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# INVARIANT $m$-CHARACTERISTICS FOR KÖTHE SPACES[1]

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**Abstract.** *New linear topological invariants for Köthe spaces are constructed.*

## 1 Introduction

The aim of this paper is to present some new results on linear topological invariants for Köthe spaces. The first non-trivial linear topological invariant in the context of Fréchet spaces was obtained by Kolmogorov [21] and Pełczyński [29] by constructing so called *approximative dimension*. A little later the approximative dimension, together with the *diametral dimension*, which was introduced by Bessaga, Pełczyński and Rolewicz [2], become, due to Mityagin [25], one of the major tools in the theory of nuclear Fréchet spaces.

The later development of the theory of linear topological invariants was mainly inspired by problems on isomorphic classification and structure of non-Schwartzian and non-regular Köthe spaces. Mityagin [26, 27, 28] obtained a new invariant that characterizes isomorphisms in the class of (non-Schwartzian) power series spaces and got important results on quasiequivalence of bases in these spaces. The developement of this approach for general Köthe spaces (see [30, 31, 32, 33]) brought to the technique of generalized linear topological invariants for Köthe spaces. Later Zahariuta [34] suggested a geometrical approach that makes the construction of generalized linear topological invariants much easier and, in the same time, obviously topological invariant. This approach was used in many papers to study isomorphic classification of different classes of Fréchet spaces (e.g. [19, 15, 16, 13, 18]).

Recently Chalov, Terzioğlu and Zahariuta [5], [6] proved the invariance of so called "$m$-rectangular characteristics" for mixed $F,DF$-spaces and for first type power Köthe spaces, answering by this positively Question 14 in [1]. Moreover, they showed by examples that the $(m+1)$-rectangle characteristic is stronger than the $m$-rectangle characteristic. Roughly speaking, each space that was considered in these papers is determined by two sequences $(\lambda_i)$ and $(a_i)$ ; for given $m$-tuple of rectangles $(\Delta_1,\dots,\Delta_m)$ in the $(\lambda,a)$-plane the corresponding $m$-rectangle characteristic is defined as a number of indices $i$ such that $(\lambda_i,a_i)\in\Delta_k$ for any $k=1,\dots,m$. Analogous $m$-rectangle characteristics were considered earlier by Chalov and Zahariuta for families of Hilbert and Banach spaces (see [4, 8, 9, 10] ).

Our aim here is to introduce and prove the invariance of an analog of $m$-rectangular characteristics: $m$-characteristics for general Köthe spaces. Let us notice that these characteristics are not a generalization of the $m$-rectangular characteristics considered in [5] and [6], because, if computed for the spaces considered in these papers they give a weaker result: $m$-rectangular characteristic with bounds on the parameters of rectangles. However the invariants we consider ($m$-characteristics for Köthe spaces) give a general point of view on

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problems of isomorphic classification.

## 2 Preliminaries

Recall that a matrix of real numbers $(a_{ip})$, $i \in I$, $p \in \mathbb{N}$, such that $0 \leq a_{ip} \leq a_{i,p+1}$ is called "Köthe matrix". Each Köthe matrix $(a_{ip})$, $i \in I$, $p \in \mathbb{N}$, determined a sequence space $K(a_{ip})$ : this is the space of all sequences $x = (x_i)$ of scalars such that $|x|_p := \sum_{i \in I} |x_i| a_{ip} < \infty$, $\forall p \in \mathbb{N}$. Regarded with the topology generated by the system of seminorms $\{|\cdot|_p : p \in \mathbb{N}\}$ it is a Fréchet space. A sequence Fréchet space $X$ is called *Köthe space* if $X = K(a_{ip})$ for some Köthe matrix $(a_{ip})$.

Further for any set $B$ we denote by $|B|$ the number of elements in $B$ if it is finite and the symbol $\infty$ if $B$ is infinite.

Suppose $X = K(a_{ip},\ i \in I)$ and $Y = K(b_{jp},\ j \in J)$ are Köthe spaces. An operator $T : X \to Y$ is *quasi-diagonal* if there exist a function $\varphi : I \to J$ and scalars $r_i$, $i \in I$ such that

$$Te_i = r_i \tilde{e}_{\varphi(i)}, \ \ i \in I,$$

where $(e_i)$ and $(\tilde{e}_j)$ are the canonical bases in the spaces $X$ and $Y$.

The next statement is well known (see, for example [35]).

**Lemma 1** *If $X$ and $Y$ are Köthe spaces such that there exist a quasi-diagonal isomorphic embedding of $X$ into $Y$, and a quasi-diagonal isomorphic embedding of $Y$ into $X$, then there is a quasi-diagonal isomorphism between $X$ and $Y$.*

Let $X, (|.|_p)$ and $Y, (||.||_p)$ be Fréchet spaces, and let let $(x_i)$ and $(y_i)$ be, respectively, some sequences of elements of $X$ and $Y$. Then:

- the sequences $(x_i)$ and $(y_i)$ are *equivalent* if

$$\forall p \, \exists q, C \qquad |x_i|_p \leq C ||y_i||_q,$$

$$\forall p \, \exists q, C \qquad ||y_i||_p \leq C |x_i|_q;$$

- the sequences $(x_i)$ and $(y_i)$ are *quasi-equivalent* if there exist a bijection $\sigma(i) : \mathbb{N} \to \mathbb{N}$ and constants $r_i > 0$ such that the sequences $(x_i)$ and $(r_i y_{\sigma(i)})$ are equivalent;
- $(x_i)$ is subordinated to $(y_i)$ if there exist a mapping $\sigma(i)$ and numbers $r_i$ such that the sequences $(x_i)$ and $(r_i y_{\sigma(i)})$ are equivalent;
- the sequences $(x_i)$ and $(y_i)$ are *weakly quasi-equivalent* if $(x_i)$ is subordinated to $(y_i)$ and $(y_i)$ is subordinated to $(x_i)$.

In general it is an open problem whether any two bases in a nuclear Fréchet space are quasi-equivalent. Crone and Robinson [12] and Kondakov [22] proved (see also [14]) that for any Fréchet space with regular absolute basis the answer is positive - any two absolute bases in such a space are quasi-equivalent. Dragilev [17] showed that the notion of weak quasi-equivalence is useful for attacking the quasi-equivalence problem and proved that any two bases in a nuclear Fréchet space are weakly quasi-equivalent.

Generalizing the results of Dragilev [17] for weak quasi-equivalence in nuclear spaces (see also Bessaga [3]) Kondakov and Zahariuta [24], (see also [23]) obtained the following result.

**Proposition 2** *If $X$ is a Köthe space then any absolute basis in a complemented subspace of $X$ is subordinated to a subsequence of the canonical basis of $X$.*

The following statement is well known (see, for example [20], Theorem 5.1.2).

**Hall-König theorem.** *Suppose $\Phi : I \to J$ is a multivalued mapping such that for any $i \in I$ $\Phi(i)$ is a finite subset of $J$. Then there exists an injective function $\varphi : I \to J$ satisfying $\varphi(i) \in \Phi(i)$ for any $i \in I$ if and only if the following condition holds:*

$$\forall M \subset I \qquad |M| \le \left| \bigcup_{i \in M} \Phi(i) \right|. \tag{1}$$

Trivial examples show that the condition $|\Phi(i)| < \infty$ $\forall i$ is essential. However we have the following proposition ([7], see also [27]).

**Proposition 3** *If $I$ is a countable set and $\Phi : I \to J$ is a multivalued mapping satisfying (1) then there exists an injective function $\varphi : I \to J$ such that*

$$\forall i \in I \qquad \varphi(i) \in \tilde{\Phi}(i) := \bigcup_{\Phi(i') \cap \Phi(i) \neq \emptyset} \Phi(i').$$

**Proof.** Consider the sets

$$I' = \{i : \Phi(i') \cap \Phi(i) \neq \emptyset \Rightarrow |\Phi(i')| < \infty\}, \quad J' = \bigcup_{i \in I'} \Phi(i).$$

By Hall-König theorem there exists a bijective function $\varphi : I' \to J'$ such that $\varphi(i) \in \Phi(i)$ for any $i \in I'$. We extend the function $\varphi$ on the set $I \setminus I' = \{i_1, i_2, \ldots, i_k, \ldots\}$ using that the set $\tilde{\Phi}(i_k) \setminus J'$ is infinite by the constuction of $J'$. Namely, we choose $\varphi(i_1) \in \tilde{\Phi}(i_1) \setminus J'$, then we choose $\varphi(i_2) \in \tilde{\Phi}(i_2) \setminus \left(J' \cup \{\varphi(i_1)\}\right)$ and so on by induction with respect to $k$. □

## 3 Characteristic β

Suppose $E$ is a linear space, $U$ and $V$ are absolutely convex sets in $E$ and $E_V$ is the set of all finite-dimensional subspaces of $E$ that are spanned on elements of $V$. We set

$$\beta(V,U) = \sup\{\dim L : \ L \in E_V, \ \ L \cap U \subset V\}.$$

It is obvious that

$$\tilde{V} \subset V, \ \ U \subset \tilde{U} \Rightarrow \beta(\tilde{V}, \tilde{U}) \le \beta(V,U)$$

and if $T$ is an injective linear operator defined on $E$ then $\beta(T(V), T(U)) = \beta(V,U)$.

Let $E$ be a Köthe space and $A$ be the set of all sequences with positive terms. We set for any $a,b \in A$

$$a \wedge b = (\min(a_i, b_i)), \ \ a \vee b = (\max(a_i, b_i)).$$

We set also for any $x = (x_i) \in E$ and $a \in A$

$$||x||_a = \sum_i |x_i| a_i, \quad B_a = \{x \in E \ : \ ||x||_a \leq 1\}.$$

It is easy to see that

$$B_{a \vee b} \subset B_a \cap B_b \subset 2B_{a \vee b}, \quad B_{a \wedge b} = \overline{\mathrm{conv}}(B_a \cup B_b).$$

The following lemma generalizes these relations. For convenience we put hereafter $a_{ip} = 0$ for $p \leq 0$.

**Lemma 4** *Let $(a_{ip})$ be a Köthe matrix such that $2^p a_{ip} \leq a_{i,p+1}$ and $U_p = \{x \in K(a_{ip}) \ : \ \sum_i |x_i| a_{ip} \ \leq 1\}$. Then for any sequence of positive numbers $t = (t_p)$ we have*

$$B_c \ \subset \ \bigcap_p t_p U_p \ \subset \ B_d, \qquad \overline{conv}(\bigcup t_p U_p) \ = \ B_g,$$

*where*

$$c = (c_i), \, c_i = \sup_p \frac{a_{ip}}{t_p}, \quad d = (d_i), \, d_i = \sup_p \frac{a_{i,p-1}}{t_p}, \quad g = (g_i), \, g_i = \inf_p \frac{a_{ip}}{t_p}.$$

**Proof.** It is obvious that $B_c \subset \bigcap t_p U_p$. If $x \in \bigcap t_p U_p$ then $|x|_p \leq t_p \ \ \forall p$, so we have

$$|x|_d = \sum_i |x_i| \sup_p \frac{a_{i,p-1}}{t_p} \leq \sum_i |x_i| \sum_p \frac{a_{i,p-1}}{t_p}$$

$$\leq \sum_p \frac{|x|_{p-1}}{|x|_p} \ \leq \ \sum_p \frac{1}{2^p} \ = \ 1.$$

Hence $\bigcap t_p U_p \subset B_d$.

Set $W = \overline{\mathrm{conv}}(\bigcup t_p U_p)$. Since obviously $W \subset B_g$ we need to prove only that $B_g \subset W$. Consider the sets of indeces $I' = \{i : \ g_i = 0\}$ and $I'' = \{i : \ g_i \neq 0\}$. Set $E' = [e_i : \ i \in I']$ and $E'' = [e_i : \ i \in I'']$, where the square brackets denote the closed linear span of the corresponding vectors; then $E = E' \oplus E''$ and $E' \subset B_g$.

It is easy to see that $W \supset E'$. Indeed, if $i \in I'$ then for any constant $c$ there exists $p_i$ such that $|c| a_{ip_i} \leq t_{p_i}$, so $ce_i \in W$, thus $[e_i] \subset W$. From here it follows immediately that $E' \subset W$.

Take any $x \in B_g$ and write it in the form $x = x' + x''$ where $x' \in E'$ and $x'' \in E''$. If $\rho > 1$ then for any $i \in I''$ there exists $p_i$ such that $a_{ip_i}/t_{p_i} \leq \rho g_i$, thus

$$\sum_{I''} \frac{a_{ip_i}}{t_{p_i}} \frac{|x_i|}{\rho} \ \leq \ \sum g_i |x_i| \leq 1.$$

Setting $\alpha_i = \frac{a_{ip_i}}{t_{p_i}} \frac{|x_i|}{\rho}$ for $i \in I''$ we have

$$\sum_{I''} \alpha_i \leq 1, \quad \frac{1}{\rho} x'' = \sum_{I''} \alpha_i v_i \quad \text{with} \quad v_i = \frac{1}{\alpha_i} \frac{x_i}{\rho} e_i \in t_{p_i} U_{p_i},$$

so $\frac{1}{\rho} x'' \in W$ for any $\rho > 1$, hence $x'' \in W$. Now for any $\varepsilon \in (0,1)$ we have that $x_\varepsilon = \varepsilon(\frac{1}{\varepsilon} x') + (1-\varepsilon) x'' \in W$, hence $x \in W$.

In order to compute $\beta(,)$ we use the following well known lemma (see for example [15]).

**Lemma 5** *If $a, b \in A$ then*

$$\beta(B_a, B_b) = |\{i : a_i / b_i \leq 1\}|.$$

□

## 4 Criterion for quasidiagonal embeddings

Here we consider a modification of the criterion for quasidiagonal isomorphism of Köthe spaces obtained in [7], see also [11]. This criterion was a starting point for us in order to define the m-characteristics for Köthe spaces. Since every Köthe space is isomorphic to some Köthe space $K(a_{ip})$ such that

$$2^p a_{ip} \leq a_{i,p+1}, \tag{2}$$

we may consider without loss of generality only Köthe spaces which matrices satisfy (2).

**Theorem 6** *Let $K(a_{ip})$ and $K(b_{jp})$ be Köthe spaces such that the corresponding matrices satisfy (2). Then the following conditions are equivalent:*

*(i) there is a quasidiagonal embedding of $K(a_{ip})$ into $K(b_{jp})$;*

*(ii) for a subsequence with respect to $p$ there exists $\ell \in \mathbb{N}$ such that for any $m \in \mathbb{N}$ and for any choice of positive numbers $t_p^k$ and $s_p^k$, $k = 1, \dots m$, such that*

$$s_p^k \leq t_p^k \leq s_{p+1}^k, \quad \forall p \tag{3}$$

$$\left| \bigcup_{k=1}^{m} \bigcap_{p,q} \left\{ i : \frac{a_{ip}}{t_p^k} \leq \frac{a_{iq}}{s_q^k} \right\} \right| \leq \left| \bigcup_{k=1}^{m} \bigcap_{p,q} \left\{ j : \frac{b_{j,p-\ell}}{t_p^k} \leq \frac{b_{j,q+\ell}}{s_q^k} \right\} \right|. \tag{4}$$

**Proof.** Suppose $T : K(a_{ip}) \to K(b_{jp})$ is an isomorphic embedding given by the formula

$$Te_i = r_i e_{j(i)}, \quad i \in \mathbb{N},$$

where $j(i) : \mathbb{N} \to \mathbb{N}$ is an injection and $r_i$, $i \in \mathbb{N}$ are positive numbers. Passing, if necessary, to a subsequence with respect to $p$ we obtain (using that $T$ is an isomorphic embedding and (2) holds) that

$$r_i b_{j(i),p-1} \leq a_{ip} \leq r_i b_{j(i),p+1} \quad \forall i \in \mathbb{N}.$$

Hence for any $t_p^k, s_p^k, k = 1, \dots m$, we have (ii) with $\ell = 1$.

Now we show that $(ii) \Rightarrow (i)$. Consider the multivalued mapping

$$\Phi(i) = \bigcap_{p,q} \left\{ j : \frac{b_{j,p-\ell}}{a_{ip}} \leq \frac{b_{j,q+\ell}}{a_{iq}} \right\}.$$

If $M = \{i_1, \ldots, i_m\}$ then by (ii) ( with $t_p^k = s_p^k = a_{i_k,p}$ ) we have

$$m = |M| \leq \left| \bigcup_{i \in M} \Phi(i) \right|,$$

i.e. the mapping $\Phi$ satisfies the Hall-König condition (1). By Proposition 3 there exists an injection $j(i) : \mathbb{N} \to \mathbb{N}$ such that $j(i) \in \tilde{\Phi}(i)$ for any $i \in \mathbb{N}$. Fix any $i$; then $j(i) \in \tilde{\Phi}(i)$ means that there exist $i_1$ and $j_1$ such that $j(i) \in \Phi(i_1)$ and $j_1 \in \Phi(i) \cup \Phi(i_1)$. Therefore we have

$$\frac{b_{j(i),p-3\ell}}{b_{j(i),q+3\ell}} \leq \frac{a_{i_1,p-2\ell}}{a_{i_1,q+2\ell}} \leq \frac{b_{j_1,p-\ell}}{b_{j_1,q+\ell}} \leq \frac{a_{i,p}}{a_{i,q}} \quad \forall p,q.$$

Setting

$$r_i = \sup_p \frac{a_{i,p}}{b_{j(i),p+3\ell}}, \qquad i \in \mathbb{N},$$

we obtain

$$r_i b_{j(i),p-3\ell} \leq a_{ip} \leq r_i b_{j(i),p+3\ell} \quad \forall i \in \mathbb{N}.$$

Hence the operator $T : K(a_{ip}) \to K(b_{jp})$, defined by

$$Te_i = r_i e_{j(i)}, \quad i \in \mathbb{N},$$

is an isomorphic quasidiagonal embedding. □

## 5 Invariant characteristics

Let $X$ be a Köthe space and $A = (a_{ip})$ be a Köthe matrix such that $X \simeq K(A)$. Consider for any $\ell = 0, 1, 2 \ldots$ and for any pair of $m$-tuples of sequences of positive numbers $s = (s_p^k)$ and $t = (t_p^k)$, $k = 1, \ldots, m$, satisfying (3), the following expression:

$$\mu_m^{A,\ell}(s,t) = \left| \bigcup_{k=1}^{m} \bigcap_{p,q} \left\{ i : \frac{a_{i,p-\ell}}{t_p^k} \leq \frac{a_{i,q+\ell}}{s_q^k} \right\} \right|.$$

Further we call this expression $m$-characteristic of the Köthe space $X$ determined by the matrix $A$. We shall prove that for any fixed $m$ the $m$-characteristics of a Köthe space $X$ define a linear topological invariant $\mu_m(X)$, namely isomorphic Köthe spaces have equivalent $m$-characteristics in the sense of the following definition.

**Definition.** *Let X and Y are Köthe spaces.*

*We say that $X$ has weaker m-characteristic than $Y$ and write $\mu_m(X) \prec \mu_m(Y)$ if there exist Köthe matrices $A$ and $B$ with $X \simeq K(A)$ and $Y \simeq K(B)$ such that*

$$\exists \ell : \quad \mu_m^{A,0}(s,t) \leq \mu_m^{B,\ell}(s,t).$$

*$\mu_m(X)$ and $\mu_m(Y)$ are equivalent (we write $\mu_m(X) \sim \mu_m(Y)$) if $\mu_m(X) \prec \mu_m(Y)$ and $\mu_m(Y) \prec \mu_m(X)$.*

The invariance of $m$-characteristics means that isomorphic Köthe spaces have equivalent characteristics. The following statement gives precise conditions for this and even proves a stronger fact: if one Köthe space is isomorphic to a complemented subspace of another Köthe space then it has a weaker $m$-characteristic.

**Theorem 7** *Suppose $X = K(a_{ip})$ and $Y = K(b_{jp})$, where $2^p a_{ip} \leq a_{i,p+1}$ and $2^p b_{ip} \leq b_{i,p+1}$. If $X$ can be embedded as a complemented subspace into $Y$ then, passing to a subsequence with respect to $p$ (i.e. reraring the corresponding systems of seminorms in $X$ and $Y$ ), one can obtain new matrices $A = (a_{ip})$ and $B = (b_{jp})$ such that*

$$\exists \ell = \ell(m) : \quad \mu_m^{A,0}(s,t) \leq \mu_m^{B,\ell}(s,t).$$

**Proof.** Let $T : X \to Y$ be an isomorphic embedding of $X$ into $Y$ as a complemented subspace. Suppose $Y = T(X) \oplus Z$. Set

$$U_p = \left\{ x \in K(a_{ip}) \, : \sum_i |x_i| a_{ip} \leq 1 \right\}, \quad V_p = \left\{ x \in K(b_{jp}) \, : \sum_j |x_j| b_{jp} \leq 1 \right\}$$

and put $W_p = Z \cap V_p$. Then, passing to subsequences 5 with respect to $p$, one can arrange the relations

$$V_p \supset T(U_p) \times W_p \supset V_{p+1}, \qquad p = 1, 2, \ldots. \tag{5}$$

First we prove the claim for $m = 1$. From (5) and the elementary properties of the characteristic $\beta$ it follows that

$$\beta(\bigcap t_p U_p, \overline{\mathrm{conv}}(\bigcup s_q U_q)) \leq$$

$$\beta(\bigcap t_p T(U_p) \times W_p, \overline{\mathrm{conv}}(\bigcup s_q T(U_q) \times W_q)) \leq$$

$$\beta(\bigcap t_p V_p, \overline{\mathrm{conv}}(\bigcup s_q V_{q+1})).$$

Using Lemma 4 and Lemma 5 we estimate, respectively, from below the first term and from above the third term in the above inequalities and obtain

$$\left| \bigcap_{p,q} \left\{ i : \frac{a_{ip}}{t_p} \leq \frac{a_{iq}}{s_q} \right\} \right| = \left| \left\{ i : \sup_p \frac{a_{ip}}{t_p} \leq \inf_q \frac{a_{iq}}{s_q} \right\} \right| \leq$$

$$\leq \left| \left\{ j : \sup_p \frac{b_{j,p-1}}{t_p} \leq \inf_q \frac{b_{j,q+1}}{s_q} \right\} \right| = \left| \bigcap_{p,q} \left\{ j : \frac{b_{j,p-1}}{t_p} \leq \frac{b_{j,q+1}}{s_q} \right\} \right|.$$

This completes the proof in the case $m = 1$.

We continue by induction with respect to $m$. Suppose $m > 1$ and the statement is true for all $m' < m$. Let $s^k$ and $t^k$, $k = 1, \dots, m$ be sequences of positive numbers satisfying (3). We call two pairs of such sequences $(s^{k_1}, t^{k_1})$ and $(s^{k_2}, t^{k_2})$ ν-equivalent, if

$$t_p^{k_1} \le s_{p+\nu}^{k_2}, \quad t_p^{k_2} \le s_{p+\nu}^{k_1} \quad \forall p.$$

Set

$$A_k = \bigcap_{p,q} \left\{ i : \frac{a_{ip}}{t_p^k} \le \frac{a_{iq}}{s_q^k} \right\} \quad B_k^\ell = \bigcap_{p,q} \left\{ j : \frac{b_{j,p-\ell}}{t_p^k} \le \frac{b_{j,q+\ell}}{s_q^k} \right\}.$$

We are to prove that there exists an $\ell$ such that

$$\left| \bigcup_{k=1}^m A_k \right| \le \left| \bigcup_{k=1}^m B_k^\ell \right|. \tag{6}$$

It is known that $|A_k| \le |B_k^1|$, $k = 1, \dots, m$, so if the sets $B_k^1$, $k = 1, \dots, m$ are disjoint then (6) holds with $\ell = 1$. Moreover, if these sets can be separated into two families with disjoint unions than by the inductive assumption the relation (6) holds with $\ell = \ell(m-1)$.

Now we consider the case where the family of sets $B_k^1, k = 1, \dots, m$ has the following **property (S)**: the intersection of the unions of any two complementary subfamilies is nonempty.

We shall show by induction with respect to $m$ that there exists an integer $\nu = \nu(m)$ and constants $C_1, \dots, C_m$ such that every two pairs of sequences among the pairs $(C_k s^k, C_k t^k)$, $k = 1, \dots, m$ are ν-equivalent. Suppose that the statement is true for all $m' < m$. Consider the family of sets $B_1^1, \dots, B_{m-1}^1$. In general this family does not have the property (S), but it can be devided into subfamilies with property (S) and with disjoint unions; in addition, the union of every such subfamily has a nonempty intersection with $B_m^1$. By the inductive assumption there are appropriate multipliers $C_k, k = 1, \dots, m-1$ such that if one of these subfamilies has $m'$ members then the pairs of sequences corresponding to this subfamily are $\nu(m')$-equivalent. Put $C_m = 1$; then the constants $C_1, \dots, C_{m-1}$ may be chosen so that to obtain the inductive assumption for $m$. Indeed, let $B_{k_1}^1, \dots, B_{k_{m_1}}^1$ be one of the subfamilies and $\nu_1 = \nu(m_1)$. At least one of the sets in this subfamily has a nonempty intersection with $B_m^1$; suppose this is so for $B_{k_1}^1$. If $j_1 \in B_{k_1}^1 \cap B_m^1$ then the following relations hold:

$$\frac{b_{j_1,p-1}}{t_p^{k_1}} \le \frac{b_{j_1,q+1}}{s_q^{k_1}}, \qquad \frac{b_{j_1,p-1}}{t_p^m} \le \frac{b_{j_1,q+1}}{s_q^m} \quad \forall p,q.$$

Set

$$D_1 = \sup_p \frac{b_{j_1,p-1}}{t_p^{k_1}}, \qquad D_2 = \sup_p \frac{b_{j_1,p-1}}{t_p^m};$$

then we have

$$b_{j_1,p-1} \le D_1 t_p^{k_1} \ \forall p, \qquad D_1 s_q^{k_1} \le b_{j_1,q+1} \ \forall q;$$
$$b_{j_1,p-1} \le D_2 t_p^m \ \forall p, \qquad D_2 s_q^m \le b_{j_1,q+1} \ \forall q.$$

From here it follows that

$$t_p^m \le s_{p+1}^m \le \frac{1}{D_2} b_{j_1,p+2} \le \frac{D_1}{D_2} t_{p+3}^{k_1} \le \frac{D_1}{D_2} s_{p+4}^{k_1},$$

$$\frac{D_1}{D_2}t_p^{k_1} \leq \frac{D_1}{D_2}s_{p+1}^{k_1} \leq \frac{1}{D_2}b_{j_1,p+2} \leq t_{p+3}^m \leq s_{p+4}^m.$$

Hence the pairs of sequences $(s^m, t^m)$ and $(Ds^{k_1}, Dt^{k_1})$, where $D = D_1/D_2$, are 4-equivalent.

Now using that the constants $C_{k_1}, \ldots, C_{m_1}$ are determined up to an arbitrary multiplier we can change them so that to have $C_{k_1} = D$. Then every one of the pairs

$$(C_{k_1}s^{k_1}, C_{k_1}t^{k_1}), \ldots (C_{k_{m_1}}s^{k_{m_1}}, C_{k_{m_1}}t^{k_{m_1}})$$

is $(\nu_1 + 4)$-equivalent to the pair $(s^m, t^m)$.

Suppose now that the constants $C_1, \ldots, C_m$ are chosen as it is described above. Then for any $k_1, k_2 < m$ the pairs $(C_{k_1}s^{k_1}, C_{k_1}t^{k_1})$ and $(s^m, t^m)$ are $(4 + \max\{\nu(m'),\ m' < m\})$-equivalent, and simultaneously, the same is true for $(C_{k_2}s^{k_2}, C_{k_2}t^{k_2})$ and $(s^m, t^m)$. Hence the pairs $(C_{k_1}s^{k_1}, C_{k_1}t^{k_1})$ and $(C_{k_2}s^{k_2}, C_{k_2}t^{k_2})$ are $\nu$-equivalent with $\nu = 8 + 2\max\{\nu(m'), m' < m\}$.

From (5) and the elementary properties of the characteristic $\beta$ it follows that

$$\beta\left(\overline{\text{conv}}\left(\bigcup_{k=1}^{m}\bigcap_{p} C_k t_p^k U_{p-m}\right), \bigcap_{n=1}^{m}\overline{\text{conv}}\left(\bigcup_{q} C_n s_q^n U_q\right)\right) \leq$$

$$\leq \beta\left(\overline{\text{conv}}\left(\bigcup_{k=1}^{m}\bigcap_{p} C_k t_p^k V_{p-m}\right), \bigcap_{n=1}^{m}\overline{\text{conv}}\left(\bigcup_{q} C_n s_q^n V_{q+1}\right)\right).$$

Estimating from below and above by using Lemma 4 and Lemma 5 and $2^m a_{ip-m} \leq a_{ip}$ we obtain that

$$\left|\bigcup_{k,n=1}^{m} A_{kn}\right| \leq \left|\left\{i : \min_k \sup_p \frac{a_{ip-m}}{C_k t_p^k} \leq \frac{1}{2^m}\max_n \inf_q \frac{a_{iq}}{C_n s_q^n}\right\}\right|$$

$$\leq \left|\left\{j : \min_k \sup_p \frac{b_{j,p-m-1}}{C_k t_p^k} \leq \max_n \inf_q \frac{b_{j,q+1}}{C_n s_q^n}\right\}\right| = \left|\bigcup_{k,n=1}^{m} B_{kn}\right|,$$

where

$$A_{kn} = \bigcap_{p,q}\left\{i : \frac{a_{ip}}{C_k t_p^k} \leq \frac{a_{iq}}{C_n s_q^n}\right\}, \qquad B_{kn} = \bigcap_{p,q}\left\{i : \frac{b_{j,p-m-1}}{C_k t_p^k} \leq \frac{b_{j,q+1}}{C_n s_q^n}\right\}.$$

Since

$$\bigcup_{k}^{m} A_k = \bigcup_{k=1}^{m} A_{kk} \subset \bigcup_{k,n=1}^{m} A_{kn}$$

in order to prove the claim (i.e. to check (6) ) it is enough to show that for some $\ell = \ell(m)$ we have

$$\bigcup_{k,n=1}^{m} B_{kn} \subset \bigcup_{k}^{m} B_k^{\ell}.$$

Fix any $k, n \leq m$. Since the pairs of sequences $(C_k s^k, C_k t^k)$ and $(C_n s^n, C_n t^n)$ are $\nu$-equivalent we have that

$$C_k s_q^k \leq C_k t_q^k \leq C_n s_{q+\nu}^n \quad \forall q.$$

Hence for any $j \in B_{kn}$ we have

$$\frac{b_{j,p-m-1}}{C_k t_p^k} \leq \frac{b_{j,q+\nu+1}}{C_n s_{q+\nu}^n} \leq \frac{b_{j,q+\nu+1}}{C_k s_q^k},$$

thus $B_{kn} \subset B_k^\ell$ with $\ell = \max(m+1, \nu+1)$. The theorem is proved. □

**Remark 8** *It is easy to see that the first step in the proof of Theorem 7 (namely, the proof at the claim for $m = 1$) provides also a proof of Proposition 2.*

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