Note di Matematica Vol. 17, 235-249 (1997)

THE PRODUCT THEORY FOR INNER PREMEASURES

HEINZ KÖNIG

Abstract. *The paper extends the product formation for inner premeasures, developed in the recent monograph of the author 1997 for the case of two factors, to arbitrary products.*

In the recent book [4] (henceforth cited as MI) the author attempted to restructure those fundamentals in measure and integration theory which serve to produce true contents and above all measures from more primitive data. The text contained basic implications from all over measure theory and beyond. One of the achievements was the resultant method of product formation for inner premeasures, which allowed to incorporate the Radon product measure into the abstract measure theory. The treatment in MI chapter VII was restricted to products of two factors. The aim of the present paper is its extension to arbitrary products.

In traditional measure theory one has on the one hand the abstract formation of arbitrary products of probability measures, with regularity not involved, and on the other hand the more or less topological product formation rooted in compactness, which is known to be quite delicate. We shall see that as before the approach in the spirit of MI leads to a unified development. The decisive new fact is that the inner tightness properties have an immediate transfer from the factors to the product formation.

On the other hand we shall be restrictive in a different respect: The second-mentioned earlier product formation rooted in compactness has been treated in the more comprehensive form of so-called projective systems; besides the standard references Bourbaki [1][2] and Schwartz [8] we refer to Lamb [6] and Stromberg [9]. For this particular extension the unified treatment in the spirit of MI turns out to produce new aspects, which call for separate treatment. Thus we shall be confined to products in the proper sense this time.

The paper consists of four sections. Section 1 extends the product of two factors to finite products, and section 2 contains some further complements to MI. Sections 3 and 4 then treat the infinite product formation. There is a natural subdivision, because in section 4 the basic assumptions will be somewhat wider than before, which in particular involves the so-called Prokhorov condition.

1 Finite Products

The present section extends the relevant portions of MI sections 20 and 21 from products of two factors to finite products. This is a routine procedure. We recall the former notations. As in MI a nonvoid set system will be called a paving. We use the multiplication on $\overline{\mathbb{R}}$ with the usual convention $0(\pm\infty) = (\pm\infty)0 := 0$. One verifies that this is an associative operation.

We assume nonvoid sets $X_1, \cdots, X_r$ and lattices $\mathfrak{S}_1, \cdots, \mathfrak{S}_r$ with $\varnothing$ in $X_1, \cdots, X_r$. Then the paving $\mathfrak{S}_1 \times \cdots \times \mathfrak{S}_r$ in $X_1 \times \cdots \times X_r$ fulfils $\cap$, and hence $(\mathfrak{S}_1 \times \cdots \times \mathfrak{S}_r)^\star$ is a lattice with $\varnothing$ in $X_1 \times \cdots \times X_r$.

Proposition 1.1. *Let $\varphi_l : \mathfrak{S}_l \to [0,\infty]$ be isotone and modular set functions with $\varphi_l(\varnothing) = 0\ \forall l = 1,\cdots,r$. Then there exists a unique isotone and modular set function $\varphi : (\mathfrak{S}_1 \times \cdots \times \mathfrak{S}_r)^\star \to [0,\infty]$ such that*

$$\varphi(S_1 \times \cdots \times S_r) = \varphi_1(S_1)\cdots\varphi_r(S_r) \quad \text{for all} \quad S_l \in \mathfrak{S}_l\ \forall l = 1,\cdots,r.$$

Of course $\varphi(\varnothing) = 0$. We write $\varphi =: \varphi_1 \times \cdots \times \varphi_r$.

Proof of existence. The proof is via induction. The case $r = 1$ is trivial, and $r = 2$ is contained in MI 20.4. For the induction step $1 \leqq r \Rightarrow r+1$ we assume $X_1,\cdots,X_r,Y$ and $\mathfrak{S}_1,\cdots,\mathfrak{S}_r,\mathfrak{T}$ and $\varphi_1,\cdots,\varphi_r,\psi$ as above, and $\varphi : (\mathfrak{S}_1 \times \cdots \times \mathfrak{S}_r)^\star \to [0,\infty]$ from the induction hypothesis. Then MI 20.4 furnishes the isotone and modular set function

$$\vartheta = \varphi \times \psi : ((\mathfrak{S}_1 \times \cdots \times \mathfrak{S}_r)^\star \times \mathfrak{T})^\star = ((\mathfrak{S}_1 \times \cdots \times \mathfrak{S}_r) \times \mathfrak{T})^\star \to [0,\infty].$$

For $S_l \in \mathfrak{S}_l\ \forall l = 1,\cdots,r$ and $T \in \mathfrak{T}$ we have

$$\vartheta((S_1 \times \cdots \times S_r) \times T) = \varphi(S_1 \times \cdots \times S_r)\psi(T) = \varphi_1(S_1)\cdots\varphi_r(S_r)\psi(T).$$

Thus under the identification $(X_1 \times \cdots \times X_r) \times Y = X_1 \times \cdots \times X_r \times Y$ and $(\mathfrak{S}_1 \times \cdots \times \mathfrak{S}_r) \times \mathfrak{T} = \mathfrak{S}_1 \times \cdots \times \mathfrak{S}_r \times \mathfrak{T}$ the set function ϑ is as required.

Proof of uniqueness. This proof is as for MI 20.5. Let $\alpha,\beta : (\mathfrak{S}_1 \times \cdots \times \mathfrak{S}_r)^\star \to [0,\infty]$ be as assumed. Fix $E \in (\mathfrak{S}_1 \times \cdots \times \mathfrak{S}_r)^\star$, that is

$$E = \bigcup_{k=1}^{n} S_1^k \times \cdots \times S_r^k \quad \text{with} \quad S_l^k \in \mathfrak{S}_l\ \forall l = 1,\cdots,r.$$

To be shown is $\alpha(E) = \beta(E)$. If the value

$$\varphi_1(S_1^k)\cdots\varphi_r(S_r^k) = \alpha(S_1^k \times \cdots \times S_r^k) = \beta(S_1^k \times \cdots \times S_r^k)$$

is $= \infty$ for some $k = 1,\cdots,n$, then $\alpha(E) = \beta(E) = \infty$. If the values are $< \infty$ for all $k = 1,\cdots,n$, then $\alpha(E),\beta(E) < \infty$. In this case we form the lattice $\mathfrak{R} := \{A \in (\mathfrak{S}_1 \times \cdots \times \mathfrak{S}_r)^\star : A \subset E\}$. From MI 2.5.1) applied to $\alpha|\mathfrak{R}$ and $\beta|\mathfrak{R}$ we obtain $\alpha(E) = \beta(E)$.

Consequence 1.2. *Assume $X_1,\cdots,X_r,Y_1,\cdots,Y_s$ and $\mathfrak{S}_1,\cdots,\mathfrak{S}_r,\mathfrak{T}_1,\cdots,\mathfrak{T}_s$ and $\varphi_1,\cdots,\varphi_r$, $\psi_1,\cdots,\psi_s$ as above. Under the identification*

$$\begin{aligned}(X_1 \times \cdots \times X_r) \times (Y_1 \times \cdots \times Y_s) &= X_1 \times \cdots \times X_r \times Y_1 \times \cdots \times Y_s,\\ (\mathfrak{S}_1 \times \cdots \times \mathfrak{S}_r) \times (\mathfrak{T}_1 \times \cdots \times \mathfrak{T}_r) &= \mathfrak{S}_1 \times \cdots \times \mathfrak{S}_r \times \mathfrak{T}_1 \times \cdots \times \mathfrak{T}_s\end{aligned}$$

we have then

$$(\varphi_1 \times \cdots \times \varphi_r) \times (\psi_1 \times \cdots \times \psi_s) = \varphi_1 \times \cdots \times \varphi_r \times \psi_1 \times \cdots \times \psi_s,$$

the domain of definition of which is

$$\begin{aligned}&((\mathfrak{S}_1 \times \cdots \times \mathfrak{S}_r)^\star \times (\mathfrak{T}_1 \times \cdots \times \mathfrak{T}_s)^\star)^\star\\ &= ((\mathfrak{S}_1 \times \cdots \times \mathfrak{S}_r) \times (\mathfrak{T}_1 \times \cdots \times \mathfrak{T}_s))^\star = (\mathfrak{S}_1 \times \cdots \times \mathfrak{S}_r \times \mathfrak{T}_1 \times \cdots \times \mathfrak{T}_s)^\star.\end{aligned}$$

This is an immediate consequence of the uniqueness assertion in 1.1. In the sequel we shall be less pedantic with obvious identifications.

Proposition 1.3. *Let $\varphi_l : \mathfrak{S}_l \to [0,\infty[$ be isotone and modular with $\varphi_l(\varnothing) = 0\ \forall l = 1,\cdots,r$, and hence*

$$\varphi = \varphi_1 \times \cdots \times \varphi_r : (\mathfrak{S}_1 \times \cdots \times \mathfrak{S}_r)^\star \to [0,\infty[.$$

0) *If $\varphi_1,\cdots,\varphi_r$ are downward $\bullet$ continuous then*

$$\varphi_\bullet(A_1 \times \cdots \times A_r) = (\varphi_1)_\bullet(A_1)\cdots(\varphi_r)_\bullet(A_r) \quad \text{for all} \quad A_l \subset X_l\ \forall l = 1,\cdots,r.$$

1) *If $\varphi_1,\cdots,\varphi_r$ are downward $\bullet$ continuous then φ is downward $\bullet$ continuous as well.* 2) *If $\varphi_1,\cdots,\varphi_r$ are $\bullet$ continuous at $\varnothing$ then φ is $\bullet$ continuous at $\varnothing$ as well.*

Proof. 1) and 2) have obvious inductive proofs from MI 21.4 and 21.5 as above. Then likewise 0) has an obvious inductive proof from MI 21.7. However, the latter MI 21.7 has been an exercise without complete proof. Since it is a basic fact, we think that we should include a proof of it. We do this with the lemma which follows.

Lemma 1.4. *Let $\mathfrak{S}$ in X and $\mathfrak{T}$ in Y be lattices with $\varnothing$, and $\varphi : \mathfrak{S} \to [0,\infty[$ and $\psi : \mathfrak{T} \to [0,\infty[$ be isotone with $\varphi(\varnothing) = \psi(\varnothing) = 0$. Let $\vartheta = \varphi \times \psi : \mathfrak{R} \to [0,\infty[$ on $\mathfrak{R} = (\mathfrak{S} \times \mathfrak{T})^\star$ be as in* MI 20.4. *If φ and ψ are downward $\bullet$ continuous then*

$$\vartheta_\bullet(A \times B) = \varphi_\bullet(A)\psi_\bullet(B) \quad \text{for all} \quad A \subset X \quad \text{and} \quad B \subset Y.$$

Proof. Fix $A \subset X$ and $B \subset Y$. $\leqq$) Let $R \in \mathfrak{R}_\bullet$ with $R \subset A \times B$. From MI 21.6.iii), and from MI 20.4.2) applied to $\varphi_\bullet$ and $\psi_\bullet$, we obtain

$$\begin{aligned}\vartheta_\bullet(R) = \int \psi_\bullet\big(R(\cdot)\big)d\varphi_\bullet \quad &\leqq \quad \int \psi_\bullet\big((A \times B)(\cdot)\big)d\varphi_\bullet \\ &= \quad (\varphi_\bullet \times \psi_\bullet)(A \times B) = \varphi_\bullet(A) \times \psi_\bullet(B).\end{aligned}$$

Since $\vartheta_\bullet$ is inner regular $\mathfrak{R}_\bullet$ the assertion follows. $\geqq$) We can assume that

$$\varphi_\bullet(A)\psi_\bullet(B) > 0,$$

and fix real t with $\varphi_\bullet(A)\psi_\bullet(B) > t > 0$. Thus $\varphi_\bullet(A), \psi_\bullet(B) > 0$, and we can find real a,b with $\varphi_\bullet(A) > a > 0, \psi_\bullet(B) > b > 0$, and $ab = t$. By inner regularity there exist

$$\begin{aligned}S \in \mathfrak{S}_\bullet \text{ with} \quad S \subset A \quad \text{and } \varphi_\bullet(S) > a,\\ T \in \mathfrak{T}_\bullet \text{ with} \quad T \subset B \quad \text{and } \psi_\bullet(T) > b.\end{aligned}$$

Then $S \times T \in \mathfrak{S}_\bullet \times \mathfrak{T}_\bullet \subset (\mathfrak{S} \times \mathfrak{T})_\bullet \subset \mathfrak{R}_\bullet$. Once more from MI 20.4.2) and MI 21.6.iii) we obtain

$$\begin{aligned}t = ab < \varphi_\bullet(S)\psi_\bullet(T) \quad &= \quad (\varphi_\bullet \times \psi_\bullet)(S \times T) \\ &= \quad \int \psi_\bullet\big((S \times T)(\cdot)\big)d\varphi_\bullet = \vartheta_\bullet(S \times T) \leqq \vartheta_\bullet(A \times B).\end{aligned}$$

The assertion follows.

We conclude with the extension of the former fundamental theorem MI 21.9.

Theorem 1.5. *Let* $\varphi_l : \mathfrak{S}_l \to [0,\infty[$ *be inner* $\bullet$ *premeasures with* $\varphi_l(\varnothing) = 0$ *and with maximal inner* $\bullet$ *extensions* $\phi_l := (\varphi_l)_\bullet|\mathfrak{C}((\varphi_l)_\bullet)$ $\forall l = 1,\cdots,r$. *Then* $\varphi = \varphi_1 \times \cdots \times \varphi_r : (\mathfrak{S}_1 \times \cdots \times \mathfrak{S}_r)^\star \to [0,\infty[$ *is an inner* $\bullet$ *premeasure, and* $\phi := \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ *is an extension of*

$$\phi_1 \times \cdots \times \phi_r : \big(\mathfrak{C}((\varphi_1)_\bullet) \times \cdots \times \mathfrak{C}((\varphi_r)_\bullet)\big)^\star \to [0,\infty].$$

Proof. Once more the proof is via induction. For the induction step $1 \leqq r \Rightarrow r+1$ we assume $X_1,\cdots,X_r,Y$ and $\mathfrak{S}_1,\cdots,\mathfrak{S}_r,\mathfrak{T}$ and $\varphi_1,\cdots,\varphi_r,\psi$ with $\phi_1,\cdots,\phi_r,\Psi$ as above. From the induction hypothesis we obtain the inner $\bullet$ premeasure $\varphi = \varphi_1 \times \cdots \times \varphi_r$ with $\phi := \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$. Then MI 21.9 says that $\vartheta := \varphi \times \psi = (\varphi_1 \times \cdots \times \varphi_r) \times \psi = \varphi_1 \times \cdots \times \varphi_r \times \psi$ is an inner $\bullet$ premeasure, and that $\theta := \vartheta_\bullet|\mathfrak{C}(\vartheta_\bullet)$ is an extension of $\phi \times \Psi$, and hence by MI 20.7 an extension of $(\phi_1 \times \cdots \times \phi_r) \times \Psi = \phi_1 \times \cdots \times \phi_r \times \Psi$. The proof is complete.

2 Preparations for Infinite Products

We start with the relevant product formations for set systems. We assume a nonvoid (for the most part infinite) index set I. Besides the common notation $\forall t \in I$ we define $\forall\!\!\!\!\forall\, t \in I$ to mean $\forall t \in I \setminus F$ with some finite $F \subset I$. Also define $\mathfrak{F}(I) \subset \mathfrak{P}(I)$ to consist of all nonvoid finite subsets of I.

We assume a family $(X_t)_{t\in I}$ of nonvoid sets X_t and put $X := \Pi_{t\in I}X_t$. For a family $(\mathfrak{A}_t)_{t\in I}$ of pavings $\mathfrak{A}_t$ in X_t we define in X the product pavings

$$\begin{aligned} \sqcap_{t\in I}\mathfrak{A}_t &:= \{\Pi_{t\in I}A_t : A_t \in \mathfrak{A}_t\ \forall t \in I\},\\ \times_{t\in I}\mathfrak{A}_t &:= \{\Pi_{t\in I}A_t : A_t \in \mathfrak{A}_t\ \forall t \in I \text{ with } A_t = X_t\ \forall\!\!\!\!\forall\, t \in I\}, \end{aligned}$$

the latter one under the assumption that $X_t \in \mathfrak{A}_t$ $\forall t \in I$. In this case we have $\times_{t\in I}\mathfrak{A}_t \subset \sqcap_{t\in I}\mathfrak{A}_t$, and the two pavings are equal when I is finite. We want to prove some useful formulas which extend MI 21.2 and 21.3.

Remark 2.1. For the product of a family $(A_t)_{t\in I}$ of subsets $A_t \subset X_t$ we have

$$(\Pi_{t\in I}A_t)' = \bigcup_{s\in I \text{ with } A_s \neq X_s} \big(A'_s \times \Pi_{t\in I\setminus\{s\}}X_t\big).$$

This is obvious. It uses the somewhat abusive but common notation

$$A'_s \times \Pi_{t\in I\setminus\{s\}}X_t := \Pi_{t\in I}B_t \quad \text{with} \quad B_s := A'_s \quad \text{and} \quad B_t := X_t\ \forall t \neq s.$$

Remark 2.2. For each paving $\mathfrak{S}$ in a nonvoid set X we have $(\mathfrak{S}^\star)_\star = (\mathfrak{S}_\star)^\star$. This is a routine verification.

Proposition 2.3. *Let* $(\mathfrak{A}_t)_{t\in I}$ *be a family of pavings* $\mathfrak{A}_t$ *in* X_t *with* $\varnothing, X_t \in \mathfrak{A}_t$ $\forall t \in I$, *and* $\bullet = \star\sigma\tau$. *Then the pavings* $\mathfrak{A} := \times_{t\in I}\mathfrak{A}_t$ *and* $\mathfrak{B} := \times_{t\in I}(\mathfrak{A}_t\perp)$ *fulfil*

$$((\mathfrak{A}_\star)^\bullet)\perp = (\mathfrak{B}^\star)_\bullet \quad \textit{and hence likewise} \quad ((\mathfrak{A}^\star)_\bullet)\perp = (\mathfrak{B}_\star)^\bullet.$$

Proof. 0) From 2.1 we have $\mathfrak{A}\perp \subset \mathfrak{B}^\star$ and $\mathfrak{B}\perp \subset \mathfrak{A}^\star$. We also recall MI 1.5.2). i) From 0) we obtain

$$((\mathfrak{A}_\star)^\bullet)\perp = ((\mathfrak{A}\perp)^\star)_\bullet \subset ((\mathfrak{B}^\star)^\star)_\bullet = (\mathfrak{B}^\star)_\bullet.$$

ii) From 0) and 2.2 we obtain

$$((\mathfrak{B}^\star)_\bullet)\perp = ((\mathfrak{B}\perp)_\star)^\bullet \subset ((\mathfrak{A}^\star)_\star)^\bullet = ((\mathfrak{A}_\star)^\star)^\bullet = (\mathfrak{A}_\star)^\bullet,$$

and hence $((\mathfrak{A}_\star)^\bullet)\perp \supset (\mathfrak{B}^\star)_\bullet$. The assertion follows.

Special Case 2.4. *Assume that the X_t are topological spaces and that X carries the product topology.* 1) *Then*

$$\mathrm{Op}(X) = \big(\times_{t\in I}\mathrm{Op}(X_t)\big)^\tau \quad \text{and} \quad \mathrm{Cl}(X) = \big((\times_{t\in I}\mathrm{Cl}(X_t))^\star\big)_\tau.$$

2) *Assume that the X_t and hence X are Hausdorff. Then*

$$\mathrm{Comp}(X) = \big((\sqcap_{t\in I}\mathrm{Comp}(X_t))^\star\big)_\tau.$$

Proof. 1) The formula for $\mathrm{Op}(X)$ is the definition of the product topology. Then from the fact that the $\mathrm{Op}(X_t)$ fulfil $\cap$ and from 2.3 we obtain

$$\mathrm{Cl}(X) = ((\mathrm{Op}(X))\perp = \big((\times_{t\in I}\mathrm{Op}(X_t))^\tau\big)\perp = \big((\times_{t\in I}\mathrm{Cl}(X_t))^\star\big)_\tau.$$

2) The inclusion $\supset$ is obvious. In order to see $\subset$ let $A \in \mathrm{Comp}(X)$. Then i) $A \in \mathrm{Cl}(X)$, and ii) A is contained in some member of $\sqcap_{t\in I}\mathrm{Comp}(X_t)$, for example in the product of its projections. From these facts and from 1) the assertion follows.

The next topic is the so-called $\bullet$ compactness for $\bullet = \star\sigma\tau$. A paving $\mathfrak{S}$ in a nonvoid set X is called $\bullet$ compact iff each paving $\mathfrak{M} \subset \mathfrak{S}$ fulfils $\varnothing \notin \mathfrak{M}_\star \Rightarrow \varnothing \notin \mathfrak{M}_\bullet$. The case $\bullet = \star$ is of course trivial. In case $\mathfrak{S}$ fulfils $\cap$ an equivalent formulation is the one in MI before 6.34, which requires that each paving $\mathfrak{M} \subset \mathfrak{S}$ of type $\bullet$ with $\mathfrak{M} \downarrow \varnothing$ has $\varnothing \in \mathfrak{M}$. We recall that to be of type $\bullet$ means finite when $\bullet = \star$, countable when $\bullet = \sigma$, and no restriction when $\bullet = \tau$.

Theorem 2.5. *If the paving $\mathfrak{S}$ in X is $\bullet$ compact, then $\mathfrak{S}_\bullet$ and $\mathfrak{S}^\star$ are $\bullet$ compact as well.*

Proof. 1) The first assertion is simple. In fact, if $\mathfrak{M} \subset \mathfrak{S}_\bullet$ is a paving with $\varnothing \notin \mathfrak{M}_\star$, then $\mathfrak{N} := \{S \in \mathfrak{S} : S \supset \text{some } M \in \mathfrak{M}\} \subset \mathfrak{S}$ is a paving with $\varnothing \notin \mathfrak{N}_\star$ as well. Now $\mathfrak{M}_\bullet \subset \mathfrak{N}_\bullet$, so that $\varnothing \notin \mathfrak{N}_\bullet$ implies that $\varnothing \notin \mathfrak{M}_\bullet$. 2) The second assertion is a well-known fundamental fact. For the sake of completeness we include a sketch of the proof attributed to Mokobodzki in Meyer [7] III.T4. Fix a paving $\mathfrak{M} \subset \mathfrak{S}^\star$ with $\varnothing \notin \mathfrak{M}_\star$. Then $\mathfrak{M}$ is contained in a maximal paving $\mathfrak{U} \subset \mathfrak{P}(X)$ with $\varnothing \notin \mathfrak{U}_\star$. 2.i) One verifies that (1) $\mathfrak{U}$ has $\cap$, and (2) if $A \subset X$ is not in $\mathfrak{U}$ then $A' \in \mathfrak{U}$. 2.ii) Each $M \in \mathfrak{M}$ contains a set $S(M) \in \mathfrak{S}$ which is in $\mathfrak{U}$. In fact, if $M = S_1 \cup \cdots \cup S_r$ with $S_1, \cdots, S_r \in \mathfrak{S}$, then one deduces from (1)(2) that at least one of the $S_1, \cdots, S_r$ must be in $\mathfrak{U}$. 2.iii) The paving $\mathfrak{N} := \{S(M) : M \in \mathfrak{M}\}$ is contained in $\mathfrak{U}$ and hence fulfils $\varnothing \notin \mathfrak{N}_\star$, and is contained in $\mathfrak{S}$ and hence fulfils $\varnothing \notin \mathfrak{N}_\bullet$. It follows that $\varnothing \notin \mathfrak{M}_\bullet$.

Consequence 2.6. *Let $(\mathfrak{A}_t)_{t\in I}$ be a family of $\bullet$ compact pavings $\mathfrak{A}_t$ in X_t. Then $\sqcap_{t\in I}\mathfrak{A}_t$ and hence $(\sqcap_{t\in I}\mathfrak{A}_t)^\star$ are $\bullet$ compact pavings in $X = \prod_{t\in I}X_t$.*

Proof. We have to prove that $\sqcap_{t\in I}\mathfrak{A}_t$ is $\bullet$ compact. Fix a paving $\mathfrak{M} \subset \sqcap_{t\in I}\mathfrak{A}_t$ with $\varnothing \notin \mathfrak{M}_\star$. Thus each $M \in \mathfrak{M}$ is nonvoid and hence has a unique representation $M = \Pi_{t\in I}M_t$ with $M_t \in \mathfrak{A}_t$. This furnishes the pavings $\mathfrak{M}_t := \{M_t : M \in \mathfrak{M}\} \subset \mathfrak{A}_t$ in X_t $\forall t \in I$. Now we have

$$\cap_{M\in\mathfrak{N}} M = \Pi_{t\in I}\big(\cap_{M\in\mathfrak{N}} M_t\big) \quad \text{for all nonvoid} \quad \mathfrak{N} \subset \mathfrak{M}.$$

Therefore $\varnothing \notin (\mathfrak{M}_t)_\star$ and hence $\varnothing \notin (\mathfrak{M}_t)_\bullet$ for all $t \in I$. It follows that $\varnothing \notin \mathfrak{M}_\bullet$.

The above consequence will be one of two sources from which we shall deduce that certain product set functions are $\bullet$ continuous at $\varnothing$. We turn to the other source, which is based on the so-called horizontal integral of MI section 11.

Lemma 2.7. *Let $\mathfrak{S}$ be a lattice in a nonvoid set X with $\varnothing \in \mathfrak{S}$, and $\varphi : \mathfrak{S} \to [0,\infty[$ be an isotone set function with $\varphi(\varnothing) = 0$ and $\sup\varphi = 1$ which is $\bullet$ continuous at $\varnothing$. Assume that $E \subset \mathrm{UM}(\mathfrak{S})$ is nonvoid of type $\bullet$ and downward directed in the pointwise order with $⨍ f d\varphi < \infty\ \forall f \in E$. If $\inf_{f\in E} ⨍ f d\varphi > \varepsilon > 0$, then there exists $a \in X$ such that $\inf_{f\in E} f(a) > \varepsilon$.*

Proof. 1) We know from MI 11.1.2) that $\{(f-\varepsilon)^+ : f \in E\} \subset \mathrm{UM}(\mathfrak{S})$. This set is nonvoid of type $\bullet$ and downward directed, and hence $\downarrow$ some $F \in [0,\infty]^X$. The claim is that F is not constant $= 0$. Let us assume that $F = 0$. 2) For $f \in E$ we have

$$⨍ f d\varphi = \int_{0\leftarrow}^{\to\infty} \varphi([f \geqq t])dt = \int_{0\leftarrow}^{\varepsilon} \varphi([f \geqq t])dt + \int_{\varepsilon\leftarrow}^{\to\infty} \varphi([f \geqq t])dt.$$

The last term is

$$= \int_{0\leftarrow}^{\to\infty} \varphi([f \geqq \varepsilon + t])dt = \int_{0\leftarrow}^{\to\infty} \varphi\big([(f-\varepsilon)^+ \geqq t]\big)dt = ⨍(f-\varepsilon)^+ d\varphi.$$

Thus $⨍ f d\varphi \leqq \varepsilon + ⨍(f-\varepsilon)^+ d\varphi$. Therefore by MI 11.22 it would follow from $F = 0$ that $\inf\{⨍ f d\varphi : f \in E\} \leqq \varepsilon$, which contradicts the hypothesis.

Consequence 2.8. *Let $\mathfrak{S}$ in X and $\mathfrak{T}$ in Y be lattices with $\varnothing$, and $\varphi : \mathfrak{S} \to [0,\infty[$ and $\psi : \mathfrak{T} \to [0,\infty]$ be isotone with $\varphi(\varnothing) = \psi(\varnothing) = 0$. Assume that $\sup\varphi = 1$, and that φ is $\bullet$ continuous at $\varnothing$. Let $\vartheta = \varphi \times \psi : (\mathfrak{S} \times \mathfrak{T})^\star \to [0,\infty]$ be as in* MI 20.4. *Assume that $\mathfrak{M} \subset (\mathfrak{S} \times \mathfrak{T})^\star$ is a paving of type $\bullet$ and downward directed with $\vartheta(M) < \infty\ \forall M \in \mathfrak{M}$. If $\inf_{M\in\mathfrak{M}} \vartheta(M) > \varepsilon > 0$, then there exists $a \in X$ such that $\inf_{M\in\mathfrak{M}} \psi(M(a)) > \varepsilon$.*

Proof. We know from MI 20.3.2) that $\psi(M(\cdot)) \in \mathrm{UM}(\mathfrak{S})$ for $M \in \mathfrak{M}$; and by definition $\vartheta(M) = ⨍ \psi(M(\cdot))d\varphi$. Thus the assertion is an immediate consequence of 2.7.

The remainder of the section consists of complements to the inner $\bullet$ extension theories of MI section 6 which will be needed in the sequel, but also deserve some interest of their own. Let X be a nonvoid set.

Remark 2.9. *Define $\mathfrak{C}(X) := \{\varnothing, X\}$, and $\varepsilon : \mathfrak{C}(X) \to [0,\infty[$ to be $\varepsilon(\varnothing) = 0$ and $\varepsilon(X) = 1$. Then $\varepsilon_\bullet(X) = 1$ and $\varepsilon_\bullet(A) = 0$ for the other $A \subset X$. Furthermore ε is an inner $\bullet$ premeasure with $\varepsilon_\bullet|\mathfrak{C}(\varepsilon_\bullet) = \varepsilon$.*

The proof consists of obvious verifications. We conclude with an extension result.

Proposition 2.10. *Let $\mathfrak{S}$ and $\mathfrak{T}$ be lattices in X with $\varnothing \in \mathfrak{S} \subset \mathfrak{T} \subset \mathfrak{S}\top\mathfrak{S}_\bullet$. Assume that $\varphi : \mathfrak{S} \to [0,\infty[$ is an inner $\bullet$ premeasure with $\varphi(\varnothing) = 0$, and that $\psi := \varphi_\bullet|\mathfrak{T} < \infty$. Then $\psi : \mathfrak{T} \to [0,\infty[$ is an inner $\bullet$ premeasure which extends φ, and $\psi_\bullet = \varphi_\bullet$.*

Proof. 1) We have even $\mathfrak{T}_\bullet \subset \mathfrak{S}\top\mathfrak{S}_\bullet$ and $\varphi_\bullet|\mathfrak{T}_\bullet < \infty$. By MI 6.7 and 6.27 $\varphi_\bullet|\mathfrak{T}_\bullet$ is downward $\bullet$ continuous; in particular ψ is downward $\bullet$ continuous. 2) ψ is supermodular by MI 6.3.5). 3) From 1) we have $\psi_\bullet = \psi = \varphi_\bullet$ on $\mathfrak{T}$. Thus $\psi_\bullet = \varphi_\bullet$ on $\mathfrak{T}_\bullet$, since both $\psi_\bullet|\mathfrak{T}_\bullet$ and $\varphi_\bullet|\mathfrak{T}_\bullet$ are downward $\bullet$ continuous by MI 6.5.iii) and 1). It follows that $\psi_\bullet = \varphi_\bullet$, since both sides are inner regular $\mathfrak{T}_\bullet$. 4) $\psi_\bullet|\mathfrak{C}(\psi_\bullet) = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ is a crude extension of $\varphi_\bullet|\mathfrak{S}\top\mathfrak{S}_\bullet$ and hence of ψ. Thus MI 6.31 says that ψ is an inner $\bullet$ premeasure.

Special Case 2.11. *Let $\mathfrak{S}$ be a lattice in X with $\varnothing \in \mathfrak{S}$, and $\varphi : \mathfrak{S} \to [0,\infty[$ be an inner $\bullet$ premeasure with $\varphi(\varnothing) = 0$ and $\sup\varphi = 1$. Define $\mathfrak{T} := \mathfrak{S} \cup \{X\}$, and $\psi : \mathfrak{T} \to [0,\infty[$ to be $\psi|\mathfrak{T} = \varphi$ and $\psi(X) = 1$. Then ψ is an inner $\bullet$ premeasure, and $\psi_\bullet = \varphi_\bullet$.*

Proof. It is obvious that $\varphi_\bullet(X) = 1$. Thus the assertion is contained in 2.10.

3 Infinite Products

We start with the standard assumptions and notations for the remainder of the paper. We assume as before a family $(X_t)_{t\in I}$ of nonvoid sets X_t and put $X := \Pi_{t\in I}X_t$. For nonvoid $U \subset I$ we put $X_U := \Pi_{t\in U}X_t$, so that $X_I = X$. In case $\varnothing \neq U \subsetneqq V \subset I$ we identify $X_U \times X_{V\setminus U}$ and X_V via the obvious canonical map, which once more is somewhat abusive but common practice.

We assume a family $(\mathfrak{T}_t)_{t\in I}$ of lattices $\mathfrak{T}_t$ in X_t with $\varnothing, X_t \in \mathfrak{T}_t$. Then $\times_{t\in I}\mathfrak{T}_t$ fulfils $\cap$, so that $\mathfrak{T} := (\times_{t\in I}\mathfrak{T}_t)^\star$ is a lattice in X with $\varnothing, X \in \mathfrak{T}$. For nonvoid $U \subset I$ likewise $\mathfrak{T}_U := (\times_{t\in U}\mathfrak{T}_t)^\star$ is a lattice in X_U with $\varnothing, X_U \in \mathfrak{T}_U$, and $\mathfrak{T}_I = \mathfrak{T}$. In case $\varnothing \neq U \subsetneqq V \subset I$ the above identification furnishes

$$\begin{aligned}\mathfrak{T}_V = (\times_{t\in V}\mathfrak{T}_t)^\star &= \big((\times_{t\in U}\mathfrak{T}_t) \times (\times_{t\in V\setminus U}\mathfrak{T}_t)\big)^\star \\ &= \big((\times_{t\in U}\mathfrak{T}_t)^\star \times (\times_{t\in V\setminus U}\mathfrak{T}_t)^\star\big)^\star = (\mathfrak{T}_U \times \mathfrak{T}_{V\setminus U})^\star.\end{aligned}$$

It will be convenient to define for nonvoid $U \subsetneqq I$ the related formation

$$\mathfrak{T}_U^I := \{A \times X_{I\setminus U} : A \in \mathfrak{T}_U\} = \mathfrak{T}_U \times \mathfrak{E}(X_{I\setminus U}) = \big(\mathfrak{T}_U \times \mathfrak{E}(X_{I\setminus U})\big)^\star,$$

which is a lattice in X with $\varnothing, X \in \mathfrak{T}_U^I$. The map $\mathfrak{T}_U \to \mathfrak{T}_U^I : A \mapsto A \times X_{I\setminus U}$ is a bijection. We put of course $\mathfrak{T}_I^I := \mathfrak{T}_I = \mathfrak{T}$. For nonvoid $U \subset I$ then

$$\mathfrak{T}_U^I = \big(\{\Pi_{t\in I}T_t : T_t \in \mathfrak{T}_t\ \forall t \in I \quad \text{with} \quad T_t = X_t\ \forall\, t \in I \quad \text{and} \quad \forall t \notin U\}\big)^\star.$$

We emphasize two obvious properties.

Properties 3.1. 1) *For nonvoid $U \subset V \subset I$ we have $\mathfrak{T}_U^I \subset \mathfrak{T}_V^I \subset \mathfrak{T}_I^I = \mathfrak{T}$.* 2) *We have $\mathfrak{T} = \bigcup_{U\in\mathfrak{F}(I)}\mathfrak{T}_U^I$, and therefore $\{\mathfrak{T}_U^I : U \in \mathfrak{F}(I)\} \uparrow \mathfrak{T}$ in the obvious sense.*

We assume at last a family $(\psi_t)_{t\in I}$ of isotone and modular set functions $\psi_t : \mathfrak{T}_t \to [0,\infty[$ with $\psi_t(\varnothing) = 0$ and $\psi_t(X_t) = 1$. Then 1.1 furnishes for each $U \in \mathfrak{F}(I)$ a unique isotone and modular set function $\psi_U : \mathfrak{T}_U \to [0,\infty[$ such that

$$\psi_U(\Pi_{t\in U}T_t) = \Pi_{t\in U}\psi_t(T_t) \quad \text{for all} \quad \Pi_{t\in U}T_t \in \times_{t\in U}\mathfrak{T}_t.$$

Of course $\psi_U(\varnothing) = 0$ and $\psi_U(X_U) = 1$. We write $\psi_U =: \times_{t\in U}\psi_t$. For $U \subsetneqq V$ in $\mathfrak{F}(I)$ we see from 1.2 that $\psi_V = \psi_U \times \psi_{V\setminus U}$. It will be convenient to define for $U \in \mathfrak{F}(I)$ the related formation $\psi_U^I := \psi_U \times \varepsilon_{I\setminus U}$, where $\varepsilon_K : \mathfrak{C}(X_K) \to [0,\infty[$ for nonvoid $K \subset I$ is the primitive set function from 2.9. By MI 20.5 this is the unique isotone and modular set function $\psi_U^I : \mathfrak{T}_U^I \to [0,\infty[$ such that $\psi_U^I(A \times X_{I\setminus U}) = \psi_U(A)$ for all $A \in \mathfrak{T}_U$. Of course $\psi_U^I(\varnothing) = 0$ and $\psi_U^I(X) = 1$. An obvious consequence is that for $U \subsetneqq V$ in $\mathfrak{F}(I)$ we have

$$\begin{aligned}\psi_U^I(A \times X_{I\setminus U}) &= \psi_U(A) = \psi_U(A)\psi_{V\setminus U}(X_{V\setminus U}) = \psi_V(A \times X_{V\setminus U})\\ &= \psi_V^I\big((A \times X_{V\setminus U}) \times X_{I\setminus V}\big) = \psi_V^I(A \times X_{I\setminus U}) \quad \forall A \in \mathfrak{T}_U,\end{aligned}$$

that is $\psi_U^I = \psi_V^I|\mathfrak{T}_U^I$. This furnishes at once the first main result of the present section.

Theorem 3.2. *There exists a unique isotone and modular set function $\psi : \mathfrak{T} \to [0,\infty[$ such that*

$$\psi(\Pi_{t\in I}T_t) = \Pi_{t\in I}\psi_t(T_t) \quad \text{for all} \quad \Pi_{t\in I}T_t \in \times_{t\in I}\mathfrak{T}_t.$$

It is defined by $\psi|\mathfrak{T}_U^I = \psi_U^I = \psi_U \times \varepsilon_{I\setminus U}$ for all $U \in \mathfrak{F}(I)$. Of course $\psi(\varnothing) = 0$ and $\psi(X) = 1$. We write $\psi =: \times_{t\in I}\psi_t$.

Likewise one has for each nonvoid $K \subset I$ a unique isotone and modular set function $\psi_K : \mathfrak{T}_K \to [0,\infty[$ such that

$$\psi_K(\Pi_{t\in K}T_t) = \Pi_{t\in K}\psi_t(T_t) \quad \text{for all} \quad \Pi_{t\in K}T_t \in \times_{t\in K}\mathfrak{T}_t.$$

In case $K \in \mathfrak{F}(I)$ this is the former one. Of course $\psi_K(\varnothing) = 0$ and $\psi_K(X_K) = 1$. We continue to write $\psi_K =: \times_{t\in K}\psi_t$. The uniqueness assertion has the immediate consequence which follows.

Consequence 3.3. *In case $\varnothing \neq U \subsetneqq V \subset I$ we have $\psi_V = \psi_U \times \psi_{V\setminus U}$.*

The main problem is of course to extend the fundamental theorem 1.5. The decisive new fact is that inner • tightness carries over from the factors ψ_t to the product formation ψ. This is the next result.

Proposition 3.4. *Assume that the ψ_t are inner • premeasures $\forall t \in I$. Then ψ is inner • tight.* Thus ψ_K for nonvoid $K \subset I$ is inner • tight as well.

Proof. Fix $A \subset B$ in $\mathfrak{T}$, and then $U \in \mathfrak{F}(I)$ such that $A, B \in \mathfrak{T}_U^I$. Thus $A = P \times X_{I\setminus U}$ and $B = Q \times X_{I\setminus U}$ with $P \subset Q$ in $\mathfrak{T}_U$. We know from 1.5 that $\psi_U : \mathfrak{T}_U \to [0,\infty[$ is inner • tight. Thus for $\varepsilon > 0$ there exists a paving $\mathfrak{M} \subset \mathfrak{T}_U$ of type • with $\mathfrak{M} \downarrow$ some $D \subset Q \setminus P$ and $M \subset Q\ \forall M \in \mathfrak{M}$ such that

$$\psi_U(Q) - \psi_U(P) - \varepsilon < \inf\{\psi_U(M) : M \in \mathfrak{M}\}.$$

It follows that $\{M \times X_{I\setminus U} : M \in \mathfrak{M}\} \subset \mathfrak{T}_U^I \subset \mathfrak{T}$ is a paving of type • with $\downarrow D \times X_{I\setminus U} \subset (Q \times X_{I\setminus U}) \setminus (P \times X_{I\setminus U}) = B \setminus A$ and all members $\subset Q \times X_{I\setminus U} = B$ such that

$$\begin{aligned}\psi(B) - \psi(A) - \varepsilon &= \psi_U(Q) - \psi_U(P) - \varepsilon < \inf\{\psi_U(M) : M \in \mathfrak{M}\}\\ &= \inf\{\psi(M \times X_{I\setminus U}) : M \in \mathfrak{M}\}.\end{aligned}$$

This proves the assertion.

Next we return to section 2 in order to obtain the conclusion that ψ is • continuous at $\varnothing$. First of all 2.6 and the trivial remark MI 6.34 have the obvious consequence which follows.

Proposition 3.5. *Assume that the $\mathfrak{T}_t$ are • compact and hence the ψ_t are • continuous at $\varnothing\ \forall t \in I$. Then $\mathfrak{T}$ is • compact and hence ψ is • continuous at $\varnothing$ as well.*

We turn to the second result which follows from 2.8.

Lemma 3.6. *Assume that the ψ_t are • continuous at $\varnothing\ \forall t \in I$. Let $\mathfrak{M} \subset \mathfrak{T}$ be a paving of type • with $\mathfrak{M} \downarrow \varnothing$ such that $\mathfrak{M} \subset \mathfrak{T}_K^I$ for some nonvoid countable $K \subset I$. Then $\inf_{M \in \mathfrak{M}} \psi(M) = 0$.*

Proof. We can assume $\bullet = \sigma\tau$ and that I is infinite, since for finite I the assertion is 1.3.2). 1) Let I be countable, so that $K = I = \mathbb{N}$. We assume that $\inf\{\psi(M) : M \in \mathfrak{M}\}$ is > 0 and hence $>$ some $\varepsilon > 0$. i) We form via induction a sequence $(a_l)_l$ of points $a_l \in X_l$ such that

$$\inf\{\psi_{\{n+1,\cdots\}}(M(a_1,\cdots,a_n)) : M \in \mathfrak{M}\} > \varepsilon \quad \text{for} \quad n \in \mathbb{N};$$

note that $M(a_1,\cdots,a_n) \in \mathfrak{T}_{\{n+1,\cdots\}}$ by MI 20.3.1). The case $n = 1$: We have $\psi = \psi_1 \times \psi_{\{2,\cdots\}}$ on $\mathfrak{T} = (\mathfrak{T}_1 \times \mathfrak{T}_{\{2,\cdots\}})^\star$. Thus from 2.8 we obtain $a_1 \in X_1$ such that $\inf\{\psi_{\{2,\cdots\}}(M(a_1)) : M \in \mathfrak{M}\} > \varepsilon$. The step $1 \leqq n \Rightarrow n+1$: We have

$$\psi_{\{n+1,\cdots\}} = \psi_{n+1} \times \psi_{\{n+2,\cdots\}} \quad \text{on} \quad \mathfrak{T}_{\{n+1,\cdots\}} = (\mathfrak{T}_{n+1} \times \mathfrak{T}_{\{n+2,\cdots\}})^\star.$$

Thus from 2.8 we obtain $a_{n+1} \in X_{n+1}$ such that

$$\inf\{\psi_{\{n+2,\cdots\}}\big(M(a_1,\cdots,a_n)(a_{n+1})\big) : M \in \mathfrak{M}\} > \varepsilon,$$

where $M(a_1,\cdots,a_n)(a_{n+1}) = M(a_1,\cdots,a_n,a_{n+1})$ is obvious from the definition. This terminates the inductive choice. ii) Now put $a := (a_l)_{l \in \mathbb{N}} \in X$. We claim that $a \in M$ for each $M \in \mathfrak{M}$; this will contradict the assumption that $\mathfrak{M} \downarrow \varnothing$. In fact, for fixed $M \in \mathfrak{M}$ there exists an $n \in \mathbb{N}$ such that $M = A \times X_{\{n+1,\cdots\}}$ with $A \in \mathfrak{T}_{\{1,\cdots,n\}}$. Then $M(a_1,\cdots,a_n) \neq \varnothing$ implies that $(a_1,\cdots,a_n) \in A$ and hence $a \in M$. 2) We turn to the case that I is uncountable. From 3.3 we have $\psi = \psi_K \times \psi_{I \setminus K}$ on $\mathfrak{T} = (\mathfrak{T}_K \times \mathfrak{T}_{I \setminus K})^\star$. By assumption each $M \in \mathfrak{M}$ is of the form $M = A(M) \times X_{I \setminus K}$ with some unique $A(M) \in \mathfrak{T}_K$. Thus $\psi(M) = \psi_K(A(M))$. Now $\{A(M) : M \in \mathfrak{M}\} \subset \mathfrak{T}_K$ is a paving of type • with $\downarrow \varnothing$. Thus from 1) we obtain

$$\inf\{\psi(M) : M \in \mathfrak{M}\} = \inf\{\psi_K(A(M)) : M \in \mathfrak{M}\} = 0.$$

The proof is complete.

Proposition 3.7. *Assume that the ψ_t are • continuous at $\varnothing\ \forall t \in I$. Then likewise ψ is • continuous at $\varnothing$*

for $\bullet = \sigma$ in all cases, and
for $\bullet = \tau$ when I is countable.

In fact, each countable paving $\mathfrak{M} \subset \mathfrak{T}$ has a nonvoid countable $K \subset I$ such that $\mathfrak{M} \subset \mathfrak{T}_K^I$. The question whether $\bullet = \tau$ does need an additional assumption must remain open.

The aim of the present section requires one more consideration.

Remark 3.8. *Assume that ψ is • continuous at $\varnothing$. For each nonvoid $K \subset I$ then ψ_K is • continuous at $\varnothing$ as well.*

Proof. Let $K \neq I$. From 3.3 we have $\psi = \psi_K \times \psi_{I \setminus K}$ on $\mathfrak{T} = (\mathfrak{T}_K \times \mathfrak{T}_{I \setminus K})^\star$. If $\mathfrak{M} \subset \mathfrak{T}_K$ is a paving of type • with $\mathfrak{M} \downarrow \varnothing$, then $\{M \times X_{I \setminus K} : M \in \mathfrak{M}\} \subset \mathfrak{T}$ is a paving of type • with $\downarrow \varnothing$ as well, and we have $\psi(M \times X_{I \setminus K}) = \psi_K(M)$ for $M \in \mathfrak{M}$. It follows that

$$\inf\{\psi_K(M) : M \in \mathfrak{M}\} = \inf\{\psi(M \times X_{I \setminus K}) : M \in \mathfrak{M}\} = 0,$$

which is the assertion.

Proposition 3.9. *Assume that the ψ_t are inner $\bullet$ premeasures $\forall t \in I$, and that ψ is $\bullet$ continuous at $\varnothing$ and hence an inner $\bullet$ premeasure as well. Then $\psi_\bullet|\mathfrak{C}(\psi_\bullet)$ is an extension of $\times_{t\in I}\big((\psi_t)_\bullet|\mathfrak{C}((\psi_t)_\bullet)\big)$.*

Proof. 1) We recall 3.2 for the family $(\vartheta_t)_{t\in I}$ of the set functions

$$\vartheta_t := (\psi_t)_\bullet|\mathfrak{C}((\psi_t)_\bullet) \quad \text{on} \quad \mathfrak{C}_t := \mathfrak{C}((\psi_t)_\bullet),$$

and form the respective $\mathfrak{C}$, and $\mathfrak{C}_U$ and $\mathfrak{C}_U^I$ for $U \in \mathfrak{F}(I)$. From 3.1.2) then $\{\mathfrak{C}_U^I : U \in \mathfrak{F}(I)\} \uparrow \mathfrak{C}$. The product $\vartheta := \times_{t\in I}\vartheta_t$ satisfies

$$\vartheta|\mathfrak{C}_U^I = (\times_{t\in U}\vartheta_t) \times \varepsilon_{I\setminus U} \quad \text{for all} \quad U \in \mathfrak{F}(I).$$

2) Fix now $U \in \mathfrak{F}(I)$. We have $\psi = \psi_U \times \psi_{I\setminus U}$ on $\mathfrak{T} = (\mathfrak{T}_U \times \mathfrak{T}_{I\setminus U})^\star$, and by 3.8 both factors are inner $\bullet$ premeasures. We use 1.15 twice. On the one hand $\psi_\bullet|\mathfrak{C}(\psi_\bullet)$ is an extension of

$$\big((\psi_U)_\bullet|\mathfrak{C}((\psi_U)_\bullet)\big) \times \big((\psi_{I\setminus U})_\bullet|\mathfrak{C}((\psi_{I\setminus U})_\bullet)\big),$$

and hence of $\big((\psi_U)_\bullet|\mathfrak{C}((\psi_U)_\bullet)\big) \times \varepsilon_{I\setminus U}$. On the other hand $(\psi_U)_\bullet|\mathfrak{C}((\psi_U)_\bullet)$ is an extension of $\times_{t\in U}\vartheta_t$, and hence $\big((\psi_U)_\bullet|\mathfrak{C}((\psi_U)_\bullet)\big) \times \varepsilon_{I\setminus U}$ is an extension of $(\times_{t\in U}\vartheta_t) \times \varepsilon_{I\setminus U} = \vartheta|\mathfrak{C}_U^I$.
3) Therefore $\psi_\bullet|\mathfrak{C}(\psi_\bullet)$ is an extension of $\vartheta|\mathfrak{C}_U^I$ for all $U \in \mathfrak{F}(I)$. The assertion follows.

The abovs results combine to the fundamental theorem which follows.

Theorem 3.10. *Assume that the ψ_t are inner $\bullet$ premeasures $\forall t \in I$. Then likewise ψ is an inner $\bullet$ premeasure*

for $\bullet = \star\sigma$ in all cases, and
for $\bullet = \tau$ when I is countable,

and also when the $\mathfrak{T}_t$ are $\bullet$ compact $\forall t \in I$ and hence $\mathfrak{T}$ is $\bullet$ compact as well. If ψ is an inner $\bullet$ premeasure, then $\psi_\bullet|\mathfrak{C}(\psi_\bullet)$ is an extension of $\times_{t\in I}\big((\psi_t)_\bullet|\mathfrak{C}((\psi_t)_\bullet)\big)$.

We continue with an important addendum.

Proposition 3.11. *Assume that ψ is $\bullet$ continuous at $\varnothing$. For each nonvoid $K \subset I$ then*

$$(\psi_K)_\bullet(A) = \psi_\bullet(A \times X_{I\setminus K}) \quad \text{for all} \quad A \subset X_K.$$

Proof. By 3.3 we have $\psi = \psi_K \times \psi_{I\setminus K}$ on $\mathfrak{T} = (\mathfrak{T}_K \times \mathfrak{T}_{I\setminus K})^\star$, and both factors are $\bullet$ continuous at $\varnothing$ by 3.8. $\geqq$) Let $\mathfrak{M} \subset \mathfrak{T}_K$ be a paving of type $\bullet$ with $\mathfrak{M} \downarrow\subset A$. Then $\{M \times X_{I\setminus K} : M \in \mathfrak{M}\} \subset \mathfrak{T}$ is a paving of type $\bullet$ with $\downarrow\subset A \times X_{I\setminus K}$, and we have $\psi(M \times X_{I\setminus K}) = \psi_K(M)$ for $M \in \mathfrak{M}$. It follows that

$$\psi_\bullet(A \times X_{I\setminus K}) \geqq \inf\{\psi(M \times X_{I\setminus K}) : M \in \mathfrak{M}\} = \inf\{\psi_K(M) : M \in \mathfrak{M}\},$$

and hence the assertion. $\leqq$) We can assume that $\psi_\bullet(A \times X_{I\setminus K}) > 0$. We fix $\varepsilon > 0$ with $\psi_\bullet(A \times X_{I\setminus K}) > \varepsilon$, and then a paving $\mathfrak{M} \in \mathfrak{T}$ of type $\bullet$ with $\downarrow\subset A \times X_{I\setminus K}$ such that $\inf\{\psi(M) : M \in \mathfrak{M}\} > \varepsilon$. Then 2.8 furnishes an $a \in X_{I\setminus K}$ such that $\inf\{\psi_K(M(a)) : M \in \mathfrak{M}\} > \varepsilon$; note that $M(a) \in \mathfrak{T}_K$ by MI 20.3.1). Now $\{M(a) : M \in \mathfrak{M}\} \subset \mathfrak{T}_K$ is a paving of type $\bullet$ which

$\downarrow$ some $D \subset X_K$. We have $D \subset A$, because each $u \in D$ fulfils $u \in M(a)$ or $(u,a) \in M$ for all $M \in \mathfrak{M}$, and hence $(u,a) \in A \times X_{I \setminus K}$ or $u \in A$. Thus by definition $(\psi_K)_\bullet(A) > \varepsilon$. The assertion follows.

The importance of the last result rests upon its connection with the so-called image measure theorem obtained in König [5] theorem 3.5. Let us recall the basic notions. We assume nonvoid sets X and Y and a map $H : X \to Y$. One defines for a set system $\mathfrak{A}$ in X the set system

$$H[\mathfrak{A}] := \{B \subset Y : H^{-1}(B) \in \mathfrak{A}\} \quad \text{in} \quad Y,$$

and for a set function $\alpha : \mathfrak{A} \to \overline{\mathbb{R}}$ the image set function $\beta = H[\alpha] : H[\mathfrak{A}] \to \overline{\mathbb{R}}$ to be $\beta(B) = \alpha(H^{-1}(B))$ for $B \in H[\mathfrak{A}]$. There are numerous properties which carry over from $\mathfrak{A}$ and α to $H[\mathfrak{A}]$ and $H[\alpha]$, for example to be a σ algebra and to be a measure, in MI called conventional measure =: measure. $H[\alpha]$ appears to be the maximal natural image of α under the map H. The above-mentioned image measure theorem relates this notion to the inner $\bullet$ extension theories of MI section 6. We combine it with the present 3.11 to obtain the result which follows.

Theorem 3.12. *Assume that the ψ_t are inner $\bullet$ premeasures $\forall t \in I$, and that ψ is $\bullet$ continuous at $\varnothing$ and hence an inner $\bullet$ premeasure as well. Let $K \subset I$ be nonvoid, and $H : X \to X_K$ denote the natural projection. Then*

$$(\psi_K)_\bullet | \mathfrak{C}((\psi_K)_\bullet) = H[\psi_\bullet | \mathfrak{C}(\psi_\bullet)].$$

Recall that ψ_K is an inner $\bullet$ premeasure as well.

Proof. [Sketch of proof] One verifies that $H(\mathfrak{T}) = \mathfrak{T}_K$ and $H^{-1}(\mathfrak{T}_K) = \mathfrak{T}_K^I \subset \mathfrak{T}$. Thus $\mathfrak{T}$ is a Lusin skeleton for H at $\mathfrak{T}$ and $\mathfrak{T}_K$ in the former sense, and it fulfils condition I) in the image measure theorem. Furthermore $\psi_K = \psi(H^{-1}(\cdot))|\mathfrak{T}_K < \infty$. Thus 3.11 says that the final condition in the image measure theorem is fulfilled as well. The assertion follows.

The present main result 3.10 contains the two most prominent traditional theorems on infinite products. For *the abstract measure* situation we refer to Hewitt-Stromberg [3] section 22 and Stromberg [9] chapter 7. Here the ψ_t are measures on σ algebras $\mathfrak{T}_t$ in X_t with $\psi_t(X_t) = 1\ \forall t \in I$. The present theorem $\bullet = \sigma$ says that ψ is an inner σ premeasure. This contains the traditional theorem which furnishes the restriction of $\psi_\sigma|\mathfrak{C}(\psi_\sigma)$ to the generated σ algebra $\mathrm{A}\sigma(\times_{t\in I}\mathfrak{T}_t) = \mathrm{A}\sigma(\mathfrak{T})$. As it is usual in the traditional abstract situation, the aspect of regularity is not considered. For the *earlier topological* situation we refer to Bourbaki [1] section III.4.6. Here the X_t are compact Hausdorff topological spaces with $\mathfrak{T}_t = \mathrm{Comp}(X_t)\ \forall t \in I$ and X carries the product topology, so that $\mathfrak{T}_\tau = \mathrm{Comp}(X)$ by 2.4. The ψ_t are Radon premeasures on X_t with (in essence) $\psi_t(X_t) = 1\ \forall t \in I$. The present theorem $\bullet = \tau$ (and even 3.4 without the subsequent results) says that ψ is an inner τ premeasure and hence $\psi_\tau|\mathfrak{T}_\tau$ is a Radon premeasure on X, and thus refurnishes the earlier result. The *more recent topological* situation of Bourbaki [2] section IX.4.3, where the X_t are arbitrary Hausdorff topological spaces $\forall t \in I$ and I is assumed to be countable, will be contained and enframed in the subsequent final section.

4 The General Situation

The standard assumptions of the present section are somewhat wider than those of the last section, but can at once be connected with the previous ones. We assume as before a family $(X_t)_{t\in I}$ of nonvoid sets X_t, and form X and the X_U for nonvoid $U \subset I$. Then we assume a family $(\mathfrak{S}_t)_{t\in I}$ of lattices $\mathfrak{S}_t$ in X_t with $\varnothing \in \mathfrak{S}_t$, and a family $(\varphi_t)_{t\in I}$ of isotone and modular set functions $\varphi_t : \mathfrak{S}_t \to [0,\infty[$ with $\varphi_t(\varnothing) = 0$ and $\sup \varphi_t = 1$. Thus we doe *not* assume that $X_t \in \mathfrak{S}_t$, and hence of course not $\varphi_t(X_t) = 1$. We are faced with the problem to extend the infinite product formation and results of the last section.

There is an obvious connection with the standard assumptions of that section: We form the family $(\mathfrak{T}_t)_{t\in I}$ of the lattices $\mathfrak{T}_t := \mathfrak{S}_t \cup \{X_t\}$ in X_t, and the family $(\psi_t)_{t\in I}$ of the set functions $\psi_t : \mathfrak{T}_t \to [0,\infty[$ defined to be $\psi_t|\mathfrak{S}_t = \varphi_t$ and $\psi_t(X_t) = 1$. This puts us into the previous situation. One notes some obvious facts, for example that $\bullet$ compactness carries over from $\mathfrak{S}_t$ to $\mathfrak{T}_t$, and that to be $\bullet$ continuous at $\varnothing$ carries over from φ_t to ψ_t, and likewise to be an inner $\bullet$ premeasure after 2.11.

From the last section we inherit the lattice $\mathfrak{T}$ and the set function $\psi : \mathfrak{T} \to [0,\infty[$, and their satellites. These formations will remain the fundamental ones in the present context. It is quite clear that there are no crude counterparts in direct terms of the new initial families $(\mathfrak{S}_t)_{t\in I}$ and $(\varphi_t)_{t\in I}$, because the formation $\times_{t\in I}\mathfrak{S}_t$ and its satellites need not be defined. The present section rather sets out to root the set function $\psi : \mathfrak{T} \to [0,\infty[$ in the family $(\mathfrak{S}_t)_{t\in I}$. We shall see that this is a somewhat delicate task, in particular when I is not countable. We emphasize that the earlier theorem 3.10 will remain the fundamental source in order to see that ψ is an inner $\bullet$ premeasure.

For the more technical part we fix a nonvoid subset $K \subset I$. We define

$$\mathfrak{P}_K := \big((\sqcap_{t\in K}\mathfrak{S}_t) \times \mathfrak{T}_{I\setminus K}\big)^{\star},$$

which in case $K = I$ is to mean $\mathfrak{P}_I := (\sqcap_{t\in I}\mathfrak{S}_t)^{\star}$; in the sequel we shall retain this convention. The basic relations between $\mathfrak{P}_K$ and $\mathfrak{T}$ are as follows.

Remark 4.1. 1) *$\mathfrak{P}_K$ is a lattice in X with $\varnothing \in \mathfrak{P}_K$ which fulfils $\mathfrak{T} \subset \mathfrak{P}_K \top \mathfrak{P}_K$. 2) If K is of type $\bullet$ then $\mathfrak{P}_K \subset \mathfrak{T}_\bullet$.*

Proof. 1) is obvious. 2) We can assume that $P \in \mathfrak{P}_K$ is of the form $P = (\Pi_{t\in K}S_t) \times T$ with $S_t \in \mathfrak{S}_t\ \forall t \in K$ and $T \in \mathfrak{T}_{I\setminus K}$. For $F \in \mathfrak{F}(K)$ then $P_F := (\Pi_{t\in F}S_t) \times X_{K\setminus F} \times T \in \mathfrak{T}$, and $\{P_F : F \in \mathfrak{F}(K)\} \subset \mathfrak{T}$ is a paving of type $\bullet$ which $\downarrow P$. The assertion follows.

We turn to the consideration of the set function $\psi : \mathfrak{T} \to [0,\infty[$.

Lemma 4.2. *Assume that ψ is downward $\bullet$ continuous, and let K be of type $\bullet$. Then $P = (\Pi_{t\in K}S_t) \times X_{I\setminus K} \in \mathfrak{P}_K$ with $S_t \in \mathfrak{S}_t\ \forall t \in K$ has*

$$\psi_\bullet(P) = \Pi_{t\in K}\varphi_t(S_t) := \inf\{\Pi_{t\in F}\varphi_t(S_t) : F \in \mathfrak{F}(K)\}.$$

Proof. For $F \in \mathfrak{F}(K)$ we have

$$P_F := (\Pi_{t\in F}S_t) \times X_{I\setminus F} \in \mathfrak{T}$$

with $\psi(P_F) = \psi_F(\Pi_{t\in F} S_t) = \Pi_{t\in F}\varphi_t(S_t)$. Then $\{P_F : F \in \mathfrak{F}(K)\} \subset \mathfrak{T}$ is a paving of type $\bullet$ which $\downarrow P$. Thus $\inf\{\psi(P_F) : F \in \mathfrak{F}(K)\} = \psi_\bullet(P)$ from MI 6.5.iii). This is the assertion.

Proposition 4.3. *Assume that ψ is downward $\bullet$ continuous, and let K be of type $\bullet$.* 0) *We have either $\psi_\bullet|\mathfrak{P}_K = 0$ or* $\sup(\psi_\bullet|\mathfrak{P}_K) = 1$.
1) *If K is countable* (which is implied when $\bullet = \star\sigma$) *then* $\sup(\psi_\bullet|\mathfrak{P}_K) = 1$. 2) *If K is uncountable* (which implies $\bullet = \tau$) *and $\varphi_t|\mathfrak{S}_t < 1\ \forall t \in K$ then $\psi_\bullet|\mathfrak{P}_K = 0$.*

Proof. 1) Fix $0 < \varepsilon < 1$, and then $0 < \varepsilon_t < 1\ \forall t \in K$ such that $\sum_{t\in K}\varepsilon_t = \varepsilon$. There exist $S_t \in \mathfrak{S}_t$ with $\varphi_t(S_t) \geqq 1 - \varepsilon_t\ \forall t \in K$. Then $P := (\Pi_{t\in K} S_t) \times X_{I\setminus K} \in \mathfrak{P}_K$ as formed in 4.2 fulfils $\psi_\bullet(P) \geqq \Pi_{t\in K}(1-\varepsilon_t) \geqq 1 - \varepsilon$. The assertion follows. 0) Assume that $\psi_\bullet|\mathfrak{P}_K$ is not $= 0$. Thus $\psi_\bullet(P) > 0$ for some $P \in \mathfrak{P}_K$. We can assume that $P = (\Pi_{t\in K} S_t) \times X_{I\setminus K}$ as formed in 4.2. 0.i) The subset $C := \{t \in K : \varphi_t(S_t) < 1\} \subset K$ must be countable. In fact, otherwise there exists $0 < \varepsilon < 1$ such that $\{t \in K : \varphi_t(S_t) \leqq \varepsilon\}$ is uncountable and in particular infinite, which implies that

$$\psi_\bullet(P) = \Pi_{t\in K}\varphi_t(S_t) = 0.$$

0.ii) Now the above proof of 1), applied to C instead of K, furnishes $\sup(\psi_\bullet|\mathfrak{P}_K) = 1$. 2) follows from the above 0.i).

Next we exhibit some properties of ψ and K which are equivalent to $\sup(\psi_\bullet|\mathfrak{P}_K) = 1$ and relevant to the present purpose.

Proposition 4.4. *Assume that the φ_t are inner $\bullet$ premeasures $\forall t \in I$, and that ψ is $\bullet$ continuous at $\varnothing$ and hence an inner $\bullet$ premeasure as well, and let K be of type $\bullet$. Then likewise $\pi_K := \psi_\bullet|\mathfrak{P}_K$ is an inner $\bullet$ premeasure. Moreover the following are equivalent.*

1) $\sup(\psi_\bullet|\mathfrak{P}_K) = 1$, *that is* $\sup\pi_K = 1$.
2) $\psi_\bullet$ *is inner regular* $(\mathfrak{P}_K)_\bullet$.
3) $(\pi_K)_\bullet = \psi_\bullet$.
4) $(\pi_K)_\bullet|\mathfrak{T} = \psi$.

Proof. i) We prove 1)$\Rightarrow$2). Fix $A \subset X$ and a real $c < \psi_\bullet(A)$. There exists a paving $\mathfrak{M} \subset \mathfrak{T}$ of type $\bullet$ with $\mathfrak{M} \downarrow\subset A$ such that $\inf\{\psi(M) : M \in \mathfrak{M}\} > c$. Fix $\varepsilon > 0$ with $\inf\{\psi(M) : M \in \mathfrak{M}\} > c + \varepsilon$. By assumption there exists $P \in \mathfrak{P}_K$ such that $\psi_\bullet(P) > 1 - \varepsilon$. Then 4.1.1) implies that $\{M \cap P : M \in \mathfrak{M}\} \subset \mathfrak{P}_K$ is a paving of type $\bullet$ with $\downarrow$ some $D \subset A$, so that $D \in (\mathfrak{P}_K)_\bullet$. For $M \in \mathfrak{M}$ we have $\psi(M) - \psi_\bullet(M \cap P) = \psi_\bullet(M \cap P') \leqq \psi_\bullet(P') < \varepsilon$, where we have used $P \in \mathfrak{T}_\bullet \subset \mathfrak{C}(\psi_\bullet)$ from 4.1.2). It follows that $\psi_\bullet(D) = \inf\{\psi_\bullet(M \cap P) : M \in \mathfrak{M}) > c$ and hence the assertion. ii) We assume 2) and prove that $\pi_K := \psi_\bullet|\mathfrak{P}_K$ is an inner $\bullet$ premeasure with 3). In fact, we see from 4.1.2) that π_K is a restriction of $\psi_\bullet|\mathfrak{T}_\bullet$ and hence of $\psi_\bullet|\mathfrak{C}(\psi_\bullet)$ with even $(\mathfrak{P}_K)_\bullet \subset \mathfrak{T}_\bullet \subset \mathfrak{C}(\psi_\bullet)$. From the assumption and MI 6.5.iii) it follows that $\psi_\bullet|\mathfrak{C}(\psi_\bullet)$ is an inner $\bullet$ extension of π_K, so that π_K is an inner $\bullet$ premeasure. Furthermore MI 6.18 implies that $(\pi_K)_\bullet = \psi_\bullet$ on $\mathfrak{T}_\bullet$ and hence on all of $\mathfrak{P}(X)$, since both sides are inner regular $\mathfrak{T}_\bullet$. iii) The implications 3)$\Rightarrow$4)$\Rightarrow$1) are obvious. iv) We have seen that the properties 1)2)3)4) are equivalent, and that in their presence the set function π_K is an inner $\bullet$ premeasure. In the opposite case 4.3.0) says that $\pi_K = 0$, so that likewise π_K is an inner $\bullet$ premeasure. This completes the proof.

We also have the related uniqueness assertion which follows.

Remark 4.5. *Assume that ψ is downward $\bullet$ continuous, and let K be of type $\bullet$. Assume that the isotone and supermodular set function $\pi : \mathfrak{P}_K \to [0,\infty[$ is downward $\bullet$ continuous and fulfils $\pi_\bullet|\mathfrak{T} = \psi$. Then $\pi = \psi_\bullet|\mathfrak{P}_K$.*

Proof. From 4.1.1) we have $\mathfrak{T} \subset \mathfrak{P}_K \top \mathfrak{P}_K$ and hence $\mathfrak{T}_\bullet \subset \mathfrak{P}_K \top (\mathfrak{P}_K)_\bullet$. Thus $\pi_\bullet|\mathfrak{T}_\bullet$ is downward $\bullet$ continuous by MI 6.7 and 6.27. By MI 6.5 therefore $\pi_\bullet = \psi_\bullet$ on $\mathfrak{T}_\bullet$. Now $\mathfrak{P}_K \subset \mathfrak{T}_\bullet$ from 4.1.2), since K is of type $\bullet$. The assertion follows.

The above results provide an answer to the present concern. Assume that the φ_t are inner $\bullet$ premeasures $\forall t \in I$, and that ψ is $\bullet$ continuous and hence an inner $\bullet$ premeasure as well. The natural idea is to ask for an inner $\bullet$ premeasure $\pi : \mathfrak{P}_I \to [0,\infty[$ which produces the basic product formation $\psi : \mathfrak{T} \to [0,\infty[$. In our frame this means that $\pi_\bullet|\mathfrak{T} = \psi$. The above 4.1.2) and its consequences recommend to assume that I be of type $\bullet$, that is finite in case $\bullet = \star$ and countable in case $\bullet = \sigma$. Then 4.5 leaves a unique candidate for π, to wit $\pi_I := \psi_\bullet|\mathfrak{P}_I$, which turns out to remain an inner $\bullet$ premeasure. But the assumptions do not suffice to ensure that $(\pi_I)_\bullet|\mathfrak{T} = \psi$; it can also happen that $\pi_I = 0$. We see from 4.4 that $(\pi_I)_\bullet|\mathfrak{T} = \psi$ is equivalent to the so-called PROKHOROV condition $\sup(\psi_\bullet|\mathfrak{P}_I) = 1$. By 4.3 the Prokhorov condition is fulfilled and hence $\pi_I : \mathfrak{P}_I \to [0,\infty[$ is the desired product formation when I is countable, but this need not be true beyond.

The results thus described are in Bourbaki [2] sections IX.4.2 and 3 in the special case that the X_t are Hausdorff topological spaces with $\mathfrak{S}_t = \mathrm{Comp}(X_t)$ and the φ_t are Radon premeasures on X_t $\forall t \in I$, and where X carries the product topology, so that $(\mathfrak{P}_I)_\tau = \mathrm{Comp}(X)$ by 2.4.2). However, this work and Schwartz [8] section I.10 treat that special case in its more comprehensive version for projective systems, a context which we decided to postpone as pointed out in the introduction. In return we perform the extension to the frame of the inner $\bullet$ extension theories of MI section 6. An additional extension is the incorporation of the nonvoid subsets $K \subset I$, to which part of the conditions can be relegated.

We conclude with the main results in separate formulations for $\bullet = \star\sigma$ and $\bullet = \tau$.

Theorem 4.6. ($\bullet = \star\sigma$). *Assume that the φ_t are inner $\bullet$ premeasures $\forall t \in I$. Then $\psi : \mathfrak{T} \to [0,\infty[$ is an inner $\bullet$ premeasure. For nonvoid $K \subset I$ of type $\bullet$ likewise $\pi_K := \psi_\bullet|\mathfrak{P}_K$ is an inner $\bullet$ premeasure, and it fulfils $(\pi_K)_\bullet = \psi_\bullet$ and hence $(\pi_K)_\bullet|\mathfrak{T} = \psi$.*

Theorem 4.7. ($\bullet = \tau$). *Assume that the φ_t are inner τ premeasures $\forall t \in I$, and that either I is countable or the $\mathfrak{T}_t$ are τ compact $\forall t \in I$. Then $\psi : \mathfrak{T} \to [0,\infty[$ is an inner τ premeasure. For nonvoid $K \subset I$ likewise $\pi_K := \psi_\tau|\mathfrak{P}_K$ is an inner τ premeasure, and it fulfils either $\pi_K = 0$ or $(\pi_K)_\tau = \psi_\tau$ and hence $(\pi_K)_\tau|\mathfrak{T} = \psi$. The latter case holds true whenever K is countable, but if K is uncountable and $\varphi_t|\mathfrak{S}_t < 1$ $\forall t \in K$ then $\pi_K = 0$.*

References

[1] N.Bourbaki, *Intégration*, Chap.1-4, 2ème Ed. Hermann, Paris, 1965.

[2] N.Bourbaki, *Intégration*, Chap.IX. Hermann, Paris, 1969.

[3] E.Hewitt and K.Stromberg, *Real and Abstract Analysis*. Springer 1965.

[4] H.König, *Measure and Integration: An Advanced Course in Basic Procedures and Applications*. Springer 1997.

[5] H.König, *Image Measures and the so-called Image Measure Catastrophe*, Positivity **1** (1997), 225-270.

[6] Ch.W.Lamb, *A Comparison of Methods for Constructing Probability Measures on Infinite Product Spaces*, Canad.Math.Bull. **30** (1987), 282-285.

[7] P.A.Meyer, *Probability and Potential*. Blaisdell 1966.

[8] L.Schwartz, *Radon Measures on Arbitrary Topological Spaces and Cylindrical Measures*. Oxford Univ.Press 1973.

[9] K.R.Stromberg, *Probability for Analysts*. Chapman & Hall 1994.

H. König
Fachbereich Mathematik
Universität des Saarlandes
D-66041 Saarbrücken
GERMANY
E-mail address: hkoenig@math.uni-sb.de