the special directions agree with the stable and unstable directions, then the contact metric structure is a 3-τ-structure. Moreover in the compact case one has that M satisfies $Q\phi = \phi Q$ and that M is a compact quotient of $\widetilde{SL}(2,\mathbb{R})$. This can be proved from properties of a

3-τ-structure [3] or seen from a result of E. Ghys [6] that if ξ is Anosov on a compact 3-dimensional contact manifold M and the stable and unstable directions are smooth, then M is a compact quotient of $\widetilde{SL}(2,\mathbb{R})$.

As an aside we note that on a compact manifold, an Anosov flow has a countable number of periodic orbits [1, Theorem 2] and if the flow admits an integral invariant, in particular if it is volume preserving, then the set of periodic orbits is dense in M [1, Theorem 3]. This in itself has some implications for contact geometry. An important conjecture of Weinstein [9] is that on a simply connected compact contact manifold ξ must have a closed orbit, so in particular the Weinstein conjecture holds for a compact contact manifold on which ξ is Anosov. There is no known example of a non-simply-connected compact contact manifold for which ξ does not have a closed orbit and the author has long felt that the Weinstein conjecture is true without the assumption of simple connectivity. The 3-dimensional torus, has a contact structure for which the set of periodic orbits is dense but no non-periodic orbit is dense in the whole manifold, see e.g. [2, p. 8].

4. 3-τ-MANIFOLDS WITH $Q\phi \neq \phi Q$

We now show the existence of a family of 3-τ-manifolds on which $Q\phi \neq \phi Q$.

Theorem. $\mathbb{R}^3$ with the standard Darboux contact form $\eta = \frac{1}{2}(dz - ydx)$ carries associated metrics giving 3-τ-structures for which $Q\phi \neq \phi Q$.

Proof. The characteristic vector field of $\eta = \frac{1}{2}(dz - ydx)$ is $\xi = 2\frac{\partial}{\partial z}$. Let f be a smooth function of x and y bounded below by a positive constant c. Then the metric given by

$$g = \frac{1}{4}\begin{pmatrix} \frac{e^{zf}+(1+f^2)e^{-zf}-2}{f^2} + y^2 & \frac{e^{zf}-1}{f} & -y \\ \frac{e^{zf}-1}{f} & e^{zf} & 0 \\ -y & 0 & 1 \end{pmatrix}$$

is an associated metric. The tensor fields ϕ and h are given by

$$\phi = \begin{pmatrix} \frac{e^{zf}-1}{f} & e^{zf} & 0 \\ -(\frac{e^{zf}+(1+f^2)e^{-zf}-2}{f^2}) & -\frac{e^{zf}-1}{f} & 0 \\ y(\frac{e^{zf}-1}{f}) & ye^{zf} & 0 \end{pmatrix},$$

$$h = \begin{pmatrix} e^{zf} & fe^{zf} & 0 \\ -(\frac{fe^{zf}+(1+f^2)(-f)e^{-zf}}{f^2}) & -e^{zf} & 0 \\ ye^{zf} & yfe^{zf} & 0 \end{pmatrix}.$$

By direct computation $\nabla_\xi h = 0$ and therefore $\mathbb{R}^3$ with this structure is a 3-τ-manifold. Also $2\lambda^2 = trh^2 = 2(1+f^2)$ and hence the positive eigenfunction of h is $\lambda = \sqrt{1+f^2} > 1$. As we remarked earlier and as was shown in [5], on a 3-dimensional contact metric manifold satisfying $Q\phi = \phi Q$, the eigenfunction λ is a constant. Thus if f is not constant, this structure on $\mathbb{R}^3$ is a 3-τ-structure satisfying $Q\phi \neq \phi Q$. □

For this structure the special directions discussed in Section 2 are given by

$$Y = f\frac{\partial}{\partial x} - \frac{\partial}{\partial y} + yf\frac{\partial}{\partial z}, \quad Z = \frac{\partial}{\partial y}.$$

To check that $\xi = 2\frac{\partial}{\partial z}$ is Anosov with respect to g, consider for simplicity just $\frac{\partial}{\partial z}$; its flow ψ_t maps a point $P_0(x, y, z)$ to the point $P(x, y, z + t)$. Now recalling that the function f was chosen to be bounded below by a positive constant c, we have for $t \leq 0$,

$$\left|\psi_{t*}\frac{\partial}{\partial y}(P_0)\right| = \left|\frac{\partial}{\partial y}(P)\right| = \frac{1}{2}e^{\frac{(z+t)f}{2}} = e^{\frac{tf}{2}}\left|\frac{\partial}{\partial y}(P_0)\right| \leq e^{\frac{ct}{2}}\left|\frac{\partial}{\partial y}(P_0)\right|.$$

Similarly for $t \geq 0$,

$$\left|\psi_{t*}\Big(f\frac{\partial}{\partial x} - \frac{\partial}{\partial y} + yf\frac{\partial}{\partial z}\Big)(P_0)\right| = \left|\Big(f\frac{\partial}{\partial x} - \frac{\partial}{\partial y} + yf\frac{\partial}{\partial z}\Big)(P)\right| = \frac{1}{2}\sqrt{1+f^2}e^{\frac{-(z+t)f}{2}}$$

$$= e^{\frac{-tf}{2}}\left|\Big(f\frac{\partial}{\partial x} - \frac{\partial}{\partial y} + yf\frac{\partial}{\partial z}\Big)(P_0)\right| \leq e^{\frac{-ct}{2}}\left|\Big(f\frac{\partial}{\partial x} - \frac{\partial}{\partial y} + yf\frac{\partial}{\partial z}\Big)(P_0)\right|.$$

Thus $\frac{\partial}{\partial z}$, equivalently ξ, is Anosov with respect to this metric; Y determines the stable subbundle and Z the unstable subbundle.

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