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THE QUASI-EQUIVALENCE PROBLEM FOR A CLASS OF KÖTHE SPACES

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Abstract. *We consider a subclass of the class of stable nuclear Fréchet-Köthe spaces, and show that quasi-equivalence property holds in this subclass.*

1 Introduction

Let E be a nuclear Fréchet space with basis; two bases (x_n) and (y_n) of E are said to be quasi-equivalent if there exists a permutation π of $\mathbb{N}$ and a sequence (γ_n) of positive scalars such that there is an isomorphism $T: E \to E$ with $T(x_n) = \gamma_n y_{\pi(n)}$. A nuclear Fréchet space E with basis is said to have the quasi-equivalence property if every two bases of E are quasi-equivalent. Quasi-equivalence was first studied by Dragilev [6]-[8] and also Mitiagin [13],[14]; Crone and Robinson [5] proved that any nuclear Fréchet space with a regular basis has the quasi-equivalence property. Further progress on this topic is due to many mathematicians, e.g. see [2]-[4],[7],[10]-[12],[18]-[22]. However the general problem whether every Fréchet nuclear space with basis has the quasi-equivalence property (the so called quasi-equivalence problem) remains open. In this note we prove that the quasi-equivalence property holds for a certain subclass of the class of stable nuclear Köthe spaces.

Let $A = (a_{i,p})_{i,p\in\mathbb{N}}$, $\mathbb{N} = \{1,2,\cdots\}$ be a matrix of non-negative real numbers such that $a_{i,p} \leq a_{i,p+1}$, then the Köthe space $K(A)$ is the Fréchet space of all sequences $x = (\eta_i)$ of scalars such that $\|x\|_p := \sum_{i\in\mathbb{N}} |\eta_i| a_{i,p} < \infty$ for all $p \in \mathbb{N}$, with the topology generated by the system of seminorms $\{\|\cdot\|_p : p \in \mathbb{N}\}$. The sequence $\{e_i\}_{i\in\mathbb{N}}$ where $e_i = (\delta_{i,j})_j$ is an absolute basis (and it is called the canonical basis) of $K(A)$.

Let $K(A)$, $K(B)$ be two Köthe spaces with canonical bases (e_i) and (f_i) respectively. A linear operator $T: K(A) \to K(B)$ is said to be quasi-diagonal (qd) if there exists a function $\sigma: \mathbb{N} \to \mathbb{N}$ and a sequence of scalars (γ_i) such that $T(e_i) = \gamma_i f_{\sigma(i)}$. We write $X \overset{qd}{\hookrightarrow} Y$ if there is a quasi-diagonal embedding $T: X \to Y$; if T is an isomorphism we say that X and Y are quasi-diagonally isomorphic. With this terminology, the quasi-equivalence problem can be stated as follows: Are isomorphic nuclear Köthe spaces quasi-diagonally isomorphic?

The following known result is very useful in this context, see [4, Proposition 3] or [17, Lemma 1.1].

Lemma 1 *Let X and Y be Köthe spaces. If $X \overset{qd}{\hookrightarrow} Y$ and $Y \overset{qd}{\hookrightarrow} X$, then X and Y are quasi-diagonally isomorphic.*

Recall that the Köthe space $K(A)$ is nuclear if and only if the so-called Grothendieck-Pietsch criterion holds, i.e. for each $p \in \mathbb{N}$ there is $q \in \mathbb{N}$ such that $\sum_i \frac{a_{i,p}}{a_{i,q}} < \infty$. In this case the topology of $K(A)$ can also be defined by the equivalent system of seminorms $\|x\|_p = \sup_i |\eta_i| a_{i,p}$.

2 Linear Topological Invariants (LTI)

Linear topological invariants (such as approximative and diametral dimensions) have been used for isomorphic classification of non-normed linear topological spaces by Pełczyński [15], Kolmogorov [10], Bessaga, Pełczynski and Rolewicz [1], Mitiagin [13] et al. In this work we consider linear topological invariants introduced by Zahariuta [20],[16]. See also [17] for an extensive consideration of these invariants.

Let X be a linear space and let U, V be absolutely convex sets in X. Then

$$\beta(V,U) = \sup_L \{\dim L : L \cap U \subset V\}$$

where the supremum is taken over all finite dimensional subspaces L of X.

It is clear from this definition that if $V_1 \subset V_2$ and $U_1 \supset U_2$ then $\beta(V_1,U_1) \leq \beta(V_2,U_2)$, and if T is an isomorphism then $\beta(T(V),T(U)) = \beta(V,U)$.

Let X be a sequence space of sequences $x = (x_i)$, $x_i \in \mathbb{C}$ with the following property: $x = (x_i) \in X$, $\forall i\, |y_i| \leq |x_i| \Rightarrow y = (y_i) \in X$. Let A be the set of all sequences such that for all $a = (a_i) \in A$, $0 \leq a_i \leq \infty$. We define

$$B(a) = B(a_i) = \{x = (x_i) \in X : \sum_{i=1}^{\infty} |x_i| a_i \leq 1\},$$
$$\tilde{B}(a) = \tilde{B}(a_i) = \{x = (x_i) \in X : \sup_i |x_i| a_i \leq 1\}.$$

As a convention we assume $0\infty = 0$ and $x\infty = \infty$ if $0 < x < \infty$. According to this convention, if $a_i = \infty$ for some i and $x \in B(a)$ or $\tilde{B}(a)$, then $x_i = 0$.

We have a suitable characterization of the function $\beta(\tilde{B}(b), B(a))$.

Lemma 2 *Let X be a sequence space with some locally convex topology for which the sequence of unit vectors (e_i) is an unconditional basis. Let $b = (b_i)$, $a = (a_i)$ be such that $0 < b_i \leq \infty$ and $0 \leq a_i < \infty$ for each $i \in \mathbb{N}$. Then*

$$|\{i : b_i \leq a_i\}| \leq \beta(\tilde{B}(b), B(a))$$

and if $B(a)$ is absorbent in X, then

$$\beta(\tilde{B}(b), B(a)) \leq |\{i : b_i \leq 2(\pi(i))^2 a_i\}|.$$

for any permutation $\pi : \mathbb{N} \to \mathbb{N}$.

Proof. Let $I = \{i : b_i < \infty\}$, $J = \{i : 0 < a_i\}$.

Let $N_1 = \{i : b_i \leq a_i\}$. Then $N_1 \subset I \cap J$, and hence $N_1 \subset N_1 \cap (I \cap J)$. Let $L_1 = \operatorname{span}\{e_i : i \in N_1\}$ where $(e_i)_{i\in\mathbb{N}}$ is the canonical basis of X. We want to show that $L_1 \cap B(a) \subset \tilde{B}(b)$ from which it follows that $|N_1| = \dim L_1 \le \beta(\tilde{B}(b), B(a))$. Now let $x = (x_i) \in L_1 \cap B(a)$. Then $x_i = 0$ if $i \notin N_1$ and $\sum_{i \in N_1} |x_i| a_i \le 1$. So, $\sup_{i \in \mathbb{N}} |x_i| b_i = \sup_{i \in N_1} |x_i| b_i \le \sup_{i \in N_1} |x_i| a_i \le \sum_{i \in N_1} |x_i| a_i \le 1$ i.e. $x \in \tilde{B}(b)$

To prove the second claim given any π define $N_2 = \{i : b_i \le (\pi(i))^2 a_i\}, L_2 = \operatorname{span}\{e_i : i \in N_2\}$. Let L be any finite dimensional subspace of X such that $L \cap B(a) \subset \tilde{B}(b)$. We will show that $\dim L \le \dim L_2 = |N_2|$. For this purpose it is enough to show that the restriction to L of the natural projection $P : X \to L_2$, $Px = \sum_{i \in N_2} x_i e_i$ is an injection.

Assume not. Then there is $y = (y_i) \in L$, $y \ne 0$ such that $Py = 0$. Thus we obtain $y_i = 0$ for all $i \in N_2$.

Since $B(a)$ is absorbent, for some $C \ne 0$, $Cy \in B(a)$ thus $Cy \in L \cap B(a) \subset \tilde{B}(b)$. So for all i, $|C||y_i|b_i \le 1$. This means that if $i \notin I$ (i.e. $b_i = \infty$) then $y_i = 0$. So $y_i \ne 0 \Rightarrow i \in (N_2 \cup I')' = N_2' \cap I$ (I' denotes the complement of the set I) and

$$|y|_{B(a)} = \sum_{i \in N_2' \cap I} |y_i| a_i \le \sum_{i \in N_2' \cap I} |y_i| \frac{b_i}{2(\pi(i))^2}$$

$$\le \Big(\sup_{i \in N_2' \cap I} |y_i| b_i\Big) \sum_{i \in N_2' \cap I} \frac{1}{2(\pi(i))^2} \le \frac{\pi^2}{12} |y|_{\tilde{B}(b)} < |y|_{\tilde{B}(b)}.$$

But $L \cap B(a) \subset \tilde{B}(b)$ is equivalent to $|y|_{\tilde{B}(b)} \le |y|_{B(a)}$ for any $y \in L$ which is a contradiction. □

Lemma 3 *Let X be a sequence space with some locally convex topology. Let $\tilde{V} = \cap_{n=1}^{\infty} \tilde{B}(b^n)$, $U = \overline{conv}(\cup_{n=1}^{\infty} B(a^n))$ where $0 < b_i^n < \infty$ and $0 < a_i^n < \infty$ for all n. Let $b_i = \sup_n b_i^n$, $a_i = \inf_n a_i^n$. Then*

$$\tilde{V} = \tilde{B}(b), \quad B(a) \subset 2U,$$

and if for some m, $B(a^m)$ is a zero neighbourhood, then $U \subset 2B(a)$.
(It could happen that $b_i = \infty$ or $a_i = 0$ for some index i.)

Proof. The proof of the equality $\tilde{V} = \tilde{B}(b)$ is trivial. For the second claim, let $I = \{i : 0 < a_i\}$, $J = \{i : a_i = 0\}$.

We show that $B(a) \subset 2U$. Fix i. We have $\frac{1}{a_i^n} e_i \in B(a^n) \subset U$ for all n. Thus, if $i \in I$, $\frac{1}{a_i} e_i \in U$ and if $i \in J$ then $\alpha e_i \in U$ for all $\alpha \in \mathbb{C}$. Now given $x = (x_i) \in B(a)$ we have

$$x = \sum_{i \in I} x_i a_i \frac{1}{a_i} e_i + \sum_{i \in J} \frac{1}{2^i} 2^i x_i e_i.$$

Since $\frac{1}{a_i} e_i \in U$ for all $i \in I$ and $\sum_{i \in I} |x_i| a_i \le 1$, we have that $\sum_{i \in I} x_i a_i \frac{1}{a_i} e_i \in U$. Similarly since $2^i x_i e_i \in U$ for all $i \in J$ and $\sum_{i \in J} \frac{1}{2^i} \le 1$, we have $\sum_{i \in J} \frac{1}{2^i} 2^i x_i e_i \in U$. So $x \in 2U$.

Finally we assume $B(a^m)$ is a zero neighbourhood. Clearly we have $B(a^n) \subset B(a)$ for all n and hence $conv(\cup_{n=1}^{\infty} B(a^n)) \subset B(a)$. Then $U = \overline{conv}(\cup_{n=1}^{\infty} B(a_n)) \subset \overline{conv}(\cup_{n=1}^{\infty} B(a^n)) + B(a^m) \subset 2B(a)$. □

3 Class C_1

Let C_1 be the class of all nuclear Köthe spaces $K(d_{i,p})$ with either **I** or **II** where
I $d_{i,p} \leq d_{i+1,p}$ for all $i,p \in \mathbb{N}$ and $\forall\, p\, \exists q, P:\ d_{2i,p} \leq P d_{i,q}$
II $d_{i+1,p} \leq d_{i,p}$ for all $i,p \in \mathbb{N}$ and $\forall\, p\, \exists q, P:\ d_{i,p} \leq P d_{2i,q}$.
Observe that If $X \in C_1$ then $X \simeq X^2$.

Theorem 4 *If $X = K(a_{i,p})$, $Y = K(b_{i,p})$ are isomorphic spaces from the class C_1 of the same type, then X is quasi-diagonally isomorphic to Y.*

To prove this, we are going to use linear topological invariants and the following Hall-Koenig Theorem:

Hall-Koenig Theorem. (see [9], Ch. 5) *Let $\mathcal{M}$, $\mathcal{N}$ be two sets and let $S: \mathcal{M} \to 2^{\mathcal{N}}$ be a map which assigns a finite set $S(m) \subset \mathcal{N}$ to each $m \in \mathcal{M}$. There exists an injection $\varphi: \mathcal{M} \to \mathcal{N}$ such that $\varphi(m) \in S(m)$ for all $m \in \mathcal{M}$ if and only if for all finite subsets $A \subset \mathcal{M}$ we have $|A| \leq |\cup_{a\in A} S(a)|$.*
Proof. [Proof of Theorem 4] We will give the proof when both spaces are of type **I** since the other case can be proved analogously. Let T be the isomorphism from X to Y. Let $\{\tilde{B}(a_{i,p})\}_{p\in\mathbb{N}}, \{B(a_{i,p})\}_{p\in\mathbb{N}}$ be the families of weighted l_∞ and l_1 balls in X respectively and let $\{\tilde{B}(b_{i,p})\}_{p\in\mathbb{N}}, \{B(b_{i,p})\}_{p\in\mathbb{N}}$ be similarly defined in Y.

If necessary, by passing to a subsequence of balls and multiplying the balls by scalars, without loss of generality we assume that

$$\forall p \quad T^{-1}(B(b_{i,p+1})) \subset B(a_{i,p}) \tag{1}$$
$$\forall q \quad T(B(a_{i,q+1})) \subset B(b_{i,q}) \tag{2}$$
$$\forall p \quad T^{-1}(\tilde{B}(b_{i,p+1})) \subset \tilde{B}(a_{i,p}) \tag{3}$$
$$\forall q \quad T(\tilde{B}(a_{i,q+1})) \subset \tilde{B}(b_{i,q}) \tag{4}$$
$$\forall p,i \in \mathbb{N} \quad a_{i,p} \leq a_{i+1,p}, \tag{5}$$
$$\forall q,i \in \mathbb{N} \quad b_{i,q} \leq b_{i+1,q}, \tag{6}$$
$$\forall p,i \in \mathbb{N} \quad a_{2i,p} \leq a_{i,p+1} \tag{7}$$
$$\forall q,i \in \mathbb{N} \quad b_{2i,q} \leq b_{i,q+1} \tag{8}$$

Also, by nuclearity (see [2]) there exists a permutation π of $\mathbb{N}$ such that

$$\forall p,i \quad 8i^2 b_{\pi(i),p} \leq b_{\pi(i),p+1}. \tag{9}$$

We will apply the Hall-Koenig theorem to the multivalued map $S: \mathbb{N} \to 2^{\mathbb{N}}$ defined by

$$S(n) = \mathcal{B}_n = \bigcap_{p,q} \{i : \frac{b_{i,p}}{(\pi^{-1}(i))^2 b_{i,q+2}} \leq 8\frac{a_{2n,p+1}}{a_{2n,q}}\}.$$

We also define

$$\mathcal{A}_n = \bigcap_{p,q} \{i : \frac{a_{i,p}}{a_{2i,q}} \leq \frac{a_{2n,p}}{a_{2n,q}}\}.$$

For any nondecreasing sequence of scalars (t_p), by (4), we have

$$\bigcap_p t_p T(\tilde{B}(a_{i,p})) \subseteq \bigcap_p t_{p+1} T(\tilde{B}(a_{i,p+1})) \subseteq \bigcap_p t_{p+1} \tilde{B}(b_{i,p})$$

and so

$$T(\bigcap_p \tilde{B}(\frac{a_{i,p}}{t_p})) \subset \bigcap_p \tilde{B}(\frac{b_{i,p}}{t_{p+1}}). \tag{10}$$

Let $n \in \mathbb{N}$ be fixed, and define $t_p = a_{2n,p}$,

$$a_{i,\infty} = \sup_p \left\{ \frac{a_{i,p}}{t_p} \right\}, \quad b_{i,\infty} = \sup_p \left\{ \frac{b_{i,p}}{t_{p+1}} \right\},$$

Then by Lemma 3 we obtain

$$\tilde{B}(a_{i,\infty}) = \bigcap_p \tilde{B}(\frac{a_{i,p}}{t_p}), \quad \tilde{B}(b_{i,\infty}) = \bigcap_p \tilde{B}(\frac{b_{i,p}}{t_{p+1}})$$

and hence (10) yields

$$T(\tilde{B}(a_{i,\infty})) \subset \tilde{B}(b_{i,\infty}). \tag{11}$$

On the other hand, by (3) and (7), for any sequence (τ_p) of scalars we have

$$\bigcup_p \frac{1}{\tau_{p+2}} T^{-1}(B(b_{i,p+2})) \subset \bigcup_p \frac{1}{\tau_{p+2}} B(a_{2i,p})$$
$$\Rightarrow T^{-1}(\bigcup_p B(\tau_{p+2} b_{i,p+2})) \subset \bigcup_p B(\tau_{p+2} a_{2i,p}). \tag{12}$$

Define now

$$b_{i,0} := \inf_p \{\tau_{p+2} b_{i,p+2}\}, \; a_{i,0} := \inf_p \{\tau_{p+2} a_{2i,p}\}$$

where $\tau_{p+2} = \frac{1}{a_{2n,p}}$ and n is the same fixed number used in the definition of t_p above. Then by Lemma 3 we obtain

$$\overline{conv}(\bigcup_p B(\tau_{p+2} a_{2i,p})) \subset 2B(a_{i,0}), \; \frac{1}{2} B(b_{i,0}) \subset \overline{conv}(\bigcup_p B(\tau_{p+2} b_{i,p+2}))$$

and hence (12) gives that

$$\frac{1}{2} B(b_{i,0}) \subset 2T(B(a_{i,0})). \tag{13}$$

Thus (11) and (13) allow us to write

$$\beta(\tilde{B}(a_{i,\infty}), B(a_{i,0})) \leq \beta(\tilde{B}(b_{i,\infty}), \frac{1}{4} B(b_{i,0})) = \beta(\tilde{B}(b_{i,\infty}), B(4b_{i,0}))$$

and therefore by Lemma 2 and the definitions of $a_{i,0}, a_{i,\infty}, b_{i,0}, b_{i,\infty}$ we obtain

$$|\mathcal{A}_n| \leq |\mathcal{B}_n|. \tag{14}$$

If i is integer such that $n \leq i \leq 2n$, then for any p and q, we have $\frac{a_{i,p}}{a_{2n,p}} \leq 1 \leq \frac{a_{2i,q}}{a_{2n,q}}$ which means, that the integers $n, n+1, \cdots, 2n$ are in $\mathcal{A}_n$.

Let $K \subset \mathbb{N}$ be an arbitrary finite set. Choose $n = \max\{k : k \in K\}$. Observe that $n \geq |K|$. So,

$$|K| \leq n \leq |\mathcal{A}_n| \leq |\mathcal{B}_n| \leq |\cup_{k\in K} \mathcal{B}_k|.$$

Thus by the Hall-Koenig Theorem, there is an injection $\varphi : \mathbb{N} \to \mathbb{N}$ such that $\varphi(n) \in \mathcal{B}_n$.

$$\frac{b_{\varphi(n),p}}{(\pi^{-1}(\varphi(n)))^2 b_{\varphi(n),q+2}} \leq 8\frac{a_{2n,p+1}}{a_{2n,q}}$$

which implies

$$\frac{b_{\varphi(n),p}}{a_{n,p+2}} \leq \frac{b_{\varphi(n),q+3}}{a_{n,q}}$$

for all p, q. Then choosing λ_n such that

$$\sup_p \left\{ \frac{b_{\varphi(n),p}}{a_{n,p+2}} \right\} \leq \lambda_n \leq \inf_q \left\{ \frac{b_{\varphi(n),q+3}}{a_{n,q}} \right\}$$

gives a quasi-diagonal embedding of X into Y.

A symmetrical argument gives a quasi-diagonal embedding of Y into X. Now the result follows from Lemma 1. □

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References

[1] C. Bessaga, A. Pełczyński, S. Rolewicz, *On diametral approximative dimension and linear homogeneity of F-spaces*, Bull. Acad. Pol. Sci. **9** (1961), 677-683.

[2] C. Bessaga, *Some remarks on Dragilev's theorem*, Studia Math. **31** (1968), 307-318.

[3] P.A. Chalov, T. Terzioğlu, V.P. Zahariuta, *First type power Köthe spaces and m-rectangular invariants*, to appear.

[4] P.A. Chalov, V.P. Zahariuta, *On quasi-diagonal isomorphisms of generalized power spaces*, Linear Topological Spaces and Complex Analysis 2, METU-TÜBİTAK, Ankara, (1995), 35-44.

[5] L. Crone, W. Robinson, *Every nuclear Fréchet space with a regular basis has the quasi-equivalence property*, Studia Math. **52** (1975), 203-207.

[6] M.M. Dragilev, *Canonical forms of bases of spaces of analytic functions*, Uspehi Mat. Nauk **15** (2) (1960), 181-188 (Russian).

[7] M.M. Dragilev, *On special dimensions defined on some classes of Köthe spaces*, Math. USSR Sbornik **9** (2) (1969), 213-228.

[8] M.M. Dragilev, *On regular bases in nuclear spaces*, Matem. Sbornik **68** (1965), 153-173 (Russian); Engl. transl., Amer. Math. Soc. Transl. **93** (1970), 61-82.

[9] M. Hall, *Combinatorics*, 1970

[10] A.N. Kolmogorov, *On the linear dimension of topological vector spaces*, Dokl. Akad. Nauk. SSSR **120** (1958), 239-341 (Russian).

[11] V.P. Kondakov, *On properties of bases in some Köthe spaces and their subspaces*, Funk. analiz i ego prilozh. **14** (1979), 58-59 (Russian).

[12] V.P. Kondakov, *On orderable absolute basis in F-spaces*, Sov. Math. Dokl. **247** (1979), 543-546.

[13] B.S. Mitiagin, *Approximative dimension and bases in nuclear spaces*, Russian Math. Surveys **16** (4) (1961), 59-127.

[14] B.S. Mitiagin, *Equivalence of bases in Hilbert scales*, Studia Math. **37** (1971), 111-137 (Russian).

[15] A. Pełczyński, *On the approximation of S-spaces by finite dimensional spaces*, Math. Z. **182** (1983), 303-310.

[16] V.P. Zahariuta, *Generalized Mitiagin invariants and continuum of pairwise nonisomorphic spaces of analytic functions*, Funk. analiz i ego prilozh. **11** (1977), 24-30 (Russian).

[17] V.P. Zahariuta, *Linear topological invariants and their applications to isomorphic classification of generalized power spaces*, Turkish J. Math. **20** (1996), 237-289.

[18] V.P. Zahariuta, P.A. Chalov, *A criterion for quasiequivalence of absolute bases in an arbitrary Fréchet space*, Izv. Severo-Kavkaz Nauchn Tsentra Vyssh Shkoly Estestv. Nauk SSSR **2** (1983), 22-24 (Russian).

[19] V.P. Zahariuta, *The quasi-equivalence of bases in finite centers of Hilbert scales*, Dokl. Akad. Nauk. SSSR **180** (1968), 786-788 (Russian).

[20] V.P. Zahariuta, *Linear Topological invariants, and the isomorphism of spaces of analytic functions*, Mathematical analysis and its applications, Vol II, pp. 3-13, Izdat. Rostov. Univ., Rostov-on-Don, 1970 (Russian).

[21] V.P. Zahariuta, *On the isomorphism of cartesian products of locally convex spaces*, Studia Math. **37** (1973), 201-221.

[22] V.P. Zahariuta, *On isomorphisms and quasi-equivalence of bases of power Köthe spaces*, Proceedings of 7th winter school in Drogobych, CEMI, Moscow, 101-126, 1976 (Russian).

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