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WHEN MAY TWO SYSTEMS OF ORTHONORMAL FUNCTIONS BE INTERCHANGED IN VECTOR-VALUED ORTHOGONAL SUMS?

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Abstract. *Given a finite orthonormal sequence $\Phi_n = (\varphi_1, \dots, \varphi_n)$ in some $L_2(\mu)$ and vectors $x_1, \dots, x_n$ in some Banach space X we are interested in the norm of the sums $\sum_{j=1}^n \varphi_j(t)x_j$ in $L_2^X(\mu)$. A constuction in [1] suggests that the system Φ_n may be replaced by the set $\Pi_n = (\pi_1, \dots, \pi_n)$ of coordinate functions $\pi_j(\sigma_1, \dots, \sigma_n) = \sigma_j$ on $\mathbb{S}^{n-1}$ viewed as an orthonormal system with respect to a suitable measure λ on $\mathbb{S}^{n-1}$. We show by a convolutional argument that after symmetrization the measure λ is uniquely determined. We also discuss related questions.*

1. INTRODUCTION

Many features in Banach space theory such as type and cotype may be stated in terms of suitable orthogonal vector-valued sums and inequalities between their L_2-norms (cf. [2], [1]). In our setting we focus on sums

$$\sum_{j=1}^{n} \varphi_j x_j$$

where the $x_1, \dots, x_n$ are vectors in some Banach space X and the n-tupel of functions $\Phi_n = (\varphi_1, \dots, \varphi_n)$ is an orthonormal system in some Hilbert space $L_2(\mu)$. We think of n and Φ_n as fixed for a moment. If $\Psi_n = (\psi_1, \dots, \psi_n)$ is another orthonormal system in some $L_2(\lambda)$ there seems to exist no general criterion whether we have for instance

$$\Big(\int \Big\| \sum_{j=1}^{n} \varphi_j(t)x_j \Big\|^2 \mu(dt)\Big)^{1/2} \leq C\Big(\int \Big\| \sum_{j=1}^{n} \psi_j(s)x_j \Big\|^2 \lambda(ds)\Big)^{1/2} \qquad (1)$$

regardless of the Banach spaces X and the vectors $x_1, \dots, x_n$ in X with some given constant $C \geq 1$. Not so, if in (1) we insist on equality and $C = 1$. We shall see that in the affirmative case the two systems in question will share the same *projective distribution.*

Before engaging in the definition, let us first fix the notation.

The scalar field will be $\mathbb{C}$. With obvious modifications the results will apply to the real case simultaneously.

We shall write $\|x\|_2$ for the euclidean norm of a vector $x = (\xi_1, \dots, \xi_n)$ in ℓ_2^n. Moreover, $x^* : \ell_2^n \to \mathbb{C}$ will be the corresponding functional and $x^* \otimes x$ the $n \times n$ matrix with entries $\overline{\xi_j}\xi_k$. The unit vectors are denoted by $e_1, \dots, e_n$.

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$\mathbb{S}^{n-1}$ is the set of all vectors in $\mathbb{C}^n$ of euclidean norm 1. The natural mappings

$$\pi_j : \mathbb{S}^{n-1} \to \mathbb{C}, \qquad s = (\sigma_1, \dots, \sigma_n) \mapsto \sigma_j \qquad (j = 1, \dots, n)$$

will play a special part in our theory. We use the symbol $\Pi_n = (\pi_1, \dots, \pi_n)$ for this system of functions.

$C(\mathbb{S}^{n-1})$ is the Banach space of all continuous complex functions on $\mathbb{S}^{n-1}$. By the Riesz representation theorem its dual is $M(\mathbb{S}^{n-1})$, the space of all complex measures on $\mathbb{S}^{n-1}$. The duality is given by

$$(f, \beta) \mapsto \int_{\mathbb{S}^{n-1}} g(s)\beta(ds).$$

On the torus $\mathbb{T}$, the group of complex numbers of modulus 1, we denote the Haar measure by $m_{\mathbb{T}}$. Similarly, on the group $\mathbf{U}_n$ of unitary $n \times n$ matrices we denote the Haar measure by $m_{\mathbf{U}_n}$. The unit matrix is I_n.

For our purposes it will be convenient not to distinguish between an n-tupel of functions $\Phi_n = (\varphi_1, \dots, \varphi_n)$ in some $L_2(\mu)$ and the Σ-measurable map given by

$$\Phi_n : \Omega \to \mathbb{C}^n \ , \ t \mapsto \Phi_n(t) = \sum_{j=1}^{n} \varphi_j(t)e_j.$$

It is important to mention that $\Phi_n = (\varphi_1, \dots, \varphi_n)$ is an orthonormal system if and only if

$$\int \Phi_n(t) * \otimes \Phi_n(t)\mu(dt) = I_n. \tag{2}$$

Note that (2) is merely shorthand for

$$\int \overline{\varphi_j(t)}\varphi_k(t)\mu(dt) = \delta_{jk} \qquad (j, k = 1, \dots, n).$$

Note that any measure μ that fulfills these n^2 conditions will turn a give map $\Omega_n : \Sigma \to \mathbb{C}^n$ into an orthonormal system. When it is advisable to be more careful about the underlying measure we rather use the symbol $[\Phi_n, \mu]$ in order to indicate the dependence.

If there are given vectors $x_1, \dots, x_n$ in some Banach space X we define

$$U : \ell_2^n \to X, e_j \mapsto x_j \quad (j = 1, \dots, n).$$

Let us denote the Banach space of square Bochner-μ-integrable X-valued functions by $L_2^X(\mu)$. Then $U\Phi_n : t \mapsto U(\Phi_n(t))$ is a member of L_2^X and we have

$$\left(\int \| \sum_{j=1}^{n} \varphi_j(t)x_j \|^2 \mu(dt) \right)^{1/2} = \|U\Phi_n\|_{L_2^X(\mu)}.$$

2. THE PROJECTIVE DISTRIBUTION

Let us fix n and an orthonormal system $\Phi_n \subset L_2(\mu)$ for the time being. Given $U : \ell_2^n \to X$ we may certainly write

$$
\begin{aligned}
\|U\Phi_n\|_{L_2^X 9\mu)} &= \left(\int_{\{\Phi_n \neq 0\}} \|U\Phi_n(t)\|^2 \mu(dt) \right)^{1/2} \\
&= \left(\int_{\{\Phi_n \neq 0\}} \left\| \frac{U\Phi_n(t)}{\|\Phi_n(t)\|_2} \right\|^2 \|\Phi_n(t)\|_2^2 \mu(dt) \right)^{1/2} \qquad (3) \\
&= \left(\int_{\mathbb{T}} \int_{\{\Phi_n \neq 0\}} \left\| U\left(\frac{\sigma\Phi_n(t)}{\|\Phi_n(t)\|_2} \right) \right\|_X^2 \|\Phi_n(t)\|_2^2 \; \mu(dt) m_{\mathbb{T}}(d\sigma) \right)^{1/2} .
\end{aligned}
$$

This observation forces our way.

Definition 1. *Let $\Phi_n \subset L_2(\mu)$ be an orthonormal system. The measure $\lambda = \lambda\,(\Phi_n, \mu)$ on $\mathbb{S}^{n-1}$ given by*

$$
\int f(s)\lambda(ds) = \int_{\mathbb{T}} \int_{\{\Phi_n \neq 0\}} f\left(\frac{\sigma\Phi_n(t)}{\|\Phi_n(t)\|_2} \right) \|\Phi_n(t)\|_2^2 \mu(dt) m_{\mathbb{T}}(d\sigma) \qquad (f \in C(\mathbb{S}^{n-1})) \qquad (4)
$$

is called the projective distribution *of Φ_n (with respect to μ).*

A note on the terminology is in order: Averaging over $\mathbb{T}$ will force λ to be $\mathbb{T}$-invariant. Thus, λ may be looked upon as a measure on the projective plane $\mathbb{C}_*^n \,/\, \mathbb{C}_* \equiv \mathbb{S}^{n-1} \,/\, \mathbb{T}$. Indeed, whithout symmetrization and with a different normalization this is exactly the construction in [1], Lemma 3.7 (1), p. 428.

Theorem 2. *Let $\Phi_n \subseteq L_2(\mu)$ be an orthonormal system. Then its projective distribution λ is uniquely determined by the following properties:*

(i) The system of projections $\Pi_n = (\pi_1, \ldots, \pi_n)$ is orthonormal with respect to λ. In particular λ has total mass n.

(ii) λ is $\mathbb{T}$-invariant

(iii) For all Banach space X und for all $U : \ell_2^n \to X$ we have

$$
\|U\Pi_n\|_{L_2^X(\lambda)} = \|U\Phi_n\|_{L_2^X(\mu)}.
$$

Proof. Clearly, λ is a positive measure on $\mathbb{S}^{n-1}$. Let us start by verifying (i) to (iii).

(i): For fixed $j, l \in \{1, \ldots, n\}$ define a continuous function g on $\mathbb{S}^{n-1}$ by $g(s) = \overline{\pi_j(s)}\ \pi_l(s)$ $(s \in \mathbb{S}^{n-1})$. Then,

$$
\begin{aligned}
\int_{\mathbb{S}^{n-1}} \overline{\pi_j(s)}\pi_l(s)\lambda(ds) &= \int\int \frac{\overline{\sigma\varphi_j(t)}}{\|\Phi_n(t)\|} \frac{\sigma\varphi_l(t)}{\|\Phi_n(t)\|} \|\Phi_n(t)\|^2 \mu(dt) m_{\mathbb{T}}(d\sigma) \\
&= \int \overline{\varphi_j(t)}\varphi_j(t)\mu(dt) \\
&= \delta_{jl}.
\end{aligned}
$$

Moreover, if we put $g(s) = \|\Pi_n(s)\|_2^2 = \sum_{j=1}^n |\sigma_j|^2 \equiv 1$ ($s \in \mathbb{S}^{n-1}$) we get

$$\lambda(\mathbb{S}^{n-1}) = \int \|\Pi_n(s)\|_2^2 \lambda(ds) = \int\int \|\zeta\Phi_n(t)\|_2^2 \mu(dt) m_{\mathbb{T}}(d\zeta) = \sum_{j=1}^n \int |\varphi_j(t)|^2 \mu(dt) = n.$$

(ii): For continuous functions g on $\mathbb{S}^{n-1}$ and complex numbers τ of modulus 1 we have

$$\begin{aligned}\int f(\tau s)\lambda(ds) &= = \int_{\mathbb{T}}\int_{\Phi_n\neq 0} f\left(\frac{\tau\sigma\Phi_n(t)}{\|\Phi_n(t)\|_2}\right)\|\Phi_n(t)\|_2^2 \mu(dt) m_{\mathbb{T}}(d\sigma)\\ &= \int_{\mathbb{T}}\int_{\Phi_n\neq 0} f\left(\frac{\sigma\Phi_n(t)}{\|\Phi_n(t)\|_2}\right)\|\Phi_n(t)\|_2^2 \mu(dt) m_{\mathbb{T}}(d\sigma) = \int f(s)\lambda(ds).\end{aligned}$$

(iii): This is the very definition of λ. Given $U : \ell_2^n \to X$, define $g \in C(\mathbb{S}^{n-1})$ by $g(s) = \|U\Pi_n(s)\|^2$ ($s \in \mathbb{S}^{n-1}$). We get

$$\begin{aligned}\|U\Pi_n\|_{L_2^X(\lambda)} &= \left(\int \|U\Pi_n(s)\|^2 \lambda(ds)\right)^{1/2}\\ &= \left(\int \|U\Phi_n(t)\|^2 \mu(dt)\right)^{1/2} = \|U\Phi_n\|_{L_2^X(\mu)}.\end{aligned}$$

As for the reverse, we shall apply a density argument.

We will construct a sequence $\{\|\cdot\|_{(k)}\}_{k=1}^\infty$ of norms on $\mathbb{C}^n$ such that any continuous and $\mathbb{T}$-invariant function g may be uniformly approximated by linear combinations of the form

$$s \mapsto \sum_{j=1}^N a_j \|V_j\Pi_n(s)\|_{(k)}^2,$$

where $a_j \in \mathbb{C}$ and $V_j \in \mathbf{U}_n$.

Indeed, if we are given two measures λ and $\tilde\lambda$ on $\mathbb{S}^{n-1}$ with the property that

$$\int \|U\Pi_n(s)\|_X^2 \lambda(ds) = \int \|U\Phi_n(t)\|_X^2 \mu(dt) = \int \|U\Pi_n(s)\|_X^2 \tilde\lambda(ds)$$

however we choose X and $U : \ell_2^n \to X$ then we may conclude by approximation that

$$\int g(s)\lambda(ds) = \int g(s)\tilde\lambda(ds)$$

for every $\mathbb{T}$-invariant $g \in C(\mathbb{S}^{n-1})$. Furthermore, if $f \in C(\mathbb{S}^{n-1})$ then

$$g(s) = \int f(\tau s) m_{\mathbb{T}}(d\tau) \qquad (s \in \mathbb{S}^{n-1})$$

is $\mathbb{T}$-invariant and by virtue of (ii)

$$\int f(s)\lambda(ds) = \int\int f(\tau s)\lambda(ds)m_{\mathbb{T}}(d\tau) = \int g(s)\lambda(ds) = \int g(s)\tilde{\lambda}(ds).$$

The same holds if we interchange $\tilde{\lambda}$ and λ, hence

$$\int f(s)\lambda(ds) = \int f(s)\tilde{\lambda}(ds).$$

Appealing to Riesz representation theorem shows that $\lambda = \tilde{\lambda}$ and thus our issue is settled. We proceed in four steps.

Step 1: Determine $r_k > 0 (k = 1, 2, \dots)$ such that

$$|\sigma_1| > r_k \quad \text{if} \quad \|s - \tau e_1\|_2 < 2^{-1} \quad \text{for some} \quad \tau \in \mathbb{T} \qquad (s = (\sigma_1, \dots, \sigma_n) \in \mathbb{S}^{n-1})$$

Define norms

$$\begin{aligned} \|\cdot\|_{(k)} : \mathbb{C}^n &\to \mathbb{R}_+ \\ x = (\xi_1, \dots, \xi_n) &\mapsto \max\left\{\frac{|\xi_1}{r_k}, \|x\|_2\right\}. \end{aligned}$$

Recall that $\|\cdot\|_2$ is the euclidean norm. $\|\cdot\|_{(k)}$ is certainly a norm again and by construction

$$\|s\|_{(k)} = 1 \quad \text{if} \quad \|s - \tau e_1\|_2 \geq 2^{-k} \quad \text{for all} \quad \tau \in \mathbb{T} \quad (s \in \mathbb{S}^{n-1}, k = 1, 2, \dots) \tag{5}$$

Step 2: For the following we denote the rotational invariant probability measure on $\mathbb{S}^{n-1}$ by m. We continue by defining a sequence $(h_k)_{k=1}^{\infty}$ of continuous non-negative functions on $\mathbb{S}^{n-1}$ by

$$h_k(s) = \frac{\|s\|_{(k)}^2 - 1}{\int \|y\|_{(k)}^2 m(dy) - 1} \quad (s \in \mathbb{S}^{n-1})$$

Note that the denominator does not vanish. Obviously, all h_k are continuous. They have the following usefull properties:

$$\int h_k(s)m(ds) = 1 \tag{6}$$

$$h_k(s) = 0 \quad \text{if} \quad \|s - \tau e_1\|_2 \geq 2^{-k} \quad \text{for all} \quad \tau \in \mathbb{T}. \tag{7}$$

$$\text{Every} \quad h_k \quad \text{is a linear combination of} \quad \|\cdot\|_{(k)}^2 \quad \text{and} \quad \|\cdot\|^2. \tag{8}$$

Step 3: Now, let g be a $\mathbb{T}$-invariant function on $\mathbb{S}^{n-1}$. Define the "convolution"

$$g_k(s) = \int g(V^{-1}e_1)h_k(Vs)m_{\mathbb{U}_n}(dV) \quad (s \in \mathbb{S}^{n-1}).$$

Claim:

$$g = \lim_{k\to\infty} g_k \qquad \text{in} \qquad C(\mathbb{S}^{n-1}). \tag{9}$$

By virtue of $\mathbb{T}$-invariance and uniform continuity, given $\varepsilon > 0$ we find $k \in \mathbb{N}$, such that for any two points s and y in $\S^{n-1}$

$$\|s - y\|_2 < 2^{-k} \quad \text{implies} \quad |g(s) - g(y)| < \varepsilon.$$

If for this particular k the term $h_k(Vs)$ is greater than 0 then (7) guaranties the existence of some $\tau \in \mathbb{T}$ such that $\|s - \tau V^{-1} e_1\|_2 = \|Vs - \tau e_1\|_2 < 2^{-k}$. The function g being $\mathbb{T}$-invariant we conclude $g(\zeta V^{-1} e_1) = g(V^{-1}e_1)$ and $|g(s) - g(V^{-1}e_1)| < \varepsilon$. Consequently, we have the following inequalities

$$\begin{aligned} |g(s) - g_k(s)| &\leq \int |g(s) - g(V^{-1}e_1)| h_k(Vs) m_{\mathbf{U}_n}(dV) \\ &\leq \varepsilon \int h_k(Ve_1) m_{\mathbf{U}_n}(dV) \\ &= \varepsilon \int h_k(s) m(ds) = \varepsilon. \end{aligned}$$

This proves claim (9) since everything applies uniformly to all $s \in \mathbb{S}^{n-1}$.

Step 4: The g_k are now going to be approximated by suitable linear combinations of squares of norms. Towards this end we take some sequence $\mathcal{F}_m$ ($m \in \mathbb{N}$) of measurable partitions of $\mathbf{U}_n$, say of cardinality m and enumerated as follows $\mathcal{F}_m = (F_{1m}, \ldots, F_{mm})$. We may require $\mathcal{F}_m$ to fulfill

$$\textit{finess}(\mathcal{F}_m) \overset{\text{def}}{=} \max_{j=1,\ldots,m} \sup_{V,W \in F_{jm}} \|V - W\|_{\mathcal{L}(\ell_2^n, \ell_2^n)} \to 0 \quad (m \to \infty) \tag{10}$$

Choose any $V_{jm} \in F_{jm}$ and let

$$g_{km}(s) = \sum_{j=1}^{m} m_{\mathbf{U}_n}(F_{jm}) g(V_{jm}^{-1} e_1) h_k(V_{jm} s) \qquad (s \in \mathbb{S}^{n-1}).$$

Claim: For all k we have:

$$g_k = \lim_{m \to \infty} g_{km} \qquad \text{in} \qquad C(\mathbb{S}^{n-1}). \tag{11}$$

Fix $k \in \mathbb{N}$ and $\varepsilon > 0$. Due to the uniform continuity of g and h_k we may choose $\delta > 0$ in such a way that for any $y_1, y_2, s_1, s_2 \in \mathbb{S}^{n-1}$ with $\|y_1 - s_1\| < \delta$ and $\|y_2 - s_2\| < \delta$ we may conclude

$$|g(y_1)h_k(s_1) - g(y_2)h_k(s_2)| < \varepsilon.$$

Now, let m_0 be sufficiently large to guarantee that *finess*($\mathcal{F}_m$) will not exceed δ for all $m \geq m_0$. Then we find uniformly in $s \in \mathbb{S}^{n-1}$

$$\begin{aligned} |g_{km}(s) - g_k(s)| &\leq \sum_{j=1}^{m} \int_{A_{jm}} |g(V^{-1}e_1)h_k(Ve_1) - g(V_{jm}^{-1}e_1)h_k(V_{jm}e_1)| m_{\mathbf{U}_n}(dV) \\ &\leq \varepsilon \sum_{j=1}^{m} m_{\mathbf{U}_n}(A_{jm} = \varepsilon. \end{aligned}$$

This proves claim (11). Recall claim (9), and we are done. Q.E.D.

3. ROTATIONAL INVARIANCE

We are going to point out the special role of rotational invariant orthonormal systems. In the real case we consider an n-dimensional Gaussian vector $\mathcal{G}_n = (g_1, \ldots, g_n)$, i.e. the g_j are i.i.d. with distribution

$$\mathbb{P}\{g_j \in F\} = \frac{1}{\sqrt{2\pi}} \int_F e^{-t^2/2} dt \qquad (F \quad \text{Borel subset of} \quad \mathbb{R}^n).$$

In the complex case the n-dimensional Gaussian vector $\mathcal{G}_n = (g_1, \ldots, g_n)$, can be obtained by setting

$$g_j = \frac{1}{\sqrt{2}} \tilde{g}_{2j-1} + \frac{i}{\sqrt{2}} \tilde{g}_{2j} \qquad (j = 1, \ldots, n)$$

provided $(\tilde{g}_1, \ldots, \tilde{g}_{2n})$ is a $2n$-dimensional real Gaussian vector (cf. [4] S12).

It is important to note that if $V \in \mathbf{U}_n$ then the distributions of $V\mathcal{G}_n$ and $\mathcal{G}_n$ are the same. This is what we call *rotational invariance*.

Definition 3. *An orthonormal system $\Phi_n \in L_2(\mu)$ is called rotational invariant, provided*

$$\mu\{V\Phi_n \in F\} = \mu\{\Phi_n \in F\} \qquad (V \in \mathbf{U}_n;\, F \quad \textit{Borel subset of } \mathbb{C}^n). \tag{12}$$

Recall that we agreed on not distinguishing between the system Φ_n and the induced $\mathbb{C}^n$-valued measurable map.

Taking into account that if Φ_n is rotational invariant then its projective distribution is rotational invariant, too, and that there is only one rotational invariant measure on $\mathbb{S}^{n-1}$ with total mass n, the following remark is obvious.

Remark 4. *Let ϖ_n be the (unique) rotational invariant measure on $\mathbb{S}^{n-1}$ with total mass n. Suppose $\Phi_n \in L_2(\mu)$ is rotational invariant then its projective distribution equals ϖ_n.*

The study of the projective distribution λ rather than that of the systems $\Phi_n \subset L_2(\mu)$ in their own right was triggered off by investigating the behaviour of a certain generalization of the absolutely-2-summing ideal norm (cf. [3]).

If $T : X \to Y$ is a bounded linear map, we define $\pi(T|\Phi_n)$ to be the smallest constant C such that

$$\|TU\Phi_n\|_{L_2^Y(\mu)} \leq C\|U\| \qquad \text{for} \quad U : \ell_2^n \to X. \tag{13}$$

We consider two special cases.

(i) If the orthonormal system $\chi_n = (\chi_1, \ldots, \chi_n)$ is given by the indicator functions $\chi_j = \mathbf{1}_{[j-1,j)} \in L_2(\mathbb{R})$ and if U is given by $Ue_j = x_j$ $(j = 1, \ldots n)$ then the left hand side in (13) computes as

$$\left(\sum_{j=1}^{n} \|Tx_j\|^2 \right)^{1/2}$$

labelled *strong* ℓ_2*-sum* of the vector tupel $(Tx_1, \ldots, Tx_n)$. The right hand side in (13) computes as

$$\sup\left\{\left(\sum_{j=1}^{n} |\langle x_j, x'\rangle|^2\right)^{1/2} : x' \in X', \|x'\| \leq 1\right\},$$

known as the *weak* ℓ_2*-sum* of the vector tupel $(x_1, \ldots, x_n)$. Accordingly, $\pi(T|\chi_n)$ coincides with the absolutely-2-summing norm computed with respect to n vectors. (cf. [4] SS18, 23-26, [3]).

(ii) If the orthonormal system is $\mathcal{G}_n = (g_1, \ldots, g_n)$, consisting of n independent Gaussian variables over some probability space $(\Omega, \Sigma, \mathbb{P})$, then $\pi(T|\mathcal{G}_n)$ coincides with the γ-summing norm computed with respect to n vectors. (cf. [4] SS12, 23-26, [3]).

In turns out that among all ideal norms built according to the above procedure, there is one which has minimal value simultaneously for all $T \in \mathcal{L}$. We formulate a somewhat more general lemma.

Lemma 5. *Let $\Phi_n \subset L_2(\mu)$ and $\tilde{\Phi}_n \subset L_2(\tilde{\mu})$ be two orthonormal systems with projective distribution λ and $\tilde{\lambda}$, respectively. Assume, there is some probability measure $\mathbb{P}$ on $\mathbf{U}_n$ such that*

$$\int_{\mathbb{S}^{n-1}} g(x)\tilde{\lambda}(dx) = \int_{\mathbf{U}_n}\int_{\mathbb{S}^{n-1}} g(Vx)\lambda(dx)\mathbb{P}(dV) \qquad \left(g \in C(\mathbb{S}^{n-1})\right). \tag{14}$$

Then

$$\pi(T|\tilde{\Phi}_n) \leq \pi(T|\Phi_n) \quad (T \in \mathcal{L}).$$

Proof. Given operators $\ell_2^n \xrightarrow{U} X \xrightarrow{T} Y$ we find

$$\|TUV\Pi_n\|_{L_2^Y(\lambda)} = \|TUV\Phi_n\|_{L_2^Y(\mu)} \leq c\|UV\| = c\|U\| \qquad (V \in \mathbf{U}_n),$$

where $c = \pi(T|\Phi_n)$. Square, integrate against $\mathbb{P}(dV)$, and take the square root, then

$$\|TU\tilde{\Phi}_n\|_{L_2^Y(\tilde{\mu})} = \|TU\Pi_n\|_{L_2^Y(\tilde{\lambda})} = \left(\int \|TUV\Pi_n\|^2_{L_2^Y(\lambda)}\mathbb{P}(dV)\right)^{1/2} \leq c\|U\|.$$

As $U : \ell_2^n \to X$ was arbitrary we have $\pi(T|\tilde{\Phi}_n) \leq c$. Q.E.D.

Note that the situation in (14) can be arranged in a simple manner: Given an orthonormal system Φ_n and some probability measure $\mathbb{P}$ on $\mathbf{U}_n$, we may *define*

$$\begin{aligned} \tilde{\Phi}_n : \Omega \times \mathbf{U}_n &\to \mathbb{C}^n, \\ (t, V) &\mapsto V\Phi_n(t) \end{aligned}$$

It is immediate to see, that if $\tilde{\lambda}$ is the projective distribution of this particular $\tilde{\Phi}_n$ indeed (14) holds with the same $\mathbb{P}$. Moreover this construction always yields an orthonormal system, since

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} \tilde{\Phi}_n(t)^* \otimes \tilde{\Phi}_n(t)\tilde{\mu}(dt) &= \int_{\mathbf{U}_n} V\left(\int_{\mathbb{S}^{n-1}} \Phi_n(t)^* \otimes \Phi_n(t)\mu(dt)\right) V^*\mathbb{P}(dV) \\ &= \int_{\mathbf{U}_n} VI_nV^*\mathbb{P}(dV) \\ &= \int_{\mathbf{U}_n} I_n\mathbb{P}(dV) = I_n.\end{aligned}$$

The case in consideration is merely specialized to the situation where $\Phi_n = \Pi_n$.

The author has the strong feeling that a converse of the above lemma holds true.

Conjecture: Given two orthonormal system Φ_n and $\tilde{\Phi}_n$ such that

$$\pi(T|\tilde{\Phi}_n) \leq \pi(T|\Phi_n) \qquad (T \in \mathcal{L}),$$

then necessarily there is a probability $\mathbb{P}$ measure on $\mathbf{U}_n$ such that (14) holds.

However, we are able proof this conjecture only in the case where Φ_n or rather its projective distribution λ are rotational invariant, i.e. $\lambda = \varpi_n$. In this case *any* probability measure $\mathbb{P}$ in (14) will again produce a rotational invariant measure $\tilde{\lambda}$ on $\mathbb{S}^{n-1}$, so that in fact $\tilde{\lambda} = \varpi_n$.

Corollary 6. *The following statements on an orthonormal system $\Psi_n \subset L_2(\nu)$ and its projective distribution $\tilde{\lambda}$ are equivalent.*

(i) $\tilde{\lambda} = \varpi_n$.

(ii) $\tilde{\lambda}$ *is rotational invariant.*

(iii) $\pi(\cdot|\Psi_n)$ *is minimal in the sense, that*

$$\pi(T|\Psi_n) \leq \pi(T|\Phi_n),$$

however we choose the operator T and the Hilbert space $L_2(\mu)$, and the orthonormal system $\Phi_n \subset L_2(\mu)$.

Proof. That (*i*) is equivalent to (*ii*) is just the fact that ϖ_n is the only rotational invariant measure on $\mathbb{S}^{n-1}$ of total mass *n*.

(*i*) $\Longrightarrow$ (*iii*): Let us consider an orthonormal system $\Phi_n \subset L_2(\mu)$ with projective distribution λ. Define

$$\begin{aligned}\tilde{\Phi}_n : \Omega \times \mathbf{U}_n &\to \mathbb{C}^n, \\ (t, V) &\mapsto V\Phi_n(t)\end{aligned}$$

regarded as an orthonormal system with respect to the product measure $\mu \otimes m_{\mathbf{U}_n}$ (see above). As a matter of fact $\tilde{\Phi}_n$ is rotational invariant, hence its projective distribution coincides with ϖ_n and we get

$$\pi(T|\tilde{\Phi}_n) = \pi(T|\Psi_n) \qquad (T \in \mathcal{L}),$$

provided (*i*) holds. By the preceeding lemma,

$$\pi(T|\tilde{\Phi}_n) \leq \pi(T|\Phi_n) \qquad (T \in \mathcal{L}).$$

Altogether we have show (*iii*).

(*iii*) $\Longrightarrow$ (*i*): We know that $\Pi_n \subset L_2(\varpi_n)$ is rotational invariant, thus $\pi(U|\,[\Pi_n, \varpi_n]) \leq \pi(U|[\Phi_n, \mu])$. Assuming (*iii*) we even have equality, i.e. $\pi(U|\,[\Phi_n, \mu]) = \pi(U|[\Pi_n, \varpi_n])$ for all $U : \ell_2^n \to X$ and all banach spaces X. By rotational invariance the map

$$\mathbf{U}_n \longrightarrow \mathbb{R}_+, V \mapsto \|UV\Pi_N\|_{L_2^X(\varpi_n)}$$

is constant equal to $\pi(U|\,[\Pi_n, \varpi_n])$. Recall that

$$\left(\int \|UV\Pi_n\|^2_{L_2^X(\lambda)} m_{\mathbf{U}_n}(dV)\right)^{1/2} = \pi(U|\,[\Pi_n, \varpi_n]).$$

Hence, assuming that the map

$$\mathbf{U}_n \to \mathbb{R}_+, V \mapsto \|UV\Pi_n\|_{L_2^X(\lambda)}$$

was *not* contant we had the contradiction

$$\pi(U|[\Phi_n, \mu]) = \sup_{V \in \mathbf{U}_n} \|UV\Pi_n\|_{L_2^X(\lambda)} > \left(\int \|UV\Pi_n\|^2_{L_2^X(\lambda)} m_{\mathbf{U}_n}(dV)\right)^{1/2} = \pi(U|[\Pi_n, \varpi_n]).$$

This proves in particular

$$\|U\Pi_n\|_{L_2^X(\lambda)} = \|U\Pi_n\|_{L_2^X(\varpi_n)}$$

and since U was arbitrary we are done by virtue of theorem 6. Q.E.D.

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