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COEFFICIENT MULTIPLIERS WITH CLOSED RANGE

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Abstract. *For two power series $f(z) = \sum_{\nu=0}^{\infty} f_\nu z^\nu$ and $g(z) = \sum_{\nu=0}^{\infty} g_\nu z^\nu$ with positive radii of convergence, the Hadamard product or convolution is defined by $f \star g\ (z) := \sum_{\nu=0}^{\infty} f_\nu g_\nu z^\nu$. We consider the problem of characterizing those convolution operators T_f acting on spaces of holomorphic functions which have closed range. In particular, we show that every Euler differential operator $\sum_{\nu=0}^{\infty} \phi_\nu (z\frac{\partial}{\partial z})^\nu$ is a convolution operator T_f and we characterize the Euler differential operators, which are surjective on the space of holomorphic functions on every domain which contains the origin.*

1 Introduction and general results

For a subset M of the Riemannian sphere S^2 let $H(M)$ be the space of all germs of holomorphic functions on M vanishing at ∞, equipped with the usual topology. (If M is open, then $H(M)$ is a Fréchet space, if M is closed, then $H(M)$ is a (DF)-space.) For $f, g \in H(\{0\})$, $f(z) = \sum_{\nu=0}^{\infty} f_\nu z^\nu$, $g(z) = \sum_{\nu=0}^{\infty} g_\nu z^\nu$ we define the Hadamard product $f \star g \in H(\{0\})$ by

$$f \star g\ (z) := \sum_{\nu=0}^{\infty} f_\nu g_\nu z^\nu.$$

It is easy to see, that for fixed f, the linear mapping $T_f : H(\{0\}) \to H(\{0\}),\ g \mapsto f \star g$ is continuous.

We consider in this paper the operators T_f acting on spaces of holomorphic functions. After introducing these operators, and stating some general facts, we give in the second chapter surjectivity results. The proofs of the results are contained in the third chapter.

Definition 1 *Let G_1, G_2 be regions in S^2, both containing the origin and with $\infty \in G_1$ if $\infty \in G_2$ and let $f \in H(\{0\})$. If $f \star g \in H(G_2)$ (i.e. $f \star g$ admits an analytic continuation to G_2) for every $g \in H(G_1)$ we call the linear mapping $T_f : H(G_1) \to H(G_2),\ g \mapsto f \star g$, a (coefficient) multiplier from $H(G_1)$ to $H(G_2)$.*

According to the closed graph theorem, a multiplier is always continuous. If T_f is a multiplier, then for every $\xi \in S^2 \setminus G_1$ the map $f(\frac{\cdot}{\xi}) = f \star \frac{\xi}{\xi - \cdot}$ is contained in $H(G_2)$ and this means $f \in H(\xi^{-1} G_2)$ for all $\xi \in S^2 \setminus G_1$. A result of J. Müller [9] shows that this condition is also sufficient for T_f to be a multiplier. We will give here an easy proof for this theorem, which shows that it is a part of a more general concept, namely of Köthe's description of linear operators between spaces of holomorphic functions.

In a second part, we include the characterization of coefficient multipliers defining a compact operator:

Theorem 2 *Let $G_1, G_2 \subset S^2$ be regions with $0 \in G_1 \cap G_2$ and with $\infty \in G_1$ if $\infty \in G_2$, and let $f \in H(\{0\})$.*
i) T_f is a continuous operator from $H(G_1)$ to $H(G_2)$ if and only if $f \in H(\xi^{-1}G_2)$ for all $\xi \in S^2 \setminus G_1$.
ii) T_f is a compact operator from $H(G_1)$ to $H(G_2)$ if and only if $f \in H(\xi^{-1}G_2)$ for all ξ in some open neighbourhood of $S^2 \setminus G_1$.

Remark 3 *i) The first part of the previous theorem, due to Müller [9], can be viewed as a strong version of the Hadamard multiplication theorem. In [9] there are examples showing that it is a proper generalization.*
ii) If $G_1, G_2 \subset \mathbb{C}$ are domains both containing the origin, then part ii) implies that every entire function defines a compact operator from $H(G_1)$ to $H(G_2)$.
iii) Multipliers are surjective if and only if they are (topological) isomorphisms, so one can use Theorem 2 (applied to the "inverse multiplier") to get characterizations of surjectivity. To obtain examples of surjective multipliers or multipliers with closed range one can use the following proposition, which can be derived from a classical theorem guaranteeing that the sum of an isomorphism and a compact operator has closed range.

Proposition 4 *Let $f, h \in H(\{0\})$ define multipliers from $H(G_1)$ to $H(G_2)$ with $f(z) = \sum_{\nu=0}^{\infty} f_\nu z^\nu$ and $h(z) = \sum_{\nu=0}^{\infty} h_\nu z^\nu$. Assume that T_f has closed range and that T_h is compact. If $f_\nu \neq 0$ whenever $h_\nu \neq 0$ for all but finitely many $\nu \in \mathbb{N}_0$, then T_{f+h} has closed range.*

A combination of Theorem 2 and Proposition 4 yields
Example. Let $0 \in G \subset \mathbb{C}$ be a simply connected domain and f a power series with radius of convergence greater than $\sup\{|\frac{z}{w}| : z \in G, w \in S^2 \setminus G\}$, then $\frac{1}{1-\cdot} + f$ defines a multiplier from $H(G)$ into itself having closed range. If every Taylor coefficient of f is different from -1, then this multiplier is even an isomorphism.

In what follows we restrict ourselves to multipliers acting on every domain $G \subset \mathbb{C}$, precisely:

Definition 5 *Let $f \in H(\{0\})$. We call T_f a common multiplier if it is a multiplier from $H(G)$ to $H(G)$ for all domains $0 \in G \subset \mathbb{C}$.*

According to Theorem 2, $f \in H(\{0\})$ defines a common multiplier if and only if $f \in H(\mathbb{C} \setminus \{1\})$. If we decompose f into the sum of a function in $H(S^2 \setminus \{1\})$ and an entire function we obtain

Proposition 6 *Let $f \in H(\{0\})$ define a common multiplier. Then there exist unique $g \in H(S^2 \setminus \{1\})$ and $h \in H(\mathbb{C})$ such that $f = g + h$ and therefore $T_f = T_g + T_h$.*

Common multipliers are strongly connected with Euler differential operators: Let ϕ be an entire function of exponential type zero with $\phi(z) = \sum_{\nu=0}^{\infty} \phi_\nu z^\nu$. For a polynomial p and $z \in \mathbb{C}$ we define

$$(\phi(z\frac{\partial}{\partial z})(p))(z) := \sum_{\nu=0}^{\infty} \phi_\nu((z\frac{\partial}{\partial z})^\nu(p))(z).$$

Using Cauchy's inequalities, we see that this definition extends to a linear and continuous map $\phi(z\frac{\partial}{\partial z}) : H(\{0\}) \to H(\{0\})$.

Definition 7 *The (linear and continuous) map $\phi(z\frac{\partial}{\partial z}) : H(\{0\}) \to H(\{0\})$ is called Euler differential operator (of infinite order).*

If $\phi(z) = \sum_{\nu=0}^{N} \phi_\nu z^\nu$ is a polynomial then $\phi(z\frac{\partial}{\partial z})$ is the Euler differential operator $\sum_{\nu=0}^{N} \phi_\nu (z\frac{\partial}{\partial z})^\nu$ (of finite order), which is clearly a continuous linear map from $H(G)$ to itself for every domain $0 \in G \subset \mathbb{C}$. In fact, the same is true for all Euler differential operators. To prove this, we use the Mellin transform $\mathcal{M}$, which is an isomorphism from the space $H(S^2 \setminus \{1\})$ onto the space of all entire functions ϕ of exponential type zero and which can be defined by $(\mathcal{M}^{-1}(\phi))(z) := \sum_{\nu=0}^{\infty} \phi(\nu) z^\nu$, where $|z| < 1$.

Proposition 8 *(cf. [6, Theorem 11.2.3.]) Let ϕ be an entire function of exponential type zero. Then $\phi(z\frac{\partial}{\partial z}) = T_{\mathcal{M}^{-1}(\phi)}$ and, in particular, $\phi(z\frac{\partial}{\partial z})$ is a common multiplier.*

If $f \in H(\{0\})$ defines a common multiplier, then, according to Propositions 6 and 8, there exist an Euler differential operator $\phi(z\frac{\partial}{\partial z})$ and an entire function g such that $T_f = \phi(z\frac{\partial}{\partial z}) + T_g$. This decomposition is unique. In particular, T_f is a differential operator with a compact perturbation.

2 Surjectivity results

We characterize those common multipliers, which are surjective on $H(G)$ for every simply connected domain $G \subset \mathbb{C}$. It is clear, that surjectivity implies that every Taylor coefficient of the function f, which defines the multiplier, has to be different from zero, i.e surjectivity implies injectivity. It turns out that this condition is not sufficient for surjectivity. We have to require, in addition, that the function $\mathcal{M}(g)$, (where $f = g + h$, $g \in H(S^2 \setminus \{1\})$), satisfies some growth condition on a certain domain. We have to introduce some notations:

Definition 9 *Let $\Omega \subset \mathbb{C}$ be an open set. An open set $\tilde{\Omega} \subset \Omega$ is called asymptotic (see [2]) to Ω if for all $\varepsilon > 0$ there exists $\rho > 1$ such that dist$(z, \mathbb{C} \setminus \Omega) < \varepsilon|z|$ for all $z \in \Omega \setminus \tilde{\Omega}$ with $|z| > \rho$.*
Let $\emptyset \neq \Omega \subset \mathbb{C}$ be open. Then we call a function $f \in H(\Omega)$ of exponential type zero on Ω, if for all $\varepsilon > 0$ there exists $C > 1$ such that $|f(z)| \leq Ce^{\varepsilon|z|}$ for all $z \in \Omega$.

Theorem 10 *Let $f \in H(\{0\})$ define a common multiplier, let $f = g + h$ be the decomposition obtained in Proposition 6 and $\phi := \mathcal{M}(g)$. Then T_f is a surjective map from $H(G)$ to itself for all simply connected domains $0 \in G \subset \mathbb{C}$ if and only if*
i) no Taylor coefficient of f vanishes and
ii) there exist an open set Ω asymptotic to $\mathbb{C} \setminus (-\infty, 0]$, such that $\phi \neq 0$ on Ω and ϕ^{-1} is of exponential type zero on Ω.
Here condition ii) can be replaced by
ii)' ϕ is a polynomial or $\lim_{z \to \infty, \phi(z)=0} \frac{z}{|z|} = -1$.

If we consider only starlike domains, we obtain the following result:

Theorem 11 *Let $f \in H(\{0\})$ define a common multiplier, let $f = g + h$ be the decomposition obtained in Proposition 6 and $\phi := \mathcal{M}(g)$. Then T_f is a surjective map from $H(G)$ to itself for all domains $0 \in G \subset \mathbb{C}$, which are starlike with respect to 0, if and only if*
i) no Taylor coefficient of f vanishes and
ii) there exist an open set Ω asymptotic to the right half plane, such that $\phi \neq 0$ on Ω and ϕ^{-1} is of exponential type zero on Ω.
Here condition ii) can be replaced by
ii)' ϕ is a polynomial or $\limsup_{z\to\infty,\phi(z)=0} Im(\frac{z}{|z|}) \leq 0$.

Conditions i) ii) are very similar to those used in [5], where a characterization of invertible elements in a Hadamard algebra is given. Moreover, our proof of the previous result will use the ideas presented in [5]. The above theorems can be applied to Euler differential operators. Using Proposition 6 and the definition of the Mellin transform, we obtain from Theorem 10 the following

Corollary 12 *Let ϕ be an entire function of exponential type zero.*
i) $\phi(z\frac{\partial}{\partial z})$ is a surjective map from $H(G)$ to itself for every simply connected domain $0 \in G \subset \mathbb{C}$ if and only if $\phi(n) \neq 0$ for all $n \in \mathbb{N}_0$ and ϕ satisfies the condition ii) (or ii)') of Theorem 10.
ii) Let, in addition, ϕ be a polynomial. Then $\phi(z\frac{\partial}{\partial z})$ is a surjective map from $H(G)$ to itself for every simply connected domain $0 \in G \subset \mathbb{C}$ if and only if $\phi(n) \neq 0$ for all $n \in \mathbb{N}_0$.

We recall that an Euler differential operator is surjective, if and only if it is a topological isomorphism. Let us compute this inverse in an easy situation:
Example. Let $\phi(z) = 1 + z$ and G be a simply connected domain containing 0. Then $\phi(z\frac{\partial}{\partial z}) = 1 + z\frac{\partial}{\partial z} = T_f$, where $f(z) = (1-z)^{-2}$, has to possess a linear and continuous inverse $\phi(z\frac{\partial}{\partial z})^{-1} : H(G) \to H(G)$, according to the Corollary. If $g(z) = -z^{-1}\log(1-z)$, we obtain $(\phi(z\frac{\partial}{\partial z}))^{-1} = T_f^{-1} = T_g$. It is clear that one can obtain this inverse also by integration.
In the next result we examine the surjectivity of the Euler differential operators on spaces of holomorphic function on starlike domains.

Corollary 13 *Let ϕ be an entire function of exponential type zero.*
$\phi(z\frac{\partial}{\partial z})$ is a surjective map from $H(G)$ to itself for every domain $0 \in G \subset \mathbb{C}$, which is starlike with respect to 0, if and only if $\phi(n) \neq 0$ for all $n \in \mathbb{N}_0$ and ϕ satisfies the condition ii) (or ii)') of Theorem 11.

For $\alpha \in \mathbb{R}$ it is also possible to characterize the Euler differential operators, which are surjective maps from $H(G)$ to itself for every domain $0 \in G \subset \mathbb{C}$, which is α-starlike. Here a domain $G \subset \mathbb{C}$ is called α-starlike, if $G \cdot \{te^{i\alpha \log t} : 0 \leq t \leq 1\} \subset G$, see e.g. [2]. 0-starlike is just starlike with respect to 0. The characteristic condition is that ϕ^{-1} is of exponential type zero in some open set asymptotic to $\{z \in \mathbb{C} : \gamma - \pi < \arg(z) < \gamma\}$, where $\gamma \in (0, \pi)$ is the root of $\cot\gamma = \alpha$. Moreover, one can proof a result analogously to Theorem 11 replacing starlike by α-starlike.
We want to give two examples showing that there exist Euler differential operators (in view of Corollary 12, ii) they are necessarily of infinite order) separating the conditions in Corollaries 12 and 13:

Example. a) Let $\phi(z) := \cosh\sqrt{z}$. Then ϕ is not a polynomial but nevertheless $\phi(z\frac{\partial}{\partial z})$ is, according to Corollary 12, a surjective Operator from $H(G)$ to itself.
b) Let $\phi(z) := \prod_{n=1}^{\infty}(1-e^{i\frac{\pi}{4}}\frac{z^2}{\lambda_n^2})$, where $1 \leq \frac{\lambda_n}{n} \to \infty$. Then ϕ is an entire function of exponential type zero, but ϕ does not satisfy condition ii) of Theorem 11. Hence, there exists a starlike domain $0 \in D \subset \mathbb{C}$, such that $\phi(z\frac{\partial}{\partial z})$ is not a surjective operator from $H(D)$ to itself. One can show that D can be chosen as $\mathbb{C}\setminus[1,\infty]$. On the other hand, $\phi(z\frac{\partial}{\partial z})$ maps $H(\mathbb{D})$ onto itself.
c) If we define $\phi(z) := \prod_{n=1}^{\infty}(1+\frac{z^2}{\lambda_n^2})$, where $1 \leq \frac{\lambda_n}{n} \to \infty$, then again ϕ is of exponential type zero and it now satisfies the conditions of Corollary 13 but not condition ii) of Theorem 10 (see [3]). Hence, $\phi(z\frac{\partial}{\partial z})$ is a surjective operator on $H(G)$ for every domain $0 \in G \subset \mathbb{C}$, which is starlike with respect to 0, but there exists a simply connected domain $0 \in D \subset \mathbb{C}$ such that $\phi(z\frac{\partial}{\partial z})$ is not surjective as a map from $H(D)$ to itself. One can show that D can be chosen as $\mathbb{C}\setminus\{te^{i\alpha\log t} : 1 \leq t < \infty\}$, where $0 \neq \alpha \in \mathbb{R}$ is arbitrary.

3 Auxilliary results and proofs

We recall Köthe's description of linear and continuous operators between spaces of holomorphic functions, see [7].
Let therefore $G_1, G_2 \subset S^2$ be domains with $\infty \in G_1$ if $\infty \in G_2$. For $i = 1,2$ choose exhaustions $(G_{i,n})_{n\in\mathbb{N}}$ of G_i by open and relatively compact domains $G_{i,n} \subset G_i$ (with $\infty \in G_{i,1}$, if $\infty \in G_i$) in such a way, that $\overline{G}_{i,n} \subset G_{i,n+1}$ and the boundary of every $G_{i,n}$ consists of finitely many, pairwise disjoint and rectifiable Jordan curves, which we assume to be oriented with respect to $G_{i,n}$. Moreover, let every component of $S^2 \setminus G_{i,n}$ contain a point of $S^2 \setminus G_i$.
For $M_1, M_2 \subset S^2$ we denote by $H(M_1 \times M_2)$ the space of all germs of holomorphic functions F on $M_1 \times M_2$ which vanish at the infinite points. Then we get $H(S^2 \setminus G_1) = \cup_{m\in\mathbb{N}} H(S^2 \setminus \overline{G}_{1,m})$ and $H((S^2 \setminus G_1) \times G_2) = \cap_{n\in\mathbb{N}} \cup_{m\in\mathbb{N}} H((S^2 \setminus \overline{G}_{1,m}) \times G_{2,n})$. Now the space $H(G_1)'$ of all continuous linear forms on $H(G_1)$ is isomorphic to $H(S^2 \setminus G_1)$ where the duality is given by

$$< \varphi, g > := \frac{1}{2\pi i}\int_{\partial G_{1,m+1}} g(\xi)\varphi(\xi)d\xi$$

where $g \in H(G_1)$ and $\varphi \in H(S^2 \setminus \overline{G}_{1,m})$.

Moreover, the space $\mathcal{L}(H(G_1), H(G_2))$ of continuous linear operators from $H(G_1)$ to $H(G_2)$ is algebraically isomorphic to $H((S^2 \setminus G_1) \times G_2)$ where the identification of an operator T with a germ is given by

$$T(g)(z) = \frac{1}{2\pi i}\int_{\partial G_{1,m+1}} g(\xi)F_T(\xi,z)d\xi = < F(\cdot,z), g >,$$

for $g \in H(G_1), F_T \in H((S^2 \setminus \overline{G}_{1,m}) \times G_{2,n})$ and $z \in G_{2,n}$.

Remark 14 *Standard arguments shows that an operator $T : H(G_1) \to H(G_2)$ is compact if and only if $F_T \in H(U \times G_2)$ for some open subset U of $S^2 \setminus G_1$.*

Proof. [Proof of Theorem 2] i) Clearly, the condition is necessary. To prove sufficiency, we show that there exist $F \in H((S^2 \setminus G_1) \times G_2)$ such that $F(\xi, z) = f(z\xi^{-1})\xi^{-1}$ for small $|z\xi^{-1}|$. Cauchy's integral formula then implies $f \star g\ (z) =< F(\cdot, z), g >$ for z in some neighbourhood of 0, and so T_f is the operator associated to F. This yields that T_f is a multiplier from $H(G_1)$ to $H(G_2)$.
We first choose exhaustions $(G_{1,n})_{n\in\mathbb{N}}$ and $(G_{2,n})_{n\in\mathbb{N}}$ of G_1 and G_2, respectively, according to the remarks at the beginning of this chapter. Without loss of generality we may assume that $0 \in G_{1,1} \cap G_{2,1}$.
For $n \in \mathbb{N}$ fix $m = m(n) > m(n-1)$ such that for all $\xi \in S^2 \setminus \overline{G}_{1,m}$ there exists $\zeta(\xi) \in S^2 \setminus G_1$ with $\xi^{-1}\overline{G}_{2,n} \subset \zeta(\xi)^{-1}G_2$. Then choose for all $\xi \in S^2 \setminus \overline{G}_{1,m}$ an $\varepsilon_\xi > 0$ such that $\eta^{-1}\overline{G}_{2,n} \subset \zeta(\xi)^{-1}G_2$ for all η with $|\eta - \xi| < \varepsilon_\xi$.
Let now $\xi_1, \xi_2 \in S^2 \setminus \overline{G}_{1,m}$ be arbitrary. Is $\eta \in S^2 \setminus \overline{G}_{1,m}$ with $|\eta - \xi_i| < \varepsilon_{\xi_i}$, $i = 1, 2$, then we have $\eta^{-1}G_{2,n} \subset \zeta(\xi_1)^{-1}G_2 \cap \zeta(\xi_2)^{-1}G_2$. Denoting by f_ζ the continuation of f to $\zeta^{-1}G_2$, we obtain $f_{\zeta(\xi_1)} \equiv f_{\zeta(\xi_2)}$ on $\eta^{-1}G_{2,n}$. This means $f_{\zeta(\xi_1)}(z\eta^{-1})\eta^{-1} = f_{\zeta(\xi_2)}(z\eta^{-1})\eta^{-1}$ for all $z \in G_{2,n}$.
Defining $F_n(\xi, z) := f_{\zeta(\xi)}(z\xi^{-1})\xi^{-1}$ for $z \in G_{2,n}, \xi \in S^2 \setminus \overline{G}_{1,m}$, we obtain $F_n(\eta, z) = f_{\zeta(\xi)}(z\eta^{-1})$ η^{-1} for all $z \in G_{2,n}$ and $\xi, \eta \in S^2 \setminus \overline{G}_{1,m}$ with $|\eta - \xi| < \varepsilon_\xi$. This implies $F_n \in H((S^2 \setminus \overline{G}_{1,m}) \times G_{2,n})$. Clearly, for small $z\xi^{-1}$, we have $F_n(\xi, z) = f(z\xi^{-1})\xi^{-1}$. The sequence $(F_n)_{n\in\mathbb{N}}$ defines now $F \in H((S^2 \setminus G_1) \times G_2)$ with the required property.
ii) If T_f is a compact operator, then, according to Remark 14, $F = F_{T_f} \in H(U \times G_2)$ for some open neighbourhood of $S^2 \setminus G_1$. This implies that $f \in H(\xi^{-1}G_2)$ for all $\xi \in U$, because $\xi F(\xi, z) = f(\xi^{-1}z)$ for small $|\xi^{-1}z|$.
Assuming the condition, then, according to part i), T_f is continuous as an operator from $H(G_{1,n})$ to $H(G_2)$ for some $n \in \mathbb{N}$. This implies that T_f is a compact operator from $H(G_1)$ to $H(G_2)$, because $H(G_2)$ is a Fréchet-Schwartz space.

Proposition 4 follows directly by the next lemma which is derived from a result due to L. Schwartz. □

Lemma 15 *Let $T, K : E \to F$ be continuous linear maps between Fréchet spaces. Assume that T has closed range and that K is compact and satisfies $L+ \ker K \supset \ker T$ for some finite dimensional $L \subset E$. Then $T + K$ has a closed range.*

Proof. We may assume that L is a finite dimensional subspace of E contained in $\ker(T)$ and $L \cap \ker(K) = \{0\}$. This subspace has a topological complement $X \supset \ker(K)$ in E. Then $\ker(T|_X) \subset \ker(K|_X)$ and the range of $T|_X$ is $T(E)$. Consider now the canonical factorisations $\widehat{T|_X}, \widehat{K|_X} : X/\ker(T|_X) \to T(E)$. Then $\widehat{T|_X}$ is a (topological) isomorphism and $\widehat{K|_X}$ is compact. We may apply a classical result, due to Schwartz, and obtain that $\widehat{T|_X} + \widehat{K|_X}$ has a closed range Y. From $(T|_X + K|_X)(X) = Y$ we conclude that $(T + K)(E) = (T + K)(X + L) = (T + K)(X) + K(L) = Y + K(L)$ is closed as the sum of a closed subspace and a finite dimensional one. □

Proof. [Proof of Proposition 8] Let $\phi(z) = \sum_{\nu=0}^{\infty} \phi_\nu z^\nu$. The polynomials are dense in $H(\{0\})$ and the maps $\phi(z\frac{\partial}{\partial z})$, $T_{\mathcal{M}^{-1}(\phi)} : H(\{0\}) \to H(\{0\})$ are linear and continuous, so it is enough to show $\phi(z\frac{\partial}{\partial z})(p) = \mathcal{M}^{-1}(\phi) \star p$ for every monomial p. If $p_k(z) = z^k$ then by definition $(\phi(z\frac{\partial}{\partial z})(p_k))(z) = \sum_{\nu=0}^{\infty} \phi_\nu((z\frac{\partial}{\partial z})^\nu(p_k))(z) = \sum_{\nu=0}^{\infty} \phi_\nu k^\nu z^k = \phi(k)z^k = \mathcal{M}^{-1}(\phi) \star p_k(z)$. □

Now, Theorem 2 implies (because of $\mathcal{M}^{-1}(\phi) \in H(S^2 \setminus \{1\})$) that $\phi(z\frac{\partial}{\partial z})$ is a common multiplier.

Before proving Theorems 10 and 11 let us give some lemmata which will be used:

Lemma 16 *Let $\Omega_i \subset \mathbb{C}$, $i = 1, \ldots, 4$, be open.*
i) If Ω_i is asymptotic to Ω_{i+1}, $i = 1, 2$, then Ω_1 is asymptotic to Ω_3.
ii) If Ω_i is asymptotic to Ω_{i+2}, $i = 1, 2$, then $\Omega_1 \cap \Omega_2$ is asymptotic to $\Omega_3 \cap \Omega_4$. In particular, if Ω_1 and Ω_2 are asymptotic to Ω_3, then $\Omega_1 \cap \Omega_2$ is also asymptotic to Ω_3.

We omit the elementary proof.

Lemma 17 *Let $\phi \neq 0$ be an entire function of exponential type zero. Then there exists an open set $\Omega \subset \mathbb{C} \setminus \{z \in \mathbb{C} : \phi(z) = 0\}$, asymptotic to $\mathbb{C} \setminus \{z \in \mathbb{C} : \phi(z) = 0\}$, such that ϕ^{-1} is of exponential type zero on Ω.*

Proof. Since ϕ is of exponential type zero, we may apply [4], p. 51, Theorem 3.7.1., and obtain the existence of an increasing sequence $(R_n)_{n\in\mathbb{N}}$ of positive numbers, such that the following holds:
For every $n \in \mathbb{N}$ and all $R \geq R_n$ we have $|\phi(z)^{-1}| \leq e^{\frac{R}{n}}$ on $|z| < R$ except on finitely many circles, whose sum of radii does not exceed $\frac{1}{2n}R$. From the proof of this result it follows, that the centers of this circles can be chosen as zeros of ϕ.
For $n \in \mathbb{N}$ define $\Omega_n := \{z : |z| > R_n,\ \mathrm{dist}(z, \{w : \phi(w) = 0\}) > \frac{1}{n}|z|\}$. Then on Ω_n we have $|\phi(z)^{-1}| \leq e^{\frac{R}{n}}$ for $|z| = R$. Let $\Omega := \cup_{n\in\mathbb{N}}\Omega_n$. Then Ω is asymptotic to $\mathbb{C} \setminus \{z \in \mathbb{C} : \phi(z) = 0\}$.
To prove that ϕ^{-1} is of exponential type zero on Ω let $n \in \mathbb{N}$ be arbitrary. Then $|\phi(z)^{-1}| \leq e^{\frac{|z|}{n}}$ on Ω_k for all $k \geq n$. But $\Omega \setminus \cup_{k\geq n}\Omega_k$ is bounded, so there exist $C > 1$ such that $|\phi(z)^{-1}| \leq Ce^{\frac{|z|}{n}}$ on Ω. □

Proposition 18 *Let ϕ be an entire function of exponential type zero and let G be an unbounded domain. Then there exist an open set Ω asymptotic to G such that ϕ^{-1} is of exponential type zero on Ω if and only if for all $\varepsilon > 0$ there is $\rho > 1$ such that for all z with $|z| > \rho$ and $\phi(z) = 0$ we have $dist(z, \mathbb{C} \setminus G) < \varepsilon|z|$.*

Proof. We may assume that ϕ has infinitely many zeros. Let Ω be asymptotic to G such that ϕ^{-1} is of exponential type zero on Ω. In particular, Ω contains no zeros of ϕ and therefore $\lim_{z\to\infty,\phi(z)=0} \frac{1}{|z|}\,\mathrm{dist}(z, \mathbb{C} \setminus G) \leq \lim_{z\to\infty, z\in G\setminus\Omega} \frac{1}{|z|}\,\mathrm{dist}(z, \mathbb{C} \setminus G) = 0$.
To prove the converse, we apply first Lemma 17 to get the existence of an open set $\tilde{\Omega}$ asymptotic to $\mathbb{C} \setminus \{z : \phi(z) = 0\}$ such that ϕ^{-1} is of exponential type zero on $\tilde{\Omega}$. Lemma 16, ii) implies that $\Omega := G \cap \tilde{\Omega}$ is asymptotic to $G \setminus \{z : \phi(z) = 0\} = G \cap (\mathbb{C} \setminus \{z : \phi(z) = 0\})$. On the other hand, $\lim_{z\to\infty,\phi(z)=0} \frac{1}{|z|}\,\mathrm{dist}(z, \mathbb{C} \setminus G) = 0$ implies that $G \setminus \{z : \phi(z) = 0\}$ is asymptotic to G. Using Lemma 16, i), we obtain that Ω is asymptotic to G, and $\Omega \subset \tilde{\Omega}$ ensures that ϕ^{-1} is of exponential type zero on Ω. □

Proof. [Proof of Theorem 10] Let $f \in H(\{0\})$ define a common multiplier, such that conditions i) and ii) are satisfied. We decompose $f = h + g$, where $h \in H(S^2 \setminus \{1\})$ and g is entire. Then $\sum_{\nu=0}^{\infty} h_\nu z^\nu = \sum_{\nu=0}^{\infty} \phi(\nu) z^\nu$, where $\phi = \mathcal{M}(h)$. Condition ii) implies now the existence of

$n \in \mathbb{N}_0$ such that $\psi := \phi(\cdot + n)^{-1}$ is holomorphic in the domain $H := \{z \in \mathbb{C} : \mathrm{Re}(z) > -\frac{1}{2}\}$ and satisfies the following condition:
For every $0 < \alpha < \pi$ there exist $\rho > 0$ such that ψ is of exponential type zero on $\{z \in \mathbb{C} : |\arg|(z) < \alpha, |z| > \rho\}$. Now a result of E. Lindelöf [8]) ensures that the power series $p(z) = \sum_{\nu=0}^{\infty} \psi(\nu) z^\nu$ has an analytic continuation to any domain contained in the Riemann surface of $\log(1-z)$ covering the origin only once. For arbitrary simply connected subdomains $0 \in G \subset \mathbb{C}$ this implies that p has an analytic continuation to $\xi^{-1}G$ for all $\xi \in S^2 \setminus G$.
Let now $0 \in G \subset \mathbb{C}$ be a simply connected domain. With p also $\sum_{\phi(\nu)\neq 0} \phi(\nu)^{-1} z^\nu = \sum_{h_\nu \neq 0} h_\nu^{-1} z^\nu =: h_{-1}(z)$ has an analytic continuation to $\xi^{-1}G$ for all $\xi \in S^2 \setminus G$. Applying Theorem 2 this shows that $T_{h_{-1}}$ is a multiplier from $H(G)$ to itself, which inverts T_h on $\{q \in H(G) : q(z) = \sum_{h_\nu \neq 0} q_\nu z^\nu\}$. Hence, this (closed) subspace is the range of T_h. Now Condition ii) implies $\lim_{\nu\to\infty} |h_\nu|^{\frac{1}{\nu}} = \lim_{\nu\to\infty} |\phi(\nu)|^{\frac{1}{\nu}} = 1$, so there exist only finitely many ν with $h_\nu = 0$ and $g_\nu \neq 0$. Applying Proposition 4 we get that $T_f = T_{h+g}$ has also a closed range. From Condition i) it follows that the polynomials are contained in the range of T_f, and Runge's theorem now shows that the range of T_f is also dense in $H(G)$. Hence T_f maps $H(G)$ onto itself.
Condition i) is clearly necessary. To prove that Condition ii) is also necessary, we decompose again $f = h + g$, where $h \in H(S^2 \setminus \{1\})$ and g is entire. Write $h(z) = \sum_{\nu=0}^{\infty} \phi(\nu) z^\nu$. Proposition 4 implies that $T_h : H(G) \to H(G)$ has a closed range for all domains $0 \in G \subset \mathbb{C}$. It is easy to see that $\phi(\nu) = 0$ for only finitely many ν. These two facts imply that $h_{-1}(z) := \sum_{\phi(\nu)\neq 0} \phi(\nu)^{-1} z^\nu$ defines a common multiplier. For $\alpha \in \mathbb{R}$ let $G_\alpha := \mathbb{C} \setminus \{t e^{i\alpha \log t} : 1 \le t < \infty\}$. Then Theorem 2 shows that $h \in H(G_\alpha)$. Let now α be fixed.
1) Let $\prod_\alpha := \{z \in \mathbb{C} : \gamma - \pi < \arg(z) < \gamma\}$, where $\gamma \in (0,\pi)$ is the root of $\cot\gamma = \alpha$. Theorem 1.2 a) in [2] guarantees the existence of $\psi \in H(\prod_\alpha)$ satisfying $\psi(\nu) = h_\nu^{-1}$ for large ν and ψ is of exponential type zero on every set $\prod_{\alpha,\varepsilon} := \{z \in \mathbb{C} : \gamma - \pi + \varepsilon < \arg(z) < \gamma - \varepsilon\}$ for every $0 < \varepsilon < \frac{\pi}{2}$. Defining $F \in H(\prod_\alpha)$ by $F(z) = \phi(z)\psi(z) - 1$, then F is also of exponential type zero on all domains $\prod_{\alpha,\varepsilon}$ for $\varepsilon > 0$ and $F(\nu) = 0$ for large ν. Following exactly the proof of Theorem 3 in [5], we obtain the existence of a domain $\Omega_\alpha \subset \prod_\alpha$, asymptotic to $\prod_\alpha$ such that $|F(z)| < \frac{1}{2}$ on Ω_α. This implies that $\phi(z) \neq 0$ on Ω_α and ϕ^{-1} is of exponential type zero on Ω_α.
2) For $\alpha \in \mathbb{R}$ choose $\rho_\alpha > 1$ such that $\phi(z)^{-1} \le e^{\frac{|z|}{|\alpha|}}$ for $z \in \Omega_{\alpha,\rho_\alpha} := \Omega_\alpha \cap \{z \in \mathbb{C} : |z| > \rho_\alpha\}$. We may assume that ρ_α is increasing in $|\alpha|$. Then $\Omega := \cup_{\alpha\in\mathbb{R}} \Omega_{\alpha,\rho_\alpha}$ is a domain asymptotic to $\mathbb{C} \setminus (-\infty, 0]$ and ϕ^{-1} is of exponential type zero on Ω.
The equivalence of the conditions ii) and ii)′ follows immediately from Proposition 18. □

Proof. [Proof of Theorem 11] Let $f \in H(\{0\})$ define a common multiplier, such that conditions i) and ii) are satisfied. We decompose $f = h + g$, where $h \in H(S^2 \setminus \{1\})$ and g is entire. Then $\sum_{\nu=0}^{\infty} h_\nu z^\nu = \sum_{\nu=0}^{\infty} \phi(\nu) z^\nu$, where $\phi = \mathcal{M}(h)$. Condition ii) implies the existence of $n \in \mathbb{N}_0$ such that $\psi := \phi(\cdot + n)^{-1}$ is holomorphic in the domain $H := \{z \in \mathbb{C} : \mathrm{Re}(z) > -\frac{1}{2}\}$ and satisfies the following condition:
For every $0 < \alpha < \frac{\pi}{2}$ there exist $\rho > 0$ such that ψ is of exponential type zero on $\{z \in \mathbb{C} : |\arg|(z) < \alpha, |z| > \rho\}$. Instead of the result of Lindelöf we now apply [2, Theorem 1.2], due to Arakelyan, ensuring that the power series $p(z) = \sum_{\nu=1}^{\infty} \psi(\nu) z^\nu$ has an analytic continuation to $\mathbb{C} \setminus [1, \infty)$. Now, Theorem 2 implies that $F \in H(\{0\})$ is a multiplier from $H(G)$ to itself

for every domain $G \subset \mathbb{C}$ which is starlike with respect to 0 if and only if $F \in H(\mathbb{C} \setminus [1,\infty))$. Therefore we may proceed as in the first part of the proof of Theorem 10 to get that T_f is surjective on $H(G)$ for all G which are starlike with respect to 0.
Clearly, i) is necessary, so it remains to show that condition ii) is also necessary. We decompose $f = h + g$ as in the proof of Theorem 10. As remarked before, we get $h_{-1} \in H(\mathbb{C} \setminus [1,\infty))$. Then part 1) (more precisely, the case $\alpha = 0$) of the proof of Theorem 10 implies that there exists a domain Ω_0 asymptotic to $\prod_0$, such that ϕ^{-1} is of exponential type zero on Ω_0.
To prove that the conditions ii) and ii)′ are equivalent, we apply Proposition 18. □

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