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LIFTING QUASIFIBRATIONS

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Abstract. *The construction of "lifting" spreads in $PG(3, q)$ to spreads in $PG(3, q^2)$ is generalized to lifting quasifibrations in $PG(3, K)$ to quasifibrations in $PG(3, K[\theta])$ for a quadratic extension $K[\theta]$ of a field K.*

1. INTRODUCTION

The process described in this paper is a construction technique called "lifting" by which spreads in $PG(3, q)$ produce spreads in $PG(3, q^2)$. This construction originated, in the odd order finite case, in Hiramine, Matsumoto, and Oyama [9]. Furthermore, the author extended this process to include the finite even order case in Johnson [13]. Also, see Johnson [14]. M. Cordero has examined the case where a semifield plane of order p^4 is lifted from a Desarguesian affine plane of order p^2 where p is a prime. See Cordero [2], [3], [4].

In this paper, we generalize the construction of "lifting" to quasifibrations in $PG(3, K)$ for an arbitrary field which has a quadratic extension field.

Since there are fields which admit infinitely many mutually nonisomorphic quadratic extension fields, there are astonishing numbers of spreads or quasifibrations which can be obtained from a given spread or quasifibration.

There are so many lifted structures that it becomes clear that there can be no general classification of such objects. Since there is a general interest in semifield spreads, there is a section devoted to the construction of semifield spreads from a given semifield spread. However, this is more to illustrate the chaos introduced into the field by this process rather that to try to provide a complete analysis of semifield planes. In fact, it might be more appropriate to treat all such lifted quasifibrations from a given quasifibration as essentially equivalent.

However, it is important to be aware that a particular spread may be a lifted version of a known spread. Moreover, it is important to give classification results which identify a lifted spread by certain group theoretic or structural properties. For example, we are able to identify lifted quasifibrations by the existence of certain elation and Baer groups.

In section 2, we give the basic construction of lifting quasifibrations in $PG(3, K)$ for K a field. In section 3, we show how to identify such structures via their collineation groups. In section 4, we discuss the corresponding "retractions". In section 5, we discuss "additive" quasifibrations and their lifted quasifibrations but we could have also discussed any particular class of lifted structures in a similar manner. Finally, in section 6, we show how chaotic the situation becomes by the construction of chains of lifts and retractions and offer a solution of sorts.

2. LIFTING

We first recall the definition of a quasifibration in $PG(3, K)$.

Definition 2.1. *Let P be a partial spread in $PG(3, K)$ for K an arbitrary skewfield. That is, P is a set of mutually disjoint left two dimensional K-vector spaces of a left 4-dimensional vector space isomorphic to $K \oplus K \oplus K \oplus K$. Let N_P denote the corresponding net. Choose two elements L, M of P and choose bases so that the elements of the vector space are written in the form (x, y) where x in L and y in M.*

If there exists a nonzero vector e in L such that the points of the translate $x = e$ of M are covered by the partial spread P, then P is called a quasifibration in $PG\,(3, K)$ with respect to (L, M).

Note that a finite quasifibration is a spread. However, there exist infinite quasifibrations which are not spreads.

Proposition 2.2. *Let S be a quasifibration in $PG(3, K)$. Then coordinates may be chosen so that the set of left 2-dimensional K-subspaces have the following form: $x = 0$, $y = x \begin{bmatrix} g(t,u) & f(t,u) \\ t & u \end{bmatrix}$ for all u, t in K and functions g and f from $K \times K$ to K.*

Furthermore, any set of matrices $x = 0$, $y = x \begin{bmatrix} g(t,u) & f(t,u) \\ t & u \end{bmatrix}$ for all t, u in K with the property that the difference of any two matrices is nonsingular or identically zero provides a quasifibration.

A quasifibration is a maximal partial spread in $PG(3, K)$.

Proof. Choose two elements of S, L, M and choose a basis for e in M so that as a two dimensional left K-space M has basis $\{s, 1\}$ so that elements of M have the form $ts + u \equiv (t, u)$ for all t, u in K and assume that $e = (0, 1)$. Furthermore, choose an analogous basis for L so that the vector space may be represented in the form $(ts + u, t^*s + u^*)$ for all t, u, t^*, u^* in K.

So, we may further consider the vectors in the form (x, y) where $x = x_1\, s + x_2 = (x_1, x_2)$ and $y = y_1\, s + y_2 = (y_1, y_2)$. It is easy to verify that the remaining elements of S are of the general form $y = xT$ where T is some 2×2 K-matrix. Hence, we may write $T = \begin{bmatrix} g(t,u) & f(t,u) \\ t & u \end{bmatrix}$ where t, u are in K and g and f are functions from $K \times K$ to K. Consider the line $x = e = (0, 1)$. A point $(0, 1, y_2, y_2)$ is incident with $y = xT$ if and only if $(t, u) = (y_1, y_2)$. Hence, for the matrices T, t, u vary independently over all elements of K.

Note that this also implies that if the difference of such matrices is nonsingular and t, u vary independently over K than the partial spread is a quasifibration. See Jha-Johnson [10] to see that a quasifibration is always a maximal partial spread.

2.1. When K admits a quadratic extension $K[\theta]$

Now assume that K admits a quadratic extension field $K[\theta]$ such that $\{1, \theta\}$ is a K basis and $\theta^2 = \theta\alpha + \beta$ for α, β in K.

Write $h(t, u) = f(t, u) + \alpha g(t, u)$ so that the quasifibration has the form:

$$x = 0, y = x \begin{bmatrix} g(t,u) & h(t,u) - \alpha g(t,u) \\ t & u \end{bmatrix}$$

for all u, t in K where x and y are K-2-vectors.

Define $F(\theta t + u) = -g(t,u)\theta + h(t,u)$. Let σ denote the automorphism of $K[\theta]$ of order two which maps $\theta t + u \rightarrow -\theta t + u$ αt for all u, t in K.

Consider the following set of 2-dimensional $K[\theta]$-subspaces:

$$x = 0, y = x\begin{bmatrix} (\theta t + u)^{\sigma} & F(\theta s + v) \\ \theta s + v & \theta t + u \end{bmatrix} \text{ for all } \theta s + v, \theta t + u \text{ of } K[\theta].$$

With the above assumption, the difference of two matrices $\begin{bmatrix} m^{\sigma} & F(z) \\ z & m \end{bmatrix} - \begin{bmatrix} n^{\sigma} & F(w) \\ w & n \end{bmatrix}$ is zero if and only if $(m-n)^{\sigma+1} = (z-w)\,(F(z) - F(w))$. Since the order of σ is two, and the mapping $\sigma + 1$ is into K, it then follows that the difference matrix is only zero provided possibly $(z-w)\,(F(z) - F(w))$ is in K. That this never occurs follows directly from the argument of Johnson [15] pp. 74-5. For example, Since $(z-w)\,(F(z) - F(w))$ for $z \neq w$ is never in K^*, then in particular, $(z-w)\,(F(z) - F(w)) \neq v^{\sigma+1}$ for any v in K^*.

Hence, we obtain a partial spread. But, also note that we do have now an example of a quasifibration. So, we obtain either a spread or a maximal partial spread. We call the constructed quasifibration a lifted quasifibration.

Proposition 2.3. *Let S be a quasifibration in $PG(3,K)$ for a field K so that there exists a quadratic extension $K[\theta]$.*

Then there is a set of quasifibrations in $PG(3,K[\theta])$ which are lifted from S.

Theorem 2.4. *Each lifted quasifibration is either a spread or a maximal partial spread.*

We now consider conditions whereby we obtain a spread.

Proposition 2.5. *A lifted quasifibration is a spread if and only if the functions G_{δ} defined by $G_{\delta}(v) = H(v)^{\sigma} - \delta v$ for each δ in K are surjective where the lifted quasifibration has the form $x = 0,\ y = x\begin{bmatrix} m^{\sigma} & H(v) \\ v & m \end{bmatrix}$ for all m, v in $K[\theta]$.*

Note that each G_{α} is injective since we have a partial spread.

Proof. Note that the above shows that the function is injective. It suffices to show that this set of subspaces forms a cover of the 4-dimensional vector space over $K[\theta] = F$.

A vector (c,d,e,f) where $(c,d) \neq (0,0)$ is contained in $y = x\begin{bmatrix} m^{\sigma} & H(v) \\ v & m \end{bmatrix}$ if and only if

$$cm^{\sigma} + dv = e \quad \text{and } cH(v) + dm = f.$$

If either c or $d = 0$ and $H(v)$ is surjective then the above system of equations may be solved uniquely.

If $cd \neq 0$ then there is a solution if and only if there exists an element v such that

$$(e - dv)\,/\,c = ((f - cH(v))\,/\,d)^{\sigma}.$$

This equation is valid if and only if

$$H(v)^{\sigma} - (d\,/\,c)^{\sigma+1}v = (cf^{\sigma} - ed^{\sigma})\,/\,c^{\sigma+1}.$$

This proves the result.

If we translate all of this into requirements for the original functions $g(t,u)$ and $f(t,u)$ describing the original functions, we obtain:

Proposition 2.6. *Let a quasifibration in $PG(3,K)$ be represented by $x=0$, $y=x\begin{bmatrix} g(t,u) & f(t,u) \\ t & u \end{bmatrix}$ for all u,t in K and functions g and f from $K\times K$ to K. Let a lifted quasifibration be represented in the form $x=0$, $y=x\begin{bmatrix} m^{\sigma} & H(v) \\ v & m \end{bmatrix}$ for all m,v in $K[\theta]$ where $\theta^2=\theta\alpha+\beta$.*

Then $H(v)^{\sigma}-\delta v$ is surjective if and only if for each fixed δ in K,

$\delta t-g(t,u)$ and $f(t,u)+\alpha g(t,u)-\delta(u+\alpha t)$ are both surjective and independent functions onto K.

Proof. Since σ has order two, $H(v)^{\sigma}-\delta v$ is surjective if and only if $H(v)-\delta v^{\sigma}$ is surjective if and only if

$$-g(t,u)\theta+(f(t,u)+\alpha g(t,u))-(\delta(\theta t+u)^{\sigma}=\delta(-\theta t+u+\alpha t))$$

is surjective.

Hence, this is equivalent to
$(\delta t-g(t,u))$ and $(f(t,u)+\alpha g(t,u)-\delta(u+\alpha t)$ both surjective and independent onto K.

Proposition 2.7. *Under the assumption of the previous proposition, assume that the initial quasifibration is a spread. Then*

$\delta t-g(t,u)$ and $f(t,u)+\alpha g(t,u)-\delta(u+\alpha t)$ are both surjective and independent functions onto K.

Proof. Since we have a spread in $PG(3,K)$, then for any vector (c,d,e,j) for $(c,d)\neq(0,0)$ there exists a component $y=x\begin{bmatrix} g(t,u) & f(t,u) \\ t & u \end{bmatrix}$ such that the vector is incident with this component. Hence, we obtain the equations:

$$cg(t,u)+dt=e \quad \text{and } cf(t,u)+du=j.$$

Note that the first equation shows that $\delta t-g(t,u)$ is onto for all δ in K.

Further assume that $c\neq 0$. Then we obtain the following:

$$g(t,u)=e/c-dt/c \text{ and } f(t,u)=j/c-du/c.$$

Since e and f are arbitrary, it follows that $g(t,u)-dt/c$ and $f(t,u)-du/c$ are completely independent and onto functions. In particular, it follows that $f(t,u)+\gamma g(t,u)-(d/c)(u+\gamma t)$ $=\gamma(e/c)+j/c$ is onto for any γ in K. Hence, this proves the assertion in the proposition.

Hence, we obtain the following theorem:

Theorem 2.8. *Let S be a spread in $PG(3,K)$ for K a field which admits a quadratic extension $K[\theta]$.*

Then there is a set of spreads in $PG(3, K[\theta])$ called the spreads lifted from S.

A more general theorem concerns quasifibrations.

Theorem 2.9. *Let S be a quasifibration in $PG(3, K)$ for K a field that admits a quadratic extension $K[\theta]$.*

(1) Then there is a set of quasifibrations in $PG(3, K[\theta])$ called the quasifibrations lifted from S.

(2) If S is a spread then all of the lifted quasifibrations are spreads.

(3) If any lifted quasifibration is a spread then S is a spread and all lifted quasifibrations are spreads.

Proof. Note that the previous results show that a quasifibration S provides a lifted quasifibration and conservely. Also, the covering requirement for a spread is equivalent to the covering requirement for a lifted spread.

Remark 1. *Given a quasifibration with respect to (L, M). We always choose coordinates as indicated above. If the quasifibration is a spread then any pair of spread lines may be chosen to represent the spread and possibly different corresponding functions $g(t, u)$ and $f(t, u)$ can be obtained which would then possibly provide different lifts from the same spread.*

3. BAER AND ELATION GROUPS

In this section, we show how to recognize lifted quasifibrations by their collineation groups. We begin by considering the effect of certain Baer and elation groups on the structure.

Theorem 3.1. *Let S be a quasifibration with respect to (L, M) in $PG(3, K[\theta])$ for K a field which admits a quadratic extension $K[\theta]$.*

Let $\pi_S = \pi$ denote the translation net associated with S.

Then the quasifibration of $PG(3, K[\theta])$ may be represented in the form

$$x = 0, y = x\begin{bmatrix} u^{\beta} & H(v) \\ v & u \end{bmatrix} \text{ for all } u, v \text{ in } K[\theta]$$

for some function H on $K[\theta]$ where β is an automorphism of $K[\theta]$

if and only if π admits a collineation group which is a semidirect product of an elation group E with axis M by a nontrivial linear Baer group B of order 2 with the following properties:

(i) some orbit of components union the axis of E is a derivable net in π

(ii) $[E, B] \neq <1>$.

In the above result, the Baer group B could be the full group with fixed point space $FixB$ or it could be larger. And, to specify the automorphism, we assume there is an embedded K-regulus net.

Theorem 3.2. *Let S be a quasifibration with respect to (L, M) in $PG(3, K[\theta])$ for K a field which admits a quadratic extension $K[\theta]$.*

Let $\pi_S = \pi$ denote the associated translation net associated with S.

Then the quasifibration of $PG(3, K[\theta])$ may be represented in the form

$$x = 0, y = x\begin{bmatrix} u & H(v) \\ v & u \end{bmatrix} \text{ for all } u, v \text{ in } K[\theta]$$

for some function F on $K[\theta]$

if and only if π admits a collineation group which is a semidirect product of an elation group E with axis M by a nontrivial linear Baer group B of order 2 with the following properties:

(i) some orbit of components union the axis of E is a derivable net which contains a K-regulus subnet in π,

(ii) $[E, B] \neq <1>$ and the full Baer group fixing FixB pointwise is B.

In this case, the quasifibration corresponds to a maximal partial flock of a quadratic cone in $PG(3, K[\theta])$.

Furthermore, in the finite case, the spread is either Desargusian or a semifield spread of Knuth type.

We consider the case when the order of B is larger than 2 in the following result.

Theorem 3.3. *Let S be a quasifibration with respect to (L, M) in $PG(3, K[\theta])$ for K a field which admits a quadratic extension $K[\theta]$.*

Let $\pi_S = \pi$ denote the associated translation net associated with S. Let σ denote the involutory automorphism of $K[\theta]$ which fixes K pointwise.

(1) Then the quasifibration of $PG(3, K[\theta])$ may be represented in the form

$$x = 0, y = x\begin{bmatrix} u^{\sigma} & H(v) \\ v & u \end{bmatrix} \text{ for all } u, v \text{ in } K[\theta]$$

for some function H on $K[\theta]$

if and only if π admits a collineation group which is a semidirect product of an elation group E with axis M by a nontrivial linear group B of order > 2 with the following properties:

(i) some orbit of components union the axis of E is a derivable net which contains a K-regulus subnet in π,·

(ii) $[E, B] \neq <1>$.

(2) If $(v - s)(H(v) - H(s))$ is never in K (for example, if $m \to m^{\sigma+1}$ is surjective) then S may be lifted from a spread in $PG(3, K)$ (equivalently, there exists a retraction spread in $PG(3, K)$) if and only if π admits a collineation group which is a semidirect product of an elation group E with axis M by a nontrivial linear group B of order > 2 with the following properties:

(i) some orbit of components union the axis of E is a derivable net which contains a K-regulus subnet in π,

(ii) $[E, B] \neq <1>$.

Proof. We first consider any quasifibration in $PG(3, K[\theta])$ of the following form: $x = 0, y = x \begin{bmatrix} m^{\sigma} & H(v) \\ v & m \end{bmatrix}$ for all m, v in $K[\theta]$ where $\theta^2 = \theta\alpha + \beta$ and σ in the involutory automorphism fixing K pointwise.

Let

$$E = < \begin{bmatrix} 1 & 0 & u^{\sigma} & 0 \\ 0 & 1 & 0 & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \forall u \epsilon K >$$

and

$$B = < \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & e & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \forall e \epsilon K - \{0\}; e^{\sigma+1} = 1 >.$$

It is easy to verify that EB satisfies the requirements stated in the theorem.

Furthermore, define $g(t, u)$ and $f(t, u)$ by

$$H(\theta t + u) = -g(t, u)\theta + (f(t, u) + \alpha g(t, u)).$$

In this case, if there is a quasifibration for which $(v - s)$ $(H(v) - H(s))$ is never in K for $v \neq s$ then, the previous arguments show that there is an associated quasifibration in $PG(3, K)$ given by

$$x = 0, y = x \begin{bmatrix} g(t, u) & f(t, u) \\ t & u \end{bmatrix} \forall t, u \epsilon K.$$

Hence, we note that any quasifibration of the form given above must be a lifted quasifibration. For example, if there is a quasifibration and $m \rightarrow m^{\sigma+1}$ is surjective then there is a retracted quasifibration.

And, any lifted quasifibration admits groups E and B as above.

It remains to show that a quasifibration in $PG(3, K[\theta])$ admitting groups E and B with the above properties has the form given.

We first give a lemma which is required for the result:

Lemma 3.4. *Let F be any field and let D be a derivable net with partial spread in $PG(3, F)$.*

Let $K(F^*)$ denote the kernel homology group of D induced by the elements of F^* as scalar mappings. Note that $K(F^*) \simeq F^*$ and leaves each component of D invariant.

If $K(F^*)$ leaves invariant two 1-dimensional Z-spaces on any component then coordinates may be chosen so that D has partial spread defined as follows:

$$x = 0, y = x \begin{bmatrix} u^{\rho} & 0 \\ 0 & u \end{bmatrix} \quad \begin{matrix} \textit{for all } u \textit{ in } F, \\ \rho \textit{ in } Gal K[\theta]. \end{matrix}$$

Proof. By Johnson [16], there exists a skewfield Z such that D corresponds to a pseudo-regulus net in $PG(3, Z)$. Hence, considered as 2×2 matrices T_{δ} with entries in F, D has the following form: $x = 0$, $y = xT_{\delta}$ for all δ in Z. Moreover, $\{T_{\delta}\}$ forms a skewfield isomorphic to Z. Under the assumptions, there are two Baer subplanes of the net which must be F-subspaces. Hence, there are two Baer subplanes which are 2-dimensional F-subspaces. We may choose coordinates so that the matrices T_{δ} are diagonal (choose a Baer subplane π_o $= \{(0, x_2, 0, y_2)$ for all x_2, y_2 in $F\}$ and $\pi_1 = \{(x_1, 0, y_1, 0)$ for all x_1, y_1 in $F\}$ and choose three components to be $x = 0$, $y = 0$, $y = x$). Hence, Z must be a field and it follows that T_{δ}

$= \begin{bmatrix} u & 0 \\ 0 & f(u) \end{bmatrix}$ for some u in F and function $f : F \to F$ such that $f(0) = 0, f(1) = 1$. It follows that since components with matrices of this form must cover π_o and π_1, then Z is isomorphic to F and f is an automorphism of F.

This proves the Lemma.

Returning to the proof of the above result (3.1), choose coordinates so that $x = 0$ represents the axis of the elation group E. We may assume that the orbit under E contains components represented in the form $y = x$ and $y = 0$.

We know the E orbit of $y = 0$ defines a derivable net in $PG(3, K[\theta] = F)$. It follows from the arguments of Jha-Johnson [11] that every E-orbit union the axis is a derivable net. And, we known that a derivable net D is a pseudo-regulus in $PG(3, Z)$ for some skewfield Z. Acting as a collineation group of the pseudo-regulus net acting in $PG(3, Z)$, E has the form:

$$\left\langle \begin{bmatrix} 1 & 0 & \delta & 0 \\ 0 & 1 & 0 & \delta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ for all } \delta \text{ in } Z \right\rangle$$

where we assume the vector space in taken as a right vector space over Z and the components have the form $x = 0$, $y = x\delta$ where $(x_1, x_2)\ \delta = (x_1\delta, x_1\delta)$. It is important to point out that the components are not all right 2-dimensional Z-spaces unless Z is a field. However, Baer subplanes incident with the zero vector are right 2-dimensional Z-subspaces and $x = 0$ may be thought of as a right Z-vector space whose intersections with the Baer subplanes incident with the zero vector are 1-dimensional right Z-subspaces. Similarly, $y = 0$ may also be considered as a right Z-space.

Now the Baer group B also acts as a collineation group of any derivable net D which contains a component fixed by B. Furthermore, B fixes the axis of E. Hence, B fixes at least one 1-dimensional Z subspace on $x = 0$. Therefore, B fixes a Baer subplane of D. Note that we may consider $x = 0$ as a right Z-subspace. Assume without loss of generality that B fixes the component $y = 0$. Assume that a 1-dimensional Z-subspace fixed by B on $x = 0$ is incident with a fixed Baer subplane of the net which intersects $y = 0$ in a B fixed 1-dimensional Z subspace. Then the 2-dimensional right Z-subspace generated by these two 1-dimensional Z-subspaces is a Baer subplane of D which is contained in $FixB$. Since Baer subplanes are maximal, it follows that $FixB$ is a Baer subplane of D. However, this implies that B and E commute contrary to assumption.

Hence, we may assume that B fixes at least two Baer subplanes of D. Now $K(F^*)$ fixes each component of D and permutes the Baer subplanes incident with the zero vector. Moreover, B is F-linear so B and $K(F^*)$ commute. Thus, $K(F^*)$ permutes the Baer subplanes of D fixed by B. Since, $K(F^*)$ is Abelian, the following situations arise:

(a) B fixes exactly two Baer subplanes of D and $K(F^*)$ fixes or interchanges the two Baer subplanes,

(b) B fixes exactly two 1-dimensional Z-subspaces on each of $x = 0$ or $y = 0$,

(c) B fixes at least three 1-dimensional Z-subspaces on a component, say $x = 0$.

Note that B and $K(F^*)$ are acting on the Desarguesian spread of right 1-dimensional Z-subspaces on $x = 0$. Hence, B is isomorphic to a subgroup of $P\Gamma L(3, Z)$ which fixes a line (infinite line) and a point. That is, B is a subgroup of $\Gamma L(2, Z)$.

On $x = 0$, choose the right 1-subspaces over Z so that $x = 0$ is decomposed via bases of these subspaces as (u, v) for all u, v in Z.

Assume in case (a) that $K(F^*)$ interchanges the two Baer subplanes fixed by B. On $x = 0$, this means that $K(F^*)$ interchanges the fixed points of B in one Z-subspace with the B fixed points in the other Z-subspace. Considered over the prime field, we have a group which is the union of two subgroups which cannot occur in this case. Hence, $K(F^*)$ must fix two Baer subplanes in case (a).

In case (c), it follows that B fixes all 1-dimensional Z-subspaces on $x = 0$ so that the elements have the general form for some δ in Z and ρ an automorphism of Z and since B also fixes nonzero points on $x = 0$, it follows that elements of B have the general form $(u, v) \to (u^\tau a, v^\tau a)$ where τ is an automorphism of Z.

However, if such an element fixed every 1-dimensional Z-subspace, it must follow that the 1-space $v = um$ for each m of Z is fixed which is equivalent to $\tau = 1$ but there are no points on $x = 0$ which are fixed by B.

In case (b), elements of B on $x = 0$ have the general form $(u, v) \to (u^\tau a, v^\tau b)$. Since B fixes exactly two 1-dimensional Z spaces on $x = 0$, it follows that we are back to case (a).

Hence, in any situation, it can be assumed that $K(F^*)$ fixes two Baer subplanes of D. By the lemma, it may be assumed that a derivable net has the form listed above.

Furthermore, since B fixes $x = 0$ and $y = 0$ and the above argument shows that without loss of generality, B fixes $\pi_o \cap (x = 0)$ and $\pi_1 \cap (y = 0)$ pointwise.

Work out the form for B using the fact that B normalizes E and E has the form listed above to obtain

$$B = < \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \delta^{-\rho} & 0 & 0 \\ 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ for } \delta \text{ in some subgroup } M \text{ of } F^* >.$$

Now consider the action on a component of the form $y = x \begin{bmatrix} G(v) & H(v) \\ v & 0 \end{bmatrix}$ where G and H are functions of F. The image is $y = x \begin{bmatrix} G(v)\delta & H(v) \\ v\delta^{\rho+1} & 0 \end{bmatrix}$. From here it follow that $G(v\delta^{\sigma+1}) = G(v)\delta$ and $H(v) = H(v\delta^{\sigma+1})$. Now subtract this matrix from $\begin{bmatrix} G(v) & F(v) \\ v & 0 \end{bmatrix}$ to note that it must be that $G(v) = G(v)\delta$ and for $\delta \neq 1$ then $G(v) = 0$ for all v in F. Similarly, it then follows that $v\delta^{\rho+1} = v$ for all v so that $\delta^{\rho+1} = 1$. Hence, it follows that the form is as maintained.

Now assume the conditions of (3.2).

Since there is an orbit under the elation group which defines a derivable net that contains a K-regulus net, it follow easily that the standard derivable net with components $x = 0$, $y = x \begin{bmatrix} u^\rho & 0 \\ 0 & u \end{bmatrix}$ contains a K-regulus net. Since with appropriate coordinate choice, we may assume that a K-regulus net has the basic form $x = 0$, $y = x \begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix}$ for all v in K, it follows that ρ is either 1 or the automorphism σ of order two which fixes K pointwise. However, if $\rho = 1$ then $\delta^{\rho+1} = \delta^2 = 1$ for all corresponding elements of B which implies that B has order 2.

Now assume that B has order two. Then $\rho = 1$ and the spread has the form:

$$x = 0, y = x\begin{bmatrix} u & H(v) \\ v & u \end{bmatrix}$$

for all u, v in $K[\theta]$. We note that the existence of the group B shows that the plane is rigid in the terminology of Jha-Johnson [12] and hence in the finite case, the plane is either Desarguesian or a semifield of Knuth type. In the infinite case, there are infinitely many examples which are not of these two basic types (e.g. Biliotti-Johnson [1]). This proves (3.2). Our remarks also prove (3.3)(1) from which (3.3)(2) is immediate.

Remark 2. *Note that if the mapping* $m \to m^{\sigma+1}$ *is surjective then* $(v - s)\,(F(v) - F(s))$ *is never in* K *for* $v \neq s$ *so a retraction may be constructed.*

Remark 3. *Note that we obtain the full Baer group of the form listed above for all* δ *such that* $\delta^{\sigma+1} = 1$ *is obtained as a collineation group of the plane provided we only assume that there exists a nontrivial Baer group of order* > 2 *of the form listed.*

Remark 4. *The mapping* $m \to m^{\sigma+1}$ *is not surjective when* K *is a subfield of the reals and* $\theta = \sqrt{-1}$ *for then* $(a + bi)^{\sigma+1} = a^2 + b^2$. *For examples, the* K*-quaternion skewfield may be represented in the form* $x = 0,\ y = x\begin{bmatrix} u^\sigma & -t^\sigma \\ t & u \end{bmatrix}$ *for all elements* u, t *in* K. *Note that we have a spread of the indicated form but there is no retraction spread since* $(t - s)\,(-(t^\sigma - s^\sigma)) = -(t - s)^{\sigma+1}$ *is always in* K *but simply not in* $K[\theta]^{\sigma+1}$.

4. RETRACTIONS

Theorem 4.1. *Let* S *be any quasifibration in* $PG(3, K[\theta])$ *where* $K([\theta]$ *is a quadratic extension of* K. *In genral,* S *has the following form* $x = 0,\ y = x\begin{bmatrix} G(t, u) & F(t, u) \\ t & u \end{bmatrix}$ *for all* t, u *in* $K[\theta]$ *and* G *and* F *are functions on* $K[\theta]$. *Let* $\theta^2 = a\theta + b$.

Assume that $f(t, u)$ *has the property that* $(t - s)\,(F(t, u) - F(s, w))$ *is never in* $K[\theta]^{\sigma+1}$ *for all* $t \neq s$ *and* u, w *in* $K[\theta]$.

(1) *Then* $x = 0,\ y = x\begin{bmatrix} u^\sigma & F(t, u) \\ t & u \end{bmatrix}$, *for all* t, u *in* $K[\theta]$ *and* σ *the automorphism of order two which fixes* K *pointwise, defines a quasifibration in* $PG(3, K[\theta])$ *called the* σ*-associated quasifibration.*

(2) *Furthermore, if* $(t - s)(F(t, u) - F(s, w))$ *is never in* K, *define* $g(\alpha, \beta)$ *and* $f(\alpha, \beta)$ *as follows:* $F(t, 0) = F^*(t = \alpha\theta + \beta) = -g(\alpha, \beta)\,\theta + (f(\alpha, \beta) + ag(\alpha, \beta))$.

Then $x = 0, y = x\begin{bmatrix} g(\alpha, \beta) & f(\alpha, \beta) \\ \alpha & \beta \end{bmatrix}$ *for all* α, β *in* K *is a quasifibration in* $PG(3, K)$ *called the retraction of the* σ*-associated quasifibration.*

(3) *The retraction in* $PG(3, K)$ *lifts to the* σ*-associated quasifibration and is a spread if and only if the* σ*-associated quasifibration is a spread. In particular, all quasifibrations are spreads in the finite case.*

5. SEMIFIELDS AND ADDITIVE QUASIFIBRATIONS

First we note that by André's results, no spread in $PG(3,K)$ can be isomorphic to a spread in $PG(3,L)$ for K and L skewfields undless L and K isomorphic. (See, e.g. Lüneburg [19]).

Since there is intrinsic interest in semifields, we construct here an infinite number of mutually nonisomorphic semifield planes. But, note that it is just as simple to construct infinite numbers of mutually nonisomorphic lifted structures of essentially any type.

Let K be any field which admits a quadratic extension $K[\theta]$ where $\theta^2 = a\theta + b$. Recall that the automorphism induced from θ maps $t\theta + u$ to $-t\theta + u + at$. Then any additive quasifibration of $PG(3,K)$ is, by definition, of the form $x = 0$, $y = x\begin{bmatrix} g(t,u) & f(t,u) \\ t & u \end{bmatrix}$ for all t, u in K where g and f are functions such that $g(t+s,\ u+v) = g(t,u) + g(s,v)$ and $f(t+s, u+v) = f(t,u) + f(s,v)$.

The quasifibration lifted from the additive quasifibration is:

$$x = 0, y = x\begin{bmatrix} v + as - s\theta & -g(t,u)\theta + f(t,u) + ag(t,u) \\ t\theta + u & v + s\theta \end{bmatrix}$$

for all t, u, s, v in K.

Hence, we see that the quasifibration lifted from an additive quasifibration is also additive.

In particular, if the original quasifibration is a semifield spread then the lifted spread is also a semifield spread.

5.1. Lifted Pappian spread

We may lift any Pappian spread in $PG(3,K)$ to a semifield spread in $PG(3,K[\theta])$.

In order that a Pappian spread exist in $PG(3,K)$ there must be a quadratic extension $K[\alpha]$ such that $\alpha^2 = c\alpha + d$ so that the Pappian spread may be represented in $PG(3,K)$ as follows:

$$x = 0, y = x\begin{bmatrix} u - ct & dt \\ t & u \end{bmatrix}.$$

Now assume that there is another quadratic extension $K[\theta]$ not necessarily distinct from $K[\alpha]$. Let $\theta^2 = a\theta + b$. Hence, we obtain:

Theorem 5.1. *If K is a field which admits quadratic extensions $K[\theta]$, $K[\alpha]$ where $\theta^2 = a\theta + b$ and $\alpha^2 = c\alpha + d$ (possibly isomorphic) then there is a semifield spread in $PG(3, K[\theta])$ defined by:*

$$x = 0, y = x\begin{bmatrix} v + as - s\theta & -(u - ct)\theta + dt + a(u - ct) \\ t\theta + u & v + s\theta \end{bmatrix}$$

for all t, u, s, v in K.

Corollary 5.2. *Let K be any field which admits infinitely many mutually nonisomorphic quadratic extensions $K[\sqrt{p}]$ where p is a nonsquare in K for p in a set λ. Fix p_o in λ. Then the following semifield spreads are mutually nonisomorphic:*

$$x = 0, y = x\begin{bmatrix} v - s\sqrt{p} & -u\sqrt{p} + p_o t \\ t\sqrt{p} + u & v + s\sqrt{p} \end{bmatrix}$$

for all t, u, s, v in K in $PG(3, K[\sqrt{p}]$.

For example, in the above corollary, one could take $K = Q$ the field of rationals.

5.2. Lifted generalized Knuth semifields

In the following, we generalize the finite Knuth semifields with spreads in $PG(3, K)$ to arbitrary fields K.

Theorem 5.3. *(see Knuth [17] when K is finite).*

Let K be a field and α, β automorphisms of $GF(q)$. Then each the following defines a quasifibration in $PG(3, q)$.

I. $x = 0,\ y = x\begin{bmatrix} u^{\alpha} + t^{\alpha}g & t^{\alpha}f \\ t & u \end{bmatrix}$ *for all u, t in K where $x^{\alpha+1} + xg - f \neq 0$ for all x in K. The quasifibration is a spread if and only if in addition the function $G(t, c) = t^{\alpha^2} f^{\alpha} - c^{\alpha+1} t - c^{\alpha} t^{\alpha} g$ is surjective for each $c \neq 0$.*

II. $x = 0, y = x\begin{bmatrix} u^{\alpha} + tg & t^{\beta}f \\ t & u \end{bmatrix}$ *for all u, t, in K where $x^{\alpha+1} - xg - f \neq 0$ for all x in K. Furthermore, this quasifibration is always a spread.*

III. $x = 0, y = x\begin{bmatrix} u^{\alpha} & t^{\beta}f \\ t & u \end{bmatrix}$ *for all u, t in K where f is a constant in K, provided $u^{\alpha+1} / t^{\beta+1} \neq f$. Furthermore, the quasifibration is a spread if and only if the function $H(t, c) = t^{\alpha\beta} f^{\alpha} + c^{\alpha+1} t$ is surjective for each $c \neq 0$.*

Proof. First we note that the differences of the matrices in each case are nonsingular or identically zero. Since the matrices are additive, this is equivalent to showing in situation *I* that $u^{\alpha+1} + t^{\alpha}ug - t^{\alpha+1}f \neq 0$ which implies if $t = 0$ that $u^{\alpha+1} = 0$ which implies that $u = 0$. Otherwise, dividing by $t^{\alpha} + 1$ and letting $(u / t) = x$, we obtain $x^{\alpha+1} + xg - f$ which is nonzero by assumption.

The argument for situation *II* is virtually identical to the above and it is trivial to note that the matrices of type *III* are always nonsingular or zero.

Now change base for type *I* by the mapping $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. It may be directly verified that the image spread has the following form:

$$x = 0, y = x\begin{bmatrix} u^{\alpha^{-1}} + t^{\alpha^{-1}} & t^{\alpha^{-1}}f \\ t & u \end{bmatrix}$$

for all u, t in K. Hence, it follows that $x^{\alpha^{-1}+1} + xg^{\alpha^{-1}} - f \neq 0$ for all x in K which is, in turn, equivalent to $x^{\alpha+1} + x^{\alpha}g - f^{\alpha} \neq 0$.

Consider type *II*. To show that we have a spread, we need to show that any vector (a, c, d, e) is in one of the components of the quasifibration. This is obvious if $a = 0$ or $c = 0$. Hence, without loss of generality, we let $a = 1$ and $c \neq 0$.

Hence, we must show that there exists u, t such that

$$\begin{aligned} u^{\alpha} + tg + ct &= d, \text{ and} \\ t^{\alpha^{-1}} f + cu &= e. \end{aligned}$$

Now solve for u to obtain the following necessary and sufficient condition:

$$((e - t^{\alpha^{-1}} f) / c)^{\alpha} = (d - t(g + c)).$$

It follows that we may meet this condition if and only if

$$t(f^{\alpha} - c^{\alpha}(g + c))$$

is a surjective function for each $c \neq 0$. However, this is merely equivalent to the condition that

$$f^{\alpha} - c^{\alpha}(g + c) = -(c^{\alpha+1} + c^{\alpha} g - f^{\alpha}) \neq 0.$$

Hence, we have a cover and a quasifibration so we have a spread in this case.

Assume the conditions of *I*. In order that we have a cover, for each vector (d, c, e, h) and not both d and $c = 0$, there must be a component $y = x \begin{bmatrix} u^{\alpha} + t^{\alpha} g & t^{\alpha} f \\ t & u \end{bmatrix}$ incident with this vector.

Clearly, this is the case if either d or c is zero. Hence, assume that $dc \neq 0$ and without loss of generality, we may assume that $d = 1$.

Thus, for any c, e, h and $c \neq 0$, there must be a solution to the following system of equations:

$$\begin{aligned} (u^{\alpha} + t^{\alpha} g) + ct &= e \\ t^{\alpha} f + eu &= h. \end{aligned}$$

Solve for u as $(h - t^{\alpha} f) / c$ so we obtain the following equivalent condition:

$$((h - t^{\alpha} f) / c))^{\alpha} = (e - cd - t^{\alpha} g).$$

Hence, we obtain:

$$t^{\alpha^2} f^{\alpha} - c^{\alpha+1} t - c^{\alpha} t^{\alpha} g = h^{\alpha} - c^{\alpha} e.$$

If the indicated function is surjective then for fixed c, and arbitrary h, e there is a solution for t and reversing the above argument, defining $u = (h - t^{\alpha} f) / c$, it follows that there is a solution (u, t) to the original system of equations and hence there is a component which covers the indicated vector.

The proof to *III* is almost exactly the same and is simply the requirement for the quasifibration to cover the vector space.

Remark 5. *Fields that satisfy type III.*

(1) *Let F be any subfield of the reals and let $K = F[\sqrt{-1}]$. Then $x = 0$, $y = x \begin{bmatrix} u^{\sigma} & -t^{\sigma} \\ t & u \end{bmatrix}$ where σ denotes complex conjugation is a spread in $PG(3, K)$ which represents the F-quaternion skew field.*

Note that the existence of the large Baer group shows that there is an automorphism group of the F-quaterions which fixes the subfield F pointwise and isomorphic to the group $\{e$ such that $e^{\sigma+1} = 1\}$ for all e in K.

For example, if F is the field of real numbers, the automorphism group of the real quaternions is well known (see e.g. Saltzmann et al [20]).

(2) *More generally, if K is any field which admits an automorphism σ such that* $m \to m^{\sigma+1}$ *is not onto and f in* $K \cap Fix\sigma - K^{\sigma+1}$ *then* $x = 0,\ y = x \begin{bmatrix} u^{\sigma} & ft^{\sigma} \\ t & u \end{bmatrix}$ *for all u, t in K forms an additive and multiplicative quasifibration which is a skewfield if and only if* $ft^{\alpha^2} - c^{\sigma+1}t$ *is a surjective function for all nonzero c in K.*

In particular, a skewfield is obtained provided $\sigma^2 = 1$.

(3) *(Jha, Johnson [10]). Let k be any field and k(x) the field of rational functions over k. Let α and β be any endomorphisms which preserves the parity of the natural valuation. Let* $f = x$ *or any element of odd valuation.*

Then $x = 0, y = x \begin{bmatrix} u^{\alpha} & xt^{\beta} \\ t & . u \end{bmatrix}$ *for all* $b, u \in k(x)$ *defines a quasifibration.*

These examples are generalized in X. Liu [18].

Proof. (2). It is easy to verify that the indicated structure is an additive and multiplicative quasifibration if and only if f is in $K \cap Fix\sigma - K^{\sigma+1}$.

Theorem 5.4. *Let K be any field which admits a quadratic extension* $K[\theta]$. *Let* $\theta^2 = a\theta + b$.

Then the lifted generalized Knuth additive quasifibrations are given as follows:

I. $x = 0, y = x \begin{bmatrix} u^{\alpha} + t^{\alpha}g & t^{\alpha}f \\ t & u \end{bmatrix}$ *for all u, t in K where* $x^{\alpha+1} + xg - f \neq 0$ *for all x in K lifts to* $x = 0,\ y = x \begin{bmatrix} (v + sa - s\theta & -(u^{\alpha} + t^{\alpha}g)\theta + t^{\alpha}f + a(u^{\alpha} + t^{\alpha}g) \\ u + t\theta & v + s\theta \end{bmatrix}$ *in* $PG(3, K[\theta])$.

II. $x = 0, y = x \begin{bmatrix} u^{\alpha} + tg & t^{\alpha^{-1}}f \\ t & u \end{bmatrix}$ *for all u, t, in K where* $x^{\alpha+1} - xg - f \neq 0$ *for all x in K lifts to* $x = 0,\ y = x \begin{bmatrix} (v + sa - s\theta & -(u^{\alpha})\theta + t^{\beta}f + au^{\alpha} \\ u + t\theta & v + s\theta \end{bmatrix}$ *in* $PG(3, K[\theta])$. *Furthermore, this quasifibration is a spread given that the original exists.*

III. $x = 0, y = x \begin{bmatrix} u^{\alpha} & t^{\beta}f \\ t & u \end{bmatrix}$ *for all u, t in K where f is a constant in K, provided* $u^{\alpha+1}/t^{\beta+1} \neq f$, *lifts to* $x = 0,\ y = x \begin{bmatrix} (v + sa - s\theta & -(u^{\alpha})\theta + t^{\beta}f + au^{\alpha} \\ u + t\theta & v + s\theta \end{bmatrix}$ *in* $PG(3, K[\theta])$.

Now we note that once an additive quasifibration is located within $PG(3, K)$, and K has a quadratic extension field $K[\theta]$ then there are many often nonisomorphic lifted additive quasifibrations.

Remark 6. *Let* $x = 0,\ y = x \begin{bmatrix} g(t,u) & f(t,u) \\ t & u \end{bmatrix}$ *be a quasifibration in* $PG(3, K)$ *where K is a field which has a quadratic extension field* $K[\theta]$ *where* $\theta^2 = a\theta + b$.

Then a lifted quasifibration with spread $x = 0,\ y = x \begin{bmatrix} w^{\sigma} & F(v) \\ v & w \end{bmatrix}$ *is never multiplicative.*

Proof. The quasifibration is multiplicative if and only if $F(v) = v^{\sigma}F(1)$ and $F(1)^{\sigma} = F(1)$ (see example (2) above). Hence, $F(t\theta + u) = -g(t,u)\,\theta + (f(t,u) + ag(t,u)) = (t\theta + u)^{\sigma}$ $F(1)$ implies that $g(t,u) = tF(1)$ which is a contradiction to the assumption that the original structure is a quasifibration.

Hence, we obtain

Remark 7. *Lifting semifields can never produce skewfields.*

Remark 8. *Let* $\begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}$ *be any basis change where* A, B, C *are* 2×2 K*-matrices.*

Then $x = 0,\ y = x\,A^{-1} \begin{bmatrix} g(t,u) & f(t,u) \\ t & u \end{bmatrix} C$ *is an additive set of matrices which may be lifted to an additive quasifibration in* $PG(3, K[\theta])$.

Note that although the original structures are isomorphic, the lifted additive quasifibrations may not be isomorphic.

5.3. Lifting Ganley semifields

The following defines an additive quasifibration in $PG(3,K)$ where K is a field of characteristic 3.

$x = 0, y = x \begin{bmatrix} u + nt^3 & nt^9 + n^3 t \\ t & u \end{bmatrix}$ for all u, t in K and n a nonsquare in K.

To see this, we need to show that $u^2 + nt^3 u - nt^{10} - n^3 t^2 \neq$ for $(u,t) \neq (0,0)$. This is equivalent to verifying that $n^2 t^6 + 4(nt^{10} + n^3 t^2)$ is a nonsquare. Since $4 \equiv 1$ and

$$\begin{aligned} n^2 t^6 + 4(nt^{10} + n^3 t^2) &= nt^2(nt^4 + t^8 + n^2) = nt^2(-2nt^4 + t^8 + n^2) \\ &= nt^2(t^4 - n)^2, \end{aligned}$$

it follows that this must always be the case. (See Cohen and Ganley [7] p. 381, example 3).

If K has a quadratic extension $K[\theta]$ where $\theta^2 = a\theta + b$, a lifted Ganley additive quasifibration is $x = 0,\ y = x \begin{bmatrix} (v + sa - s\theta & -(u + nt^3)\theta + nt^9 + n^3 t + a(u + nt^3) \\ u + t\theta & v + s\theta \end{bmatrix}$ for all u, t, v, s in K.

6. LIFTING LIFTS - CHAINS

Note that any lift can be relifted. Actually, a lift can be recoordinatized and relifted assuming there is a convenient quadratic extension.

A *chain of lifts* is a set of lifts which are constructed from a given initial spread or quasifibration and a set of quadratic extensions by multiple lifting or lifting lifts. As noted above, such chains do not have to be direct chains in that one lifts the lifts without recoordinatization.

To bring a bit of order into this, we consider *regular chains* where lifts are lifted directly without coordinatization.

As an example of an infinite regular chain of lifted lifts, we consider an infinite sequence of quadratic extensions of quadratic extensions of the rational numbers Q. consider a sequence $\{\delta_n$ where $n = 1, 2, \ldots, \}$. Define $Q[\delta_1] = Q_1, Q_1[\delta_2] = Q_2, \ldots, Q_k[\delta_{k+1}] = Q_{k+1}$ for $k = 1, 1, \ldots$ where $Q = Q_o$ and where it is assumed that $Q_k[\delta_{d+1}]$ is a quadratic extension of Q_k. Let $\delta_k^2 = a_k \delta_k + b_k$ for $k = 1, 2, \ldots$. Let σ_k denote the involutory automorphism of Q_{k+1} which fixes Q_k pointwise.

We recall the previous theorem on lifting Pappian spreads:

Theorem 6.1. *If K is a field which admits quadratic extensions $K[\theta]$, $K[\alpha]$ where $\theta^2 = a\theta + b$ and $\alpha^2 = c\alpha + d$ (possibly isomoprhic) then there is a semifield spread in $PG(3, K[\theta])$ defined by:*

$$x = 0, y = x\begin{bmatrix} v + as - s\theta & -(u - ct)\theta + dt + a(u - ct) \\ t\theta + u & v + s\theta \end{bmatrix}$$

for all t, u, s, v in K.

For example, we could start with a Pappian spread in $PG(3, Q)$ defined by any quadratic extension $Q[\alpha]$ of Q and form a regular infinite chain of semifield spread.

Theorem 6.2. *Let S_o denote the initial Pappian spread in $PG(3, Q)$. If $\alpha^2 = a_o\alpha + b_o$ then the spread has the form $x = 0$, $y = x\begin{bmatrix} u - a_o t & b_o t \\ t & u \end{bmatrix}$ for all u, t in Q.*

Let S_{k+1} denote the semifield in $PG(3, Q_k[\delta_{k+1}])$ constructed by lifting the semifield spread in $PG(3, Q_k)$.

Represent S_k by $x = 0$, $y = x\begin{bmatrix} g_k(t_k, u_k) & f_k(t_k, u_k) \\ t_k & u_k \end{bmatrix}$ for all t_k, u_k in Q_k. Note that the functions g_k and f_k are well defined by induction.

Then S_{k+1} is $x = 0$,

$$y = x\begin{bmatrix} w_{k+1}^{\sigma_{k+1}} & -g_k(t_k, u_k)\delta_{k+1} + f_k(t_k, u_k) + a_{k+1}g_k(t_k, u_k) \\ t_{k+1} = t_k\delta_{k+1} + u_k & w_{k+1} \end{bmatrix}$$

for all t_k, u_k in Q_k and for all w_{k+1} in Q_{k+1}.

Hence, we obtain an infinite sequence of semifield spreads $\{S_i \text{ for } 0 \le i < \infty\}$ where S_o is a Pappian spread and each of the other spreads are non-Desarguesian semifield spreads.

Remark 9. *We have constructed but a few of the many variations of lifted spreads and lifted semifield spreads and chains of lifted spreads that can be obtained by lifting.*

In fact, the situation is completely chaotic. For example, there are so many semifield spreads that there is really no hope in forming a classification. There are infinitely many mutually nonisomorphic semifields spreads which lie in quadratic extensions fields of the rational field. There are infinitely many mutually nonisomorphic semifields spreads which lie in regular chains.

Furthermore, one may form a chain of lifts, recoordinatize, form a chain of retractions, and then lift again etc.

Therefore, we formulate the following equivalence relation.

Definition 6.3. *Two spreads or quasifibrations S and R in two three dimensional projective spaces over fields L and K respectively will be said to be lift equivalent if and only if there is a chain of quadratic extensions from one field to the other such that S may be constructed from R by a sequence of lifts and retractions.*

Remark 10. *Hence, the spreads in three dimensional projective spaces over fields are in disjoint classes under lift equivalency.*

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