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EXTENSION OF ANALYTICITY FOR SOLUTIONS OF PARTIAL DIFFERENTIAL OPERATORS [1]

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Abstract. *We introduce a quantitative version of the complement of the analytic wave front set and study its extension for solutions of partial differential operators. This quantitative result can be applied in the study of surjective partial differential operators on spaces of real analytic functions.*

In the study of surjective partial differential operators on spaces of rea1 analytic functions (Langenbruch [18]) and of elliptic systems of partial differential operators on nonconvex sets (Langenbruch [19]) a centra1 idea is to apply arguments coming from the theory of analytic wave front sets to rea1 analytic functions. This seems to be useless since the classica1 analytic wave front set of a rea1 analytic function is void. We in fact use a quantified version of the (complement of the) analytic wave front set (called regularity set) which is nontrivial also for rea1 analytic functions and we have to know how the regularity set extends for solutions of partial differential equations. The introduction of this regularity set and the study of its extension properties is the main aim of the present paper.

The paper is organized as follows: in section 1 we introduce the regularity set $\operatorname{reg}_L(f)$ of $f \in C^\infty(\Omega)$ by means of a quantitative version of the estimates used to define the analytic wave front set of distributions (see Definition 1.1). We also introduce hyperfunctions as forma1 boundary values of harmonic functions and, correspondingly, the notion of the uniform regularity set of a harmonic function (see Definition 1.3). In Proposition 1.4 we then show that the regularity set of $f \in C^\infty(\Omega)$ can be described by the uniform regularity set of a harmonic representing function u_f for f. We thus can use the theory of boundary values of harmonic functions to study the extension of the regularity set of C^∞-functions.

Let P(D) always be a partial differential operator with constant coefficients in n variables. The extension of C^∞-regularity for solutions of P(D) has been characterized by Hörmander ([11], see also [12, section 11.31) using a sequence of distributional parametrices which are regular on sufficiently large sets. Correspondingly, in section 2 we will construct a sequence of regular generalized elementary solutions for P(D) (see Theorem 2.3). The elementary solutions are harmonic functions in $(n+1)$ variables defined outside thin strips near $\mathbb{R}^n$ and thus can be considered as generalized hyperfunctions.

By means of a suitable duality (see Lemma 3.1) the regular elementary solutions from section 2 are then used in section 3 to extend the uniform regularity set of harmonic functions (see Theorem 3.3; this is simiiar to the use of distributional parametrices with small C^∞ singular support to extend C^∞-regularity (see Hörmander [12, section 11.3])). The main result of this paper is given in Theorem 3.4, where we prove that the regularity set of $f \in$ C"(R) extends in cones with polynomial bounds on the regularity parameter L. This centra1 result

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is needed in the study of partial differential operators which are surjective on rea1 analytic functions (see Langenbruch [18]).

In section 4 we finally obtain as an easy consequence of Theorem 3.4 a Holmgren type theorem for the analytic wave front set of hyperfunction solutions of P(D) which essentially is a special case of Sjostrand [24, Theorem 15.11 and then prove some of its consequences. The extension of analytic regularity has been studied (usually for operators with variable coefficients) by many authors. A selection of corresponding papers is contained in the references (J.M. Bony [3, 4], J.M. Bony, P. Schapira [5], A. Grigis, P. Schapira, J. Sjostrand [6], N. Hanges [7], N. Hanges, J. Sjostrand [8], L. Hormander [10], M. Kashiwara, T. Kawai [14], P. Laubin [20, 21], 0. Liess [22, 23], J. Sjostrand [24], the reader is also referred to the literature cited in these papers).

1 Regularity sets

In this section hyperfunctions are introduced as forma1 boundary values of harmonic functions (Bengel [2], Hormander [12, chapter IX]). Correspondingly, we introduce the notion of the regularity set of C^∞–functions (see Definition 1.1) which is a quantitative decomposition of the complement of the analytic wave front set $WF_A(f)$ for $f \in C^\infty(\Omega)$ and which can be described by means of the uniform regularity set of a defining function u_f (see Proposition 1.4). The regular generalized elementary solution constructed in section 2 can thus be used to extend the complement of the analytic wave front set of zerosolutions in section 4.

In this paper, $n \in \mathbb{N}$ is always at least 2. A point in $\mathbb{R}^{n+1}$ is usually written as $(x,y) \in \mathbb{R}^n \times \mathbb{R}$. Open Euclidean balls in $\mathbb{R}^n$ are denoted by $U_\varepsilon(\xi)$ and $U_\varepsilon := U_\varepsilon(0)$. Let S^n be the Euclidean unit sphere in $\mathbb{R}^n$ and

$$\langle x,\xi\rangle := \sum x_j\xi_j \text{ for } x,\xi \in \mathbb{C}^k.$$

$\Delta = \sum_{k<n} (\partial/\partial x_k)^2 + (\partial/\partial y)^2$ is the Laplace operator on $\mathbb{R}^{n+1}$ and the harmonic functions on an open set $V \subset \mathbb{R}^{n+1}$ are denoted by $C_\Delta(V)$. For a subset $A \subset \mathbb{R}^{n+1}$, the space of harmonic germs near A is denoted by $C_\Delta(A)$. By $\widetilde{C}_\Delta$ we denote the corresponding spaces of harmonic functions which are even with respect to y.

In the following, Ω always is an open set in $\mathbb{R}^n$. As a definition of the hyperfunctions $\mathfrak{B}(\Omega)$ we set (see Bengel [2] and Hormander [12, chapter IX])

$$\mathfrak{B}(\Omega) := \widetilde{C}_\Delta\Big(\Omega \times (\mathbb{R}\setminus\{0\})\Big) \Big/ \widetilde{C}_\Delta(\Omega \times \mathbb{R})$$

The elements of $[u] \in \mathfrak{B}(\Omega)$ are called defining functions for $[u]$. Restrictions of a hyperfunction are defined via defining functions. For a closed set $S \subset \mathbb{R}^n$ let $A(S)$ denote the germs of rea1 analytic functions near S. For an analytic functional $T \in A(K)'$, $K \subset \mathbb{R}^n$ compact, we define a hyperfunction via the defining function

$$u_T(x,y) := \langle_\xi T, E(x-\xi,y)\rangle,\ (x,y) \in \mathbb{R}^{n+1} \setminus (K \times \{0\}) \tag{1.1}$$

where E is the canonica1 elementary solution of Δ defined by

$$E(x,y) := -|(x,y)|^{1-n} / \big((n-1)c_{n+1}\big)$$

(c_{n+1} is the area of the unit sphere $S^{n+1} \subset \mathbb{R}^{n+1}$, see e.g. Hormander [12, Theorem 3.3.2] and notice that $(n+1) \geq 3$). In this way $A(\mathbb{R}^n)'$ is embedded into the hyperfunctions and coincides with the hyperfunctions with compact support. Thus, also the distributions with compact support are embedded into hyperfunctions. More generally, $[u_T] \in \mathfrak{B}(\Omega)$ represents a distribution $T \in D(\Omega)'$ iff u_T can be extended to a distribution $\bar{u}_T \in D(\Omega \times \mathbb{R})'$ such that

$$\Delta \bar{u}_T = T \otimes \delta_y \tag{1.2}$$

(compare Langenbruch [16]). u_T is called a representing function for T.

To prepare the notion of the regularity sets we now introduce the class $A_{C,\Omega}$ which will serve as "analytic cut off functions" as in the theory of wave front sets for distributions (see e.g. Hormander [12, Lemma 8.4.4]). This class is defined as follows (for $\Omega \subset \mathbb{R}^n$ open and $C \geq 1$):

$$\begin{aligned} A_{C,\Omega} := \Big\{ (\varphi_k) \in D(\Omega)^{\mathbb{N}} \;\Big|\; & \forall d \in \mathbb{N}\; \exists C_d \geq 1 \forall k \in \mathbb{N} : \\ & \|\varphi_k^{(\alpha+\beta)}\|_\infty \leq C_d (kC)^{|\alpha|} \text{ if } |\alpha| \leq k \text{ and } |\beta| \leq d \Big\}. \end{aligned} \tag{1.3}$$

Some useful technical results follow: by Leibniz' formula we have

$$(\varphi_k h_k) \in A_{C+B,\Omega} \text{ if } (\varphi_k) \in A_{C,\Omega} \text{ and } (h_k) \in A_{B,\Omega}. \tag{1.4}$$

There is $B_1 > 0$ such that the following holds: for $K \subset\subset \Omega$ and $\delta := \operatorname{dist}(K, \partial\Omega)$

$$\text{there is } (\varphi_k) \in A_{B_1/\delta,\Omega} \text{ such that } \varphi_k = 1 \text{ near } K \text{ for each } k. \tag{1.5}$$

To see this, we set $\varphi_k := g_k * h$ where $h \in D(U_{\delta/4})$ satisfies $\int h(\xi)\, d\xi = 1$ and $g_k \in D(K + U_{3\delta/4})$ is chosen by Hormander [12, Theorem 1.4.2] (with $d_j = \delta/(8k)$ for $1 \leq j \leq k$) such that $g_k = 1$ near $K + \overline{U}_{\delta/2}$. h is needed to estimate the β-derivatives in (1.3).

The Fourier transforms of functions in $A_{C,\Omega}$ satisfy the following typical estimates: there is $B_2 \geq 1$ such that for $(\varphi_k) \in A_{C,U_1}$ we have

$$(1+|s|)^d |\hat{\varphi}_k(s)| \leq B_2 C_d \big(B_2 kC/(1+|s|)\big)^j \text{ if } j \leq k \text{ and } s \in \mathbb{R}^n. \tag{1.6}$$

One reason to include the β-derivatives in (1.3) is the fact that then $(\hat{\varphi}_k)$ is bounded in $L_1(\mathbb{R}^n)$ for $(\varphi_k) \in A_{C,\Omega}$ (see also the proof of Remark 1.2). Obviously, $(\psi_k) = (v)$ satisfies the estimates for $A_{L_1,\Omega}, L_1 > L_0$, if v satisfies the Cauchy estimates

$$|v^{(a)}(x)| \leq C(L_0|a|)^{|a|} \text{ on } \Omega. \tag{1.7}$$

We thus get by (1.4) and (1.6): there is $B_3 \geq 1$ such that

$$(1+|s|)^d |(\varphi_k v)\hat{}\,(s)| \leq C_d B_3 \big(B_3 k(L_0+C)/(1+|s|)\big)^k \text{ on } \mathbb{R}^n \tag{1.8}$$

if $(\varphi_k) \in A_{C,U_1}$ and v satisfies (1.7). This motivates the following definition of regularity sets for C^∞–functions which corresponds to an estimate like (1.8) on cones. This notion will also be used in the study of partial differential operators which are surjective on real analytic functions (Langenbruch [18]). For $\Theta \in S^n$ let

$$\Gamma_b(\Theta) := \{ s \in \mathbb{R}^n \mid |s/|s| - \Theta| < b \}.$$

Definition 1.1 *Let $\Omega \subset \mathbb{R}^n$ be open, $\Theta \in S^n$ and $L = (L_0, L_1, L_2) \in [1,\infty[^3$. Let $f \in C^\infty(\Omega)$.* We *say that* $\Omega \times \{\Theta\} \subset \operatorname{reg}_L(f)$ *iff for any* $C \geq 1$ and *any* $(\varphi_k) \in A_{C,\Omega}$ *there is* $C_1 \geq 1$ *such that*

$$|(f\varphi_k)\hat{\ }(s)| \leq C_1\left((L_0 + L_1 C)k/(1+|s|)\right)^k \text{ if } s \in \Gamma_{1/L_2}(\Theta). \tag{1.9}$$

Except for Theorem 2.3 below we will only use $L = (L_0, L_1) \in [1,\infty[^2$ and

$$\operatorname{reg}_{(L_0,L_1)}(f) := \operatorname{reg}_{(L_0,L_1,L_1)}(f)$$

in this paper.
Definition 1.1 is a quantitative version of the estimates needed to define the analytic wave front set, that is, $(x,\Theta) \notin WF_A(f)$ if there is $L \geq 1$ such that $U_{1/L}(x) \times \{\Theta\} \subset \operatorname{reg}_{(L,L)}(f)$ (Hormander [12, Lemma 8.4.4]).

If u and C are fixed in (1.9) and if $\operatorname{supp} \varphi_k \subset K \subset\subset \Omega$ for any k, the closed graph theorem implies that the constant

$$C_1 \text{ in (1.9) only depends on the sequences } (C_d) \text{ for } (\varphi_k) \text{ in (1.3).} \tag{1.10}$$

If $f \in C^\infty(\Omega)$ and $\Omega \times \{\Theta\} \subset \operatorname{reg}_L(f)$, then

$$\Omega \times \{\Theta\} \subset \operatorname{reg}_{\tilde{L}}(\partial_x^\beta f) \text{ if } L_0 < \tilde{L}_0 \text{ and } L_1 < \tilde{L}_1. \tag{1.11}$$

We must prove this only for the case that $\beta = e_j$ is a canonica1 unit vector. But then (1.11) easily follows from the product rule (notice that $(D_j\varphi_k) \in A_{C,\Omega}$ and $(\varphi_{k-1})_k \in A_{C,\Omega}$ if $(\varphi_k) \in A_{C,\Omega}$).

In the calculations with the cones $\Gamma_b(\Theta)$ we will often use the following fact: let $0 < b \leq 1$. Then

$$s \in \Gamma_b(\Theta) \text{ if } \xi \in \Gamma_{b/2}(\Theta) \text{ and } |\xi - s| < b|\xi|/4. \tag{1.12}$$

In fact,
$|s/|s| - \Theta| \leq |s/|s| - s/|\xi|| + |s-\xi|/|\xi| + |\xi/|\xi| - \Theta| < ||\xi| - |s||/|\xi| + 3b/4 < b.$

Remark 1.2 There is $B_4 \geq 1$ such that the following holds:

a) If $(\varphi_k) \in A_{C,U_1}$ with $\sup_k ||\varphi_k|| =: C_0 < \infty$ and if $(v_k) \in D(\mathbb{R}^n)^{\mathbb{N}}$ satisfies

$$\sup_k ||v_k||_1 = C_1 < \infty \text{ and } |\hat{v}_k(s)| \leq \left(L_0 k/(1+|s|)\right)^k \text{ if } s \in \Gamma_{1/L_1}(\Theta)$$

then

$$\left|(\varphi_k v_k)\hat{\ }(s)\right| \leq (C_0 + C_1)\left((2L_0 + B_4 L_1 C)k/(1+|s|)\right)^k \text{ if } s \in \Gamma_{1/(2L_1)}(\Theta).$$

b) If for $f \in C^\infty(\mathbb{R}^n)$ there is $\{f_k \; k \in \mathbb{N}\}$ bounded in $D(U_1)$ such that $f_k(x) = f(x)$ for $x \in \Omega \subset U_1$ and

$$|\hat{f}_k(s)| \leq \left(L_0 k/(1+|s|)\right)^k \text{ if } s \in \Gamma_{1/L_1}(\Theta),$$

then $\Omega \times \{\Theta\} \subset \operatorname{reg}_{(2L_0, B_4 L_1)}(f)$.

Proof. Using (1.12) we get the following estimate for cp, $v \in D(\mathbb{R}^n)$, $k \in \mathbb{N}$, $b \in]0,1]$ and $s \in \Gamma_{b/2}(\Theta)$ (by Hörmander [12, (8.1.3')] with $M = 0$, $C = \|v\|_1$ and $c = b/4$):

$$(1+|s|)^k |(\varphi v)^\frown(s)| \leq 2^k \|\widehat{\varphi}\|_1 \sup\{ |\widehat{v}(\eta)|(1+|\eta|)^k \quad \eta \in \Gamma_b(\Theta)\} \\ +(5/b)^k \|v\|_1 \int |\widehat{\varphi}(\eta)|(1+|\eta|)^k \, d\eta. \tag{1.13}$$

a) This follows from (1.6) and (1.13) (with $B_4 = 5B_2$).

b) This directly follows from a) and Definition 1.1. □

We finally show the basic fact that the regularity set of a C^∞–function f can be characterized by a uniform regularity estimate (1.9) valid for any defining function u_f (see (1.2)). Of course, the wave front set of u_f is void since u_f is rea1 analytic. We introduce the appropriate notion: let

$$BC_\Delta(\Omega \times (\mathbb{R} \setminus \{0\})) := \Big\{ u \in C_\Delta(\Omega \times (\mathbb{R}\setminus\{0\})) \quad \forall K \subset\subset \Omega, a \in \mathbb{N}_0^n :$$

$$\sup\{ |\partial_y^d \partial_x^a u(x,y)| \quad x \in K, 0 < |y| \leq 1, d = 1,2\} < \infty \Big\}.$$

Definition 1.3 *Let $L \in [1,\infty[^2$ and let $u \in BC_\Delta(\Omega \times (\mathbb{R} \setminus \{0\}))$. We say that $\Omega \times \{\Theta\} \subset$ UReg $_L(u)$ iff for any $C \geq 1$ und any $(\varphi_k) \in A_{C,\Omega}$ there is $C_1 \geq 1$ such that for $d = 0, 1$*

$$\left|(\partial_y^d u(\ ,y)\varphi_k)^\frown(s)\right| \leq C_1\left((L_0 + L_1 C)k/(1+|s|)\right)^k \\ \text{if } s \in \Gamma_{1/L_1}(0) \text{ und } 0 < |y| \leq 1/L_1. \tag{1.14}$$

Proposition 1.4 *Let $f \in C^\infty(\Omega)$ and let u_f be a defining function of f.*

a) $u_f \in BC_\Delta(\Omega \times (\mathbb{R} \setminus \{0\}))$

b) There is $B_5 \geq 1$ such that the following holds:

i) Let $\omega \subset \mathbb{R}^n$ be open und $\omega + U_\varepsilon \subset\subset \Omega$. If $\Omega \times \{\Theta\} \subset$ reg $_L(f)$, then $\omega \times \{\Theta\} \subset$ UReg $_{B_5(L_0+1/\varepsilon, L_1)}(u_f)$

ii) If $\Omega \times \{\Theta\} \subset$ UReg $_L(u_f)$, then $\Omega \times \{\Theta\} \subset$ reg $_L(f)$.

Proof. a) To prove this we can assume that $f \in D(\Omega)$ and that $u_f = E * f$. One easily sees that for any $K \subset\subset \mathbb{R}^n$ there is $C_1 \geq 1$ such that

$$\int_K \left|\partial_y^d E(x,y)\right| dx \leq C_1 \text{ for } 0 < |y| \leq 1. \tag{1.15}$$

This implies the claim.

b)i) Let $(\varphi_k) \in A_{C,\omega}$. Choose $\psi \in D(Q)$ such that $\psi \equiv 1$ on $\omega_1 := \omega + U_{\varepsilon/2}$. Let $U_{\psi f}$ be the representing function of ψf defined by (1.1). Then

$$u_f = U_{\psi f} + v \text{ on } \omega_1 \times \mathbb{R} \text{ for some } v \in C_\Delta(\omega_1 \times \mathbb{R}).$$

Since the Laplacean is elliptic, there is $B \geq 1$ such that

$$\left|\partial_y^d \partial_x^a v(x,y)\right| \leq C_2 (B|a|/\varepsilon)^{|a|} \text{ for } (x,y) \in \omega \times [-1,1]. \tag{1.16}$$

$(\varphi_k v(\ ,\ y))\hat{}$ thus satisfies the required estimates by (1.8).

To prove (1.14) for $U_{\psi f}$ we choose two sequences of functions $(g_k), (h_k) \in A_{8B_1,\mathbb{R}^n}$ in the following way by (1.5): $g_k(x) \equiv 1$ for $|x| \leq \varepsilon/8$ and supp $(g_k) \subset U_{\varepsilon/4}$, $h_k(x) \equiv 1$ on $\omega -$ supp (ψ) and supp $(h_k) \subset K := \omega -$ supp $\psi + U_1$.

For $d = 0, 1, y \neq 0$ and $x \in \omega$ we then have

$$\partial_y^d U_{\psi f}(x,y) = (\psi f) * \left(g_k \partial_y^d E(\ ,y)\right)(x) + \psi f * \left((1-g_k) h_k \partial_y^d E(\ ,y)\right)(x). \tag{1.17}$$

Since E satisfies (1.16) (with new B) for $|x| \geq \varepsilon/8$, $x \in K$, and $|y| \leq 1$, we get

$$\begin{aligned} &\left(f\psi * \left((1-g_k)h_k \partial_y^d E(\ ,y)\right)\hat{}(s)\right. \\ &\leq \|f\psi\|_1 \left|\left((1-g_k)h_k\partial_y^d E(\ ,y)\right)\hat{}(s)\right| \leq C_3\Big(B_5 k/\left(\varepsilon(1+|s|)\right)\Big)^k \end{aligned} \tag{1.18}$$

for some $B_5 \geq 1$ by (1.8). Since the last term in (1.17) is bounded in $D(\mathbb{R}^n)$ (uniformly in y), (1.14) follows for this term by (1.18) and Remark 1.2. Since $\Omega \times \{\Theta\} \subset \operatorname{reg}_L(f)$, we get for $s \in \Gamma_{1/L_1}(0)$ and $0 < |y| \leq 1$ by (1.15) and (1.10)

$$\begin{aligned} \left|\left(\varphi_k\left[(\psi f) * g_k\partial_y^d E(\ ,y)\right]\right)\hat{}(s)\right| &\leq \left\|g_k \partial_y^d E(\ ,y)\right\|_1 \sup_{|\xi|\leq\varepsilon/4}\left|(\varphi_k(\cdot+\xi)f)\hat{}(s)\right| \\ &\leq C_1\left((L_0+L_1 C)k/(1+|s|)\right)^k. \end{aligned} \tag{1.19}$$

ii) Let (1.14) hold for u_f. Since u_f satisfies (1.2), the distributional boundary value of $\partial_y u_f$ is f by Langenbruch [16, Satz 1.21. Since u_f is even w.r.t. y, this means that for $(\varphi_k) \in A_{C,\Omega}$

$$\begin{aligned} |(f\varphi_k)\hat{}(s)| &= |\langle f, \varphi_k e^{-i\langle\ ,s\rangle}\rangle| = 2\lim_{y\to 0}|\langle\partial_y u_f(\ ,y),\ \varphi_k e^{-i\langle\ ,s\rangle}\rangle \\ &= 2\lim_{y\to 0}\ \left(\partial_y u_f(\ ,y)\varphi_k\right)\hat{}(s)\Big| \leq C_1\left((L_0+L_1C)k/(1+|s|)\right)^k \text{ if } s \in \Gamma_{1/L_1}(\Theta) \end{aligned}$$

by (1.14). The proposition is proved. □

2 Regular elementary solutions

In the remaining part of this paper P(D) = $P(D_x)$ always is a partial differential operator in n (x -) variables with constant coefficients and degree m. P_m denotes the principal part of P. Also, Θ and N are always vectors in the unit sphere $S^n \subset \mathbb{R}^n$.

To show that the regularity set of harmonic zerosolutions of $P(D_x)$ extends in certain directions we need to construct (generalized) elementary solutions for $P(D_x)$ which have large regular sets. This construction is given in this section. The elementary solutions will be defined in the space $\widetilde{C}_\Delta\left(\Omega \times (\mathbb{R} \setminus [-c,c])\right)$, c > 0, which can be considered as defining functions of a sheaf more general than hyperfunctions (these correspond to the case c = 0). $E \in \widetilde{C}_\Delta\left(\Omega \times (\mathbb{R}\setminus[-c,c])\right)$ is canonically written as $E(x,y) = E_+(x,|y|)$ with $E_+ \in C_\Delta(\Omega \times]c, \infty[)$. The appropriate notion of an elementary solution for P(D) on Ω now is the following (compare the embedding of distributions into hyperfunctions in (1.2)):

Definition 2.1 *Let $0 \in \Omega$. $E \in \widetilde{C}_\Delta\left(\Omega \times (\mathbb{R} \setminus [-c,c])\right)$ is called an elementary solution for P(D) on Ω if $P(D)E$ can be extended to $\Omega \times \mathbb{R}$ as a distribution H such that $\Delta H = \delta$.*

The existence of regular elementary solutions can be shown if there are sufficiently large regions in $\mathbb{C}^n$ where $P(z)$ does not vanish. This can be proved under weak assumptions (see Lemma 2.2 below).

For $P_m(0) = 0$ let $P_{m,\Theta}$ be the localization of P_m at Θ defined as follows: let

$$q_\Theta := \min\{k \in \mathbb{N} \ \exists\ \beta \in \mathbb{N}_0^n : |\beta| = k \text{ and } D^\beta P_m(0) \neq 0\}$$

be the order of the root Θ of P_m. Now,

$$P_{m,\Theta}(\xi) := \sum_{|\alpha|=q_\Theta} P_m^{(\alpha)}(\Theta)\,\xi^\alpha/\alpha!. \tag{2.1}$$

Alternatively,

$$P_{m,\Theta}(x) = \lim_{s\longrightarrow 0} \left(P_m(\Theta+sx)s^{-q_\Theta}\right), \tag{2.1'}$$

where s^{q_Θ} is the lowest order term of the expansion of $P_m(\Theta + sx)$. For $\Theta = e_1$ this means that

$$P_m(x) = P_{m,\Theta}(x')x_1^{m-q_\Theta} + \sum_{0\leq k<m-q_\Theta} Q_k(x')x_1^k \tag{2.2}$$

if $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$ where the Q_k are homogeneous polynomials and $Q_k = 0$ or $\deg(Q_k) = m-k$. Let

$$\widetilde{P}(x,t) := \sup\{\,|P(x+\xi)| \ \ |\xi| \leq t\} \text{ and } \widetilde{P}_{\langle N\rangle}(x,t) := \sup\{\,|P(x+\tau N)| \ \ |\tau| \leq t\}.$$

It is well-known (Hörmander [12, Lemma 10.4.2]) that there is $C \geq 1$ such that

$$\widetilde{P}(x,t) \leq \widetilde{P}(x,ts) \leq C\widetilde{P}(x,t)s^m \text{ for any } t \geq 0 \text{ and } s \geq 1 \tag{2.3}$$

and this also holds for $\widetilde{P}_{\langle N\rangle}$.

By means of a linear change of coordinates we will mainly be concerned with the standard case $\Theta = e_1$ and $N = e_n$ and then write $x = (x_1, x'', x_n) \in \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R}$. We will finally need the following unsymmetric cones $\widetilde{\Gamma}_t(\rho, \varkappa)$ for $\rho \geq 1$ and $0 < \varkappa \leq 1$: for $1 \geq t > 0$ let

$$\begin{aligned}\widetilde{\Gamma}_t(\rho,\varkappa) &:= \left\{\,\xi \in \mathbb{R}^n \quad \left|\left(\xi_1 - |\xi|_\infty, \xi'', \rho t^{\varkappa}\xi_n\right)\right|_\infty < t|\xi|_\infty\right\}\\ &= \left\{\,\xi \in \mathbb{R}^n \ \ \xi_1 = |\xi|_\infty,\ |\xi''|_\infty < t|\xi_1|,\ |\xi_n| < t^{1-\varkappa}|\xi_1|/\rho\right\}.\end{aligned}$$

To see this equality we notice that the second set is obviously contained in the first, and the opposite inclusion follows since the assumptions $|\xi|_\infty = |\xi''|_\infty$, $|\xi|_\infty = |\xi_n|$ and $|\xi|_\infty = -\xi_1$ directly lead to obvious contradictions. Except for Theorem 2.3 we will always have $\varkappa = 1$ in this paper.

Lemma 2.2 *Let $P_{m,e_1}(e_n) \neq 0$. There is $\rho \geq 1$ such that for any $\lambda \geq 1$ there are $b > 0$ and $0 < \gamma \leq 1$ such that for any $0 < t \leq 1/(\rho\lambda)$ there is $C = C(t) \geq 1$ such that for any $\xi \in \widetilde{\Gamma}_{\lambda t}(\rho, 1)$ with $|\xi| \geq C$ there is $\vartheta \in \mathbb{R}$ with $|\vartheta| \leq t|\xi|/2$ such that*

$$\left|P\left(\xi + (it|\xi| + z\vartheta)e_n + \zeta\right)\right| \geq b\widetilde{P}(\xi, t|\xi|) \tag{2.4}$$

for any $z \in \mathbb{C}$ with $|z| = 1$ and any $\zeta \in \mathbb{C}^n$ with $|\zeta| < 2\gamma t|\xi|$.

Proof. We will show that there is $\rho \geq 1$ such that for any $\lambda \geq 1$ there is $b_1 \geq 1$ such that for any $0 < \delta \leq 1/(\rho\lambda)$

$$\widetilde{P}(\xi, t|\xi|) \leq b_1 \widetilde{P}_{\langle e_n \rangle}(\xi, t|\xi|) \text{ if } \xi \in \widetilde{\Gamma}_{\lambda t}(\rho, 1) \text{ and } |\xi| \geq C(t). \tag{2.5}$$

This implies the claim by Hormander [12, Lemma 11.3. 10] (with the constant $\varkappa$ in loc. cit. chosen as $1/2$, $V' = (e_n)$ and $\eta^0 = e_n$). To show (2.5) we use the form (2.2) for P_m. In the proof below the constants A_k can be chosen independently of λ. Let $\Theta := e_1$.

i) There are $\rho \geq 1$ such that for any $\lambda \geq 1$ there is $b_2 \geq 1$ such that

$$(P_m)\widetilde{\ }((1, \eta'', \tau), t) \leq b_2 (P_m)\widetilde{\ }_{\langle e_n \rangle}((1, \eta'', \tau), t) \tag{2.6}$$

if $|\tau| \leq 1/\rho$, $|\eta''|_\infty \leq \lambda t$ and $0 < t \leq 1/(\rho\lambda)$.

Proof. Let $\eta'' \in \mathbb{R}^{n-2}$ with $|\eta''|_\infty \leq \lambda t \leq 1/2$ and $\tau \in \mathbb{R}$ with $|\tau| \leq 1/2$. Then by (2.2)

$$\begin{aligned}(P_m)\widetilde{\ }((1, \eta'', \tau), t) &\leq (P_{m,\Theta})\widetilde{\ }((\eta'', \tau), t) 2^{m - q_\Theta} + \textstyle\sum_{k < m - q_\Theta} \widetilde{Q}_k((\eta'', \tau), t) 2^k \\ &\leq A_1 \textstyle\sum_{k \leq m - q_\Theta} \left((\lambda + 1)t + |\tau|\right)^{m-k} \leq A_2 (\lambda t + |\tau|)^{q_\Theta}.\end{aligned} \tag{2.7}$$

For $1/2 \geq \mu t \geq \lambda t$ we get similarly using (2.3) first

$$\begin{aligned}(P_m)\widetilde{\ }_{\langle e_n \rangle}((1, \eta'', \tau), t) &\geq C_\mu (P_m)\widetilde{\ }_{\langle e_n \rangle}((1, \eta'', \tau), \mu t) \\ &\geq C_\mu (P_{m,\Theta})\widetilde{\ }_{\langle e_n \rangle}((\eta'', \tau), \mu t) - C_\mu A_3 (\mu t + |\tau|)^{q_\Theta + 1}.\end{aligned} \tag{2.8}$$

We have

$$P_{m,\Theta}(x'', y_n) = \sum_{j \leq q_\Theta} H_j(x'') x_n^j$$

where $H_{q_\Theta}(x'') \equiv c \neq 0$, H_j are homogeneous polynomials and $H_j = 0$ or $\deg(H_j) = q_\Theta - j$. This shows that for $\mu \geq \lambda$

$$\begin{aligned}(P_{m,\Theta})\widetilde{\ }_{\langle e_n \rangle}((\eta'', \tau), \mu t) &\geq \left|P_{m,\Theta}(\eta'', \tau + \operatorname{sgn}(\tau)\mu t)\right| \\ &\geq c\left(|\tau| + \mu t\right)^{q_\Theta} - A_4 \lambda t \left(\mu t + |\tau|\right)^{q_\Theta - 1} \geq c\left(|\tau| + \mu t\right)^{q_\Theta}/2\end{aligned} \tag{2.9}$$

if also $\mu \geq 2A_4\lambda/c$. We now fix $\mu := \max(1, 2A_4/c)\lambda$ and get by (2.8) and (2.9)

$$(P_m)\widetilde{\ }_{\langle e_n \rangle}((1, \eta'', \tau), t) \geq C_\mu c\left(|\tau| + \lambda t\right)^{q_\Theta}/4$$

if $|\tau| + \mu t \leq c/(4A_3)$. Together with (2.7) this shows (2.6).

ii) Let $P = \sum P_k$ be the expansion of P in homogeneous polynomials. For ρ from (2.6) and $\xi \in \widetilde{\Gamma}_{\lambda t}(\rho, 1)$ we have $\zeta := \xi/|\xi|_\infty = (1, \xi''/|\xi|_\infty, \xi_n/|\xi|_\infty)$ with $|\xi''|_\infty/|\xi|_\infty < \lambda t$ and $|\xi_n|/|\xi|_\infty < 1/\rho$ by the definition of $\widetilde{\Gamma}_{\lambda t}(\rho, 1)$. We can thus apply (2.6) if $0 < t \leq 1/(\rho\lambda)$ and get (using also (2.3))

$$\begin{aligned}\widetilde{P}(\xi, t|\xi|)/(Cn^{m/2}) &\leq (P_m)\widetilde{\ }(\xi, t|\xi|_\infty) + \textstyle\sum_{k<m} (P_k)\widetilde{\ }(\xi, t|\xi|_\infty) \\ &\leq |\xi|_\infty^m (P_m)\widetilde{\ }(\zeta, t) + A_6 |\xi|_\infty^{m-1} \leq C_1 |\xi|_\infty^m (P_m)\widetilde{\ }_{\langle e_n \rangle}(\zeta, t) + A_6 |\xi|_\infty^{m-1} \\ &\leq C_1 \widetilde{P}_{\langle e_n \rangle}(\xi, t|\xi|) + A_7 |\xi|_\infty^{m-1} \text{ for } \xi \in \widetilde{\Gamma}_{\lambda t}(\rho, 1).\end{aligned}$$

This shows (2.5) since $\widetilde{P}(\xi, t|\xi|) \geq A_8 (t|\xi|_\infty)^m$. □

The following Theorem 2.3 now states the existence of appropriately regular (generalized) elementary solutions. It is formulated only for $\Theta = e_1$ and $N = e_n$. We have included parameter dependent polynomials P_t for later purposes. In this paper we will only use the case where P_t is independent of t.

The claims in Theorem 2.3 i) and ii) are stated in the form needed to prove the main extension result for the regularity set in section 3 (see Theorem 3.3). There we will need simultaneous estimates as satisfied by the following regular cut off functions

$$B_{C,\Omega} := \{ (\varphi_{k,\nu}) \in D(\Omega)^{\mathbb{N}\times\mathbb{N}} \quad \forall d \in \mathbb{N}\ \exists\, C_d \geq 1 \forall k,\nu \in \mathbb{N} :$$

$$\|\varphi_{k,\nu}^{(\alpha+\gamma+\beta)}\|_\infty \leq C_d (kC)^{|\alpha|}(\nu C)^{|\gamma|} \text{ if } |\alpha| \leq k,\ |\gamma| \leq \nu \text{ and } |\beta| \leq d\}$$

and also the following unisotropic variant (for $I \geq 1$) :

$$\widetilde{B}_{C,\Omega}(I) := \{ (\varphi_{k,\nu}) \in D(\Omega)^{\mathbb{N}\times\mathbb{N}} \quad \forall d \in \mathbb{N}\ \exists\, C_d \geq 1 \forall k,\nu \in \mathbb{N} :$$

$$\|\varphi_{k,\nu}^{(\alpha+\gamma+\beta)}\|_\infty \leq D_d (kC)^{|\gamma|}(\nu C)^{|\alpha|} I^{\alpha_n+\gamma_n} \text{ if } |\alpha| \leq k,\ |\gamma| \leq \nu \text{ and } |\beta| \leq d\}.$$

The following Paley-Wiener estimates hold (compare (1.6)): there exists $B_2 \geq 1$ such that $(\varphi_{k,\nu}) \in \widetilde{B}_{C,U_\varepsilon}(I)$ satisfies

$$|\widehat{\varphi}_{k,\nu}(z)| \leq C_0 e^{\varepsilon|\operatorname{Im} z|}\left(B_2 Ck/(1+|z|^{\sim})\right)^k \text{ for } z \in \mathbb{C}^n \tag{2.10}$$

and

$$|\widehat{\varphi}_{k,\nu}(z)| \leq C_0 e^{\varepsilon|\operatorname{Im} z|}\left(B_2 Ck/(1+|z|^{\sim})\right)^k \left(B_2 C\nu/(1+|z|^{\sim})\right)^\nu \text{ for } z \in \mathbb{C}^n \tag{2.10'}$$

where $|z|^{\sim} := |(z', z_n/I)|$. Let $W_\varepsilon := \{ (x', x_n) \in \mathbb{R}^{n-1}\times\mathbb{R} \quad |x'| < \varepsilon, |x_n| < \varepsilon \}$.

Theorem 2.3 *There exists $A_1 \geq 1$ such that the following holds for any polynomial $0 \neq P_t$ in n variables with $\deg P_t \leq m$: assume that there are $\rho \geq 1$ and $0 < \varkappa \leq 1$ such that for any $\lambda \geq 1$ there are $b > 0$, $0 < \delta \leq 1$ and $0 < \gamma \leq 1$ such that for any $0 < t \leq \delta$ there is $C \geq 1$ such thatfor any $\xi \in \widetilde{\Gamma}_{\lambda t}(\rho, \varkappa)$ with $|\xi| \geq C$ there is 6 $\in$ IR with $|\vartheta| \leq t|\xi|/2$ such that*

$$\left|P_t\left(\xi + (it|\xi| + z\vartheta)e_n + \zeta\right)\right. \geq b\widetilde{P}_t(\xi, t|\xi|) \tag{2.11}$$

for any $z \in \mathbb{C}$ with $|z| = 1$ und any $\zeta \in \mathbb{C}^n$ with $|\zeta| < 2\gamma t|\xi|$.

Then there are $A_2, A_3 \geq 1$ such thatfor any $L_2 \geq \rho$, $0 < \varepsilon \leq 1$ and $0 < t \leq 1/A_2$ there is un elementary solution $E = E_{\varepsilon,t,L_2} \in \widetilde{C}_\Delta\left(W_{2\varepsilon A_3} \times (\mathbb{R}\setminus[-T/2, T/2])\right)$, $T := 64A_3\varepsilon t$, for $P_t(D)$ such that E can be written as $E = F + G$ with $F, G \in \widetilde{C}_\Delta(W_{2\varepsilon A_3} \times (IR\setminus[-T/2, T/2]))$, where G_+ can be extended as a harmonic function to

$$X_\varepsilon := \{ (x,y) \in W_{2\varepsilon A_3} \times \mathbb{R} \qquad (|x'| > A_3\varepsilon, |x_n| < \varepsilon, y > -\varepsilon t \text{ or } (x_n \geq 0, y > -x_n t/8) \}.$$

Moreover, we have the following estimates (for $T := 64A_3\varepsilon t$):

i) If $\omega \subset W_{2\varepsilon A_3}$ *und if* $\omega \times \{e_1\} \subset \mathrm{reg}_L(h)$ *for* $L = (L_0, L_1, L_2)$ *und* $h \in C^\infty(\omega)$, *then*

$$\sum_\nu \left| \left((\partial_y^{\nu+d} G_+(\,,T) * \varphi_{k,\nu} h)\psi_{\nu,k}\right)\hat{}\,(s)\right| T^\nu/\nu! \\ \le C\big(k/(T(1+|s|))\big)^k \text{ for } s \in \mathbb{R}^n \text{ and } d = 0,1$$

if $(\varphi_{k,\nu}) \in B_{K_1,\omega}, (\psi_{k,\nu}) \in B_{K_1,W_{2\varepsilon A_3}}$, $K_1 \ge 1/\varepsilon$ *and* $t \le \min\{1/(A_1L_2), 1/(\varepsilon A_3 A_1(L_0+K_1L_1))\}$.

ii) For any $(\psi_{k,\nu}) \in \widetilde{B}_{16B_1/(\varepsilon A_3),W_{2\varepsilon A_3}}(A_3 I)$, $I \ge 1$, *with* B_1 *taken from (1.5) and uny bounded set* $B \subset D(W_{2\varepsilon A_3})$ *there is* $C \ge 1$ *such thutfor any* $g \in B$

$$\sup_{g\in B} \sum_\nu \left| \left((\partial_y^{\nu+d} F_+(\,,T) * P_t(D)g)\psi_{\nu,k}\right)\hat{}\,(s)\right| T^\nu/\nu! \le C\Big(k/\big(T(1+|s|)\big)\Big)^k \\ \text{if } s \in \Gamma_t(e_1),\ d=0,1 \text{ and if } t \le 1/\big(A_1(L_2A_3I)^{1/\varkappa}\big) \tag{2.12}$$

Proof. 1) The definition and the properties of G+ are prepared in a) – c):

a) Fix $\lambda, \rho, \varkappa, \delta, b$ and γ as above and let $L_2 \ge \rho$. Let

$$\widehat{\Gamma}_{\lambda t} := \widetilde{\Gamma}_{\lambda t}(4n^{1/2}L_2,\varkappa) = \Big\{\xi = (\xi_1,\xi'',\xi_n) \in \mathbb{R}^n$$

$$|\xi_1| = |\xi|_\infty, |\xi''|_\infty < \lambda t|\xi_1|, |\xi_n| < (\lambda t)^{1-\varkappa}|\xi_1|/(4n^{1/2}L_2)\Big\}$$

Let $0 < t \le \delta$ and $0 < \varepsilon \le 1$. In the proof below the constants A_k are independent of ε, t and L_2 but may depend on $\lambda, \rho, \varkappa, \delta$, b and γ.

There are $A_k \ge 1$ such that for any $1 \ge \varepsilon > 0$ and any $0 < t < 1/(2\gamma)$ there are $j_0 \in$ N, C^∞–functions $\{\chi_j\}$ and points $\xi_j \in \mathbb{R}^n$ such that $\sum \chi_j = 1$ on $\mathbb{R}^n \setminus U_1$ and

$$\operatorname{supp}\chi_j \subset B_j := \big\{\xi \;\big|\; |\xi-\xi_j| \le \gamma t_j\big\} \subset \mathbb{R}^n \setminus U_4 \text{ if } j \ge j_0\ (t_j := t|\xi_j|), \tag{2.13}$$

the intersection of more than A_1 balls B_j is empty and

$$|D^\alpha \chi_j(\xi)| \le A_2^{|\alpha|+1}\varepsilon^{|\alpha|}, \text{ if } |\alpha| \le \varepsilon t_j. \tag{2.14}$$

This is proved similarly as Hormander [12, Lemma 11.3.1 1] by application of Hormander [12, Theorem 1.4.10] to $\|\ \|_y := |/(\gamma t|y|)$ which is a uniformly slowly varying metric on $\mathbb{R}^n \setminus \{0\}$.

With C = $C(t)$ from (2.11) let

$$J := \{j \ge j_0 \quad \operatorname{supp}\chi_j \subset \{x \in \widehat{\Gamma}_{\lambda t} \quad |x| \ge C\}\}.$$

From now on let always $j \in$ J. $\widehat{\Gamma}_{\lambda t}$ is contained in $\widetilde{\Gamma}_{\lambda t}(\rho,\varkappa)$ since $L_2 \ge \rho$. For $0 < t \le 1/(2\gamma)$ we can thus choose ϑ_j for ξ_j by (2.11) with $|\vartheta_j| \le t|\xi_j|/2 = t_j/2$. For $x \in \mathbb{C}^n$ we set

$$Q(x) := (x,x) = |\operatorname{Re} x|^2 - |\operatorname{Im} x|^2 + 2i\langle \operatorname{Re} x, \operatorname{Im} x\rangle. \tag{2.15}$$

$|\xi|$ can be extended by $(Q(\xi))^{1/2}$ as a holomorphic function on

$$W := \{\xi \in \mathbb{C}^n \mid |\operatorname{Re}\xi| > |\operatorname{Im}\xi|\}$$

since Re $Q(\xi) > 0$ for $\xi \in W$ by (2.15). We will denote the extension of $|\xi|$ by $\langle\xi\rangle$. For $(x, y) \in \mathbb{R}^{n+1}$ we want to define

$$u_j(x,y) = (2\pi)^{-n} \int_{|z|=1} \int \chi_j(\xi) \exp\left(i\langle x, \zeta(\xi) + z\vartheta_j e_n\rangle - y\langle\zeta(\xi) + z\vartheta_j e_n\rangle\right) \times \\ \times 1/\left(P_t(\zeta(\xi) + z\vartheta_j e_n)\langle\zeta(\xi) + z\vartheta_j e_n\rangle\right) d\zeta\, dz/(4\pi i z) \tag{2.16}$$

with $\zeta(\xi) = \xi + it|\xi|e_n$ and $d\zeta = (1 + it\xi_n/|\xi|)d\xi$.

When proving the existence and estimates for (2.16) we will consider also complex ξ in the integrand. This is needed in part b) of this proof. Let

$$D_j := \{\xi \in \mathbb{C}^n \mid |\xi - \xi_j| \leq 3\gamma t_j/2\}.$$

D_j is contained in W for $t < 1/3$ since

$$|\operatorname{Re}\xi| \geq |\xi_j| - |\xi - \xi_j| \geq |\xi_j|(1 - 3t/2) > 3t|\xi_j|/2 \geq |\xi - \xi_j| \geq |\operatorname{Im}\xi|$$

for $\xi \in D_j$. Hence $\langle\xi\rangle$ and $\zeta(\xi)$ are defined and holomorphic on D_j.

For $0 < t \leq \tau_1 := 1/12$ and $\xi \in D_j$ we have

$$|Q(\xi_j) - Q(\xi)| = |2\langle\xi_j, \xi_j - \xi\rangle - Q(\xi - \xi_j)| \leq (3\gamma t + (3\gamma t/2)^2)|\xi_j|^2 < \gamma|\xi_j|^2/2$$

and thus for $\tau \in [Q(\xi_j), Q(\xi)] := \operatorname{conv}\left(Q(\xi_j), Q(\xi)\right)$

$$|\tau| \geq |\xi_j|^2 - |Q(\xi) - Q(\xi_j)| \geq |\xi_j|^2/4.$$

Since Re $(Q(\xi)) > 0$ for $\xi \in D_j \subset W$, this implies for $\xi \in D_j$ and $0 < t \leq \tau_1$:

$$|\langle\xi\rangle - |\xi_j|| \leq |Q(\xi) - Q(\xi_j)| \sup\{|\tau|^{-1/2}/2 \mid \eta \in [Q(\xi_j), Q(\xi)]\} < \gamma|\xi_j|/2 \tag{2.17}$$

and thus

$$|\zeta(\xi) - \xi_j - it|\xi_j|e_n| \leq |\xi - \xi_j| + t|\langle\xi\rangle - |\xi_j|| < 2\gamma t|\xi_j|. \tag{2.18}$$

By (2.11) and (2.13) we thus have for $\xi \in D_j$

$$|P_t(\zeta(\xi) + z\vartheta_j e_n)| \geq b\widetilde{P}_t(\xi_j, t|\xi_j|) \geq C_1(t|\xi_j|)^{\deg P_t} \geq C_1 t. \tag{2.19}$$

$\left(\zeta(\xi) + z\vartheta_j e_n\right) \in W$ for $\xi \in D_j, |z| \leq 1$ and $0 < t \leq \tau_1$ since by (2.17)

$$\begin{aligned} |\operatorname{Re}\left(\zeta(\xi) + z\vartheta_j e_n\right)| &= |\operatorname{Re}\xi + \operatorname{Re}(it\langle\xi\rangle)e_n + \vartheta_j e_n \operatorname{Re} z| \\ &\geq |\xi_j| - |\xi - \xi_j| - t|\langle\xi\rangle| - t|\xi_j| \geq \left(1 - t(3/2 + 2\gamma)\right)|\xi_j| \\ &> t\left(3/2 + 2\gamma\right)|\xi_j| \geq t\left(|\langle\xi\rangle| + |\xi_j|/2\right) \geq |\operatorname{Im}\left(\zeta(\xi) + z\vartheta_j e_n\right)|. \end{aligned}$$

Thus $\langle\zeta(\xi) + z\vartheta_j e_n\rangle$ is defined and holomorphic on D_j. Since Re $Q(\eta) > 0$ for $\eta \in W$, we also have

$$|\operatorname{Im}\langle\zeta(\xi) + z\vartheta_j e_n\rangle| \leq \operatorname{Re}\langle\zeta(\xi) + z\vartheta_j e_n\rangle$$

and since $|\xi_j| \le 2|\xi|$ for $\xi \in D_j$, *we* get by (2.17) for $\xi \in D_j$

$$\begin{aligned}\text{Re}\,\langle\zeta(\xi)+z\vartheta_j e_n\rangle &\ge 2^{-1/2}\left|\langle\zeta(\xi)+z\vartheta_j e_n\rangle\right. \\ &\ge 2^{-1/2}\big(|\xi|-t|\langle\xi\rangle|-t|\xi_j|/2\big) \ge |\xi|/4.\end{aligned} \tag{2.20}$$

By (2.19) and (2.20) the denominator in (2.16) is bounded from below near supp χ_j by (2.13). u_j is thus defined and infinitely differentiable on $\mathbb{R}^{n+1}$. Obviously,

$$u_j \in C_\Delta(\mathbb{R}^{n+1}). \tag{2.21}$$

For $\xi \in \text{supp}\,\chi_j$ CC D_j we have by (2.20) (since then $|\xi_j| \le 12|\xi|/11$ for $t \le \tau_1$):

$$\begin{aligned}&\text{Re}\,\big(i\langle x,\zeta(\xi)+z\vartheta_j e_n\rangle - y\langle\zeta(\xi)+z\vartheta_j e_n\rangle\big) \\ &\le -x_n t|\xi| - \text{Im}\,(z)x_n\vartheta_j - y|\xi|/4 \le \big(t(2|x_n|/3-x_n)-y/4\big)\,|\xi| \text{ for } y>0.\end{aligned} \tag{2.22}$$

and since $\left|\langle\zeta(\xi)+z\vartheta_j e_n\rangle\right. \le \left|\zeta(\xi)+z\vartheta_j e_n\right. \le 2|\xi|$

$$\begin{aligned}&\text{Re}\,\big(i\langle x,\zeta(\xi)+z\vartheta_j e_n\rangle - y\langle\zeta(\xi)+z\vartheta_j e_n\rangle\big) \\ &\le (-x_n t/3+2|y|)|\xi| \text{ for } x_n>0 \text{ and } y\in \text{IR}.\end{aligned} \tag{2.22'}$$

We now set

$$u := \sum_{j\in J} u_j. \tag{2.23}$$

By (2.19), (2.20), (2.22) and (2.22') this sum converges in *C"(V)* for

$$\text{v} \;:=\Big\{\,(x,y)\in\mathbb{R}^{n+1}\;\; y>8t|x_n| \text{ or } \big(x_n>0 \text{ and} y>-x_n t/8\big)\Big\}$$

and $u \in C_\Delta(V)$ by (2.21).

b) There is $A_3 \ge 1$ such that u can be extended as a harmonic function CO

$$Y_\varepsilon := \big\{\,(x,y)\in\mathbb{R}^n \text{ x } \mathbb{R}\;\; |x'|>\varepsilon A_3,\, |x_n|<\varepsilon,\, y>-\varepsilon t\big\}.$$

Proof. Let $|x_n| < \varepsilon$ and $|y| < 9\varepsilon t$.

$$\begin{aligned}u_j(x,y) &= (2\pi)^{-n}\exp(-t_j x_n + i\langle x,\xi_j+z\vartheta_j e_n\rangle)\times \\ &\times \int_{|z|=1}\int_{|\xi|\le\gamma} t_j^n \exp(i\langle t_j x,\xi\rangle)\chi_j(\xi_j+t_j\xi)F_{j,z}(\xi_j+t_j\xi)\,d\xi\,dz/(4\pi i z)\end{aligned}$$

where

$$\begin{aligned}F_{j,z}(\xi) := \;&\exp\big(x_n t(|\xi_j|-\langle\xi\rangle)-y\langle\zeta(\xi)+z\vartheta_j e_n\rangle\big)\times \\ &\times\big(1+it\xi_n/\langle\xi\rangle\big)/\big(P_t(\zeta(\xi)+z\vartheta_j e_n)\langle\zeta(\xi)+z\vartheta_j e_n\rangle\big).\end{aligned}$$

By (2.17), (2.19) and (2.20) the denominator in $F_{j,z}$ is bounded from below on D_j ($|\xi| \ge 1$ by (2.13)). Thus $F_{j,z}$ is holomorphic on D_j. By (2.17) $F_{j,z}$ can be estimated for $0 < t \le \tau_1$:

$$|F_{j,z}(\xi)| \le C_2\exp\big((|x_n|\gamma t/2+2|y|)|\xi_j|\big) \le C_2\exp(19\varepsilon t|\xi_j|) \text{ for } \xi\in D_j$$

since $|\langle\zeta(\xi)+z\vartheta_j e_n\rangle| \le 2|\xi_j|$ for $\xi \in D_j$. By Cauchy's estimate with (poly)radius $\gamma t_j/(4n^{1/2})$ we get for $|\delta| \le \varepsilon t_j$ and real ξ with $|\xi-\xi_j| \le \gamma t_j$

$$|D^\delta F_{j,z}(\xi)| \le C_2 A_4^{|\delta|}\delta!\,t_j^{-|\xi|}\exp\big(19\varepsilon t|\xi_j|\big) \le C_2(A_4\varepsilon)^{|\delta|}\exp(19\varepsilon|\xi_j|). \tag{2.24}$$

By partial integration, (2.14) and (2.24) we get for $|\beta| \le \varepsilon t_j$

$$\left|x^{\beta}u_j(x,y)\right| \le C_3(A_5\varepsilon)^{|\beta|}|\xi_j|^n \exp\left(21\varepsilon t|\xi_j|\right). \tag{2.25}$$

Let $|x'| \ge A_6\varepsilon \ge n^{1/2}A_5e^{22}\varepsilon$. We then set $\beta| = [\varepsilon t_j]$ in (2.25) and get

$$|u_j(x,y)| \le C_4|\xi_j|^n \exp(-\varepsilon t|\xi_j|).$$

The sum (2.23) defining u on V thus converges locally uniformly on $\{(x,y)\ |x'| \ge A_6\varepsilon, |x_n < \varepsilon$ and $|y| < 9\varepsilon t\}$ and it defines a harmonic function by (2.21). Since $u \in C_\Delta(V)$ by a) the claim of b) follows.

c) The constant A_3 from b) will be fixed from now on. We now prove the estimate corresponding to i): let $\omega\ c\ W_{2\varepsilon A_3}$ and $h \in$ C"(o) with ω x $\{\Theta\}$ c reg $_L(h)$. Let $(\varphi_{k,\nu}) \in B_{K_1,\omega}$ and $(\psi_{k,\nu}) \in B_{K_1,W_{2\varepsilon A_3}}$. Then for $d = 0,1$

$$\left|\left((\partial_y^{\nu+d}u(\ ,y) * h\varphi_{k,\nu})\psi_{k,\nu}\right)\widehat{}(s)y^{\nu}/\nu!\right|$$
$$\le C_5\sum_{j\in J}\int_{|z|=1}\int\chi_j(\xi)\langle\zeta(\xi)+z\vartheta_je_n\rangle^{\nu+d-1}(h\varphi_{k,\nu})\widehat{}(\zeta(\xi)+z\vartheta_je_n)\times$$
$$\widehat{\psi}_{k,\nu}(s-\zeta(\xi)-z\vartheta_je_n)/P_t(\zeta(\xi)+z\vartheta_je_n)\exp\left(-y\langle\zeta(\xi)+z\vartheta_je_n\rangle\right)d\zeta\,dz/z\Big||y|^{\nu}/\nu!$$
$$\le C_6\sum_j\int\chi_j(\xi)\left(2|\xi||y|\right)^{\nu}/\nu!e^{-y|\xi|/4}\times$$
$$\sup_{|z|=1}\ (h\varphi_{k,\nu})\widehat{}(\zeta(\xi)+z\vartheta_je_n)\widehat{\psi}_{k,\nu}(s\quad\zeta(\xi)\quad z\vartheta_je_n)\big|d\zeta.$$

If $t < 1/(4L_2\lambda n^{1/2})$, then

$$\widehat{\Gamma}_{\lambda t} \subset \Gamma_{1/(2L_2)}(e_1).$$

Indeed, if $\xi \in \widehat{\Gamma}_{\lambda t}$, then

$$|(\xi'',\xi_n)| \le n^{1/2}|(\xi'',\xi_n)|_\infty \le n^{1/2}\max\left(\lambda t,\ 1/(4n^{1/2}L_2)\right)|\xi_1| \le |\xi_1|/(4L_2)$$

and

$$|\xi_1 - |\xi|\ = |\xi| - \xi_1 \le |\xi_1|\left(\left(1+1/(4L_2)^2\right)^{1/2}\right) \le |\xi_1|/(4L_2)^2$$

and therefore

$$\left|\xi - |\xi|e_1\right| \le |\xi_1|/(2L_2) \le |\xi|/(2L_2)$$

and $\xi \in \Gamma_{1/(2L_2)}(e_1)$. For $\xi \in \widehat{\Gamma}_{\lambda t}$ we thus get by (1.12)

$$\mathit{Re}\ \left(\zeta(\xi)+z\vartheta_je_n\right) \in \Gamma_{1/L_2}(e_1) if also t < 1/(12L_2).$$

Since $(\varphi_{k,\nu}) \in B_{K_1,\omega}$, ω **c** $W_{2A_3\varepsilon}$ and $K_1 \ge 1/\varepsilon$, we see by (1.4) and Cauchy's estimate that $(\varphi_{k,\nu}\exp(\langle\ \mathrm{Im}\,\eta_n,\ \rangle - 3\varepsilon A_3|\ \mathrm{Im}\,\eta_n|))_\nu \in A_{K_1+1/\varepsilon,\omega}$ **c** $A_{2K_1,\omega}$ for $\eta \in \mathbb{R}^{n-1}$ x $\mathbb{C}$ with constants C_d which are uniform w.r.t. $\mathrm{Im}\,\eta_n$ and k. Since ω x $\{e_1\}$ **c** reg $_L(h)$, we thus get by (1.10) for $\xi \in \widehat{\Gamma}_{\lambda t}$ and $|s-\xi| \ge |s|/2$

$$\sup_{|z|=1}\left|(h\varphi_{k,\nu})\widehat{}\left(\zeta(\xi)+z\vartheta_je_n\right)\widehat{\psi}_{k,\nu}(s-\zeta(\xi)-z\vartheta_je_n)\right.$$
$$\le C_7\left(\nu(L_0+2L_1K_1)/(1+\left|\ \mathrm{Re}\ \left(\zeta(\xi)+z\vartheta_je_n\right)\right|)\right)^{\nu}\times$$
$$\times\left(kB_2K_1/(1+|s-\zeta(\xi)-z\vartheta_je_n|)\right)^k\exp\left(5\varepsilon A_3|\ \mathrm{Im}\ (\zeta_n(\xi)+z\vartheta_j)|\right)$$
$$\le C_7\left(4\nu(L_0+L_1K_1)/|\xi|\right)^{\nu}\left(4kB_2K_1/(1+|s|)\right)^k\exp\left(10\varepsilon A_3t|\xi|\right)$$

if $t < \tau_1$ (use also (2.20)), since then

$$|s-\zeta(\xi)-z\vartheta_j e_n| \geq |s-\xi|-2t|\xi| \geq |s-\xi|(1-6t) \geq |s-\xi|/2 \geq |s|/4.$$

Similarly as above we get

$$\begin{aligned}&\left\| \left(\varphi_{k,\nu} \exp\left(\langle \operatorname{Im} \eta_n, \rangle - 3\varepsilon A_3 |\operatorname{Im} \eta_n|\right)\right)^{(\alpha+\beta)} \right\|_\infty \\ &\leq C_d \left(2K_1(k+\nu)\right)^{|\alpha|} \text{ if } |\alpha| \leq (k+\nu) \text{ and } |\beta| \leq d.\end{aligned}$$

Since $\omega \times \{e_1\} \subset \operatorname{reg}_L(h)$, we thus get for $\xi \in \widehat{\Gamma}_{\lambda t}$ and $|s-\xi| \leq |s|/2$ (and hence $|\xi| \geq |s|/2$) again by (1.10)

$$\begin{aligned}&\sup_{|z|=1} \left|(h\varphi_{k,\nu})\widehat{\ }\left(\zeta(\xi)+z\vartheta_j e_n\right)\widehat{\psi}_{k,\nu}\left(s-\zeta(\xi)-z\vartheta_j e_n\right)\right. \\ &\leq C_8\left(2(\nu+k)(L_0+L_1K_1)/(1+|\operatorname{Re}\left(\zeta(\xi)+z\vartheta_j e_n\right)|)\right)^{\nu+k} \times \\ &\quad\times \exp\left(5A_3\varepsilon|\operatorname{Im}\left(\zeta_n(\xi)+z\vartheta_j\right)|\right) \\ &\leq C_9\left(8e\nu(L_0+L_1K_1)/|\xi|\right)^\nu \left(16ek(L_0+L_1K_1)/(1+|s|)\right)^k \exp\left(10\varepsilon t A_3|\xi|\right).\end{aligned}$$

Here we have also used the trivial estimate

$$(j+d)^{j+d} < e^{j+d}(j+d)! \leq e^{j+d} j!\,d! \binom{j+d}{j} \leq (2ej)^j(2ed)^d \text{ if } j,d \in \mathbb{N}_0. \tag{2.26}$$

Summarizing we have proved that for $s \in \mathbb{R}^n$ and $d = 0.1$

$$\begin{aligned}&\left(\left(\partial_y^{\nu+d} u(\,,y) * h\varphi_{k,\nu}\right)\psi_{k,\nu}\right)\widehat{\ }(s) y^\nu/\nu! \leq \\ &\leq C_{10}\left(16e^2(L_0+L_1K_1)|y|\right)^\nu \left((16e+4B_2)k(L_0+L_1K_1)/(1+|s|)\right)^k \times \\ &\quad\times \int \exp\left((10\varepsilon t A_3 - y/4)|\xi|\right) d\xi \\ &\leq C_{11} 2^{-\nu}\left(A_7(L_0+L_1K_1)k/(1+|s|)\right)^k \\ &\text{if } y > 40\varepsilon t A_3 \text{ and } y < 1/\left(32e^2(L_0+L_1K_1)\right)\end{aligned} \tag{2.27}$$

II) The definition and properties of F are now prepared in d) and e). The choice of λ will be fixed in e).

d) We now set

$$\widetilde{L} := \left\{\ell \geq j_0 \mid \ell \notin J,\ \operatorname{supp} \chi_\ell \subset \mathbb{R}^n \setminus U_{C(t)},\ \operatorname{supp} \chi_\ell \not\subset \widehat{\Gamma}_{\lambda t}\right\}$$

(compare the definition of J in a)) and define v_ℓ for $\ell \in \widetilde{L}$ by a modification of the construction of Hormander [12, section 7.3] as follows: let $\Phi \in C^\infty(\operatorname{Pol}^0(m) \times \mathbb{C}^n)$ be chosen from [12, Lemma 7.3.12] such that $\Phi(H,w) = 0$ for $|w| \geq 1$. With the path $\zeta(\xi)$ and $\langle\xi\rangle$ defined as above we set for $(x,y) \in \mathbb{R}^{n+1}$ and $4 \in \widetilde{L}$

$$\begin{aligned}v_\ell(x,y) = (2\pi)^{-n} \int\int \chi_\ell(\xi)\Phi\left(P_t(\zeta(\xi)+\cdot),w\right) \exp\left(i\langle x,\zeta(\xi)+w\rangle - y\langle\zeta(\xi)+w\rangle\right) \times \\ \times 1/\left(2P_t(\zeta(\xi)+w)\langle\zeta(\xi)+w\rangle d\zeta\, dw.\right.\end{aligned} \tag{2.28}$$

Φ is constructed such that for some $C_1 > 0$ we have for any ξ and w

$$\left|\Phi(P_t(\zeta(\xi)+\cdot),w)\right|/\left|P_t(\zeta(\xi)+w)\right| \leq C_1. \tag{2.29}$$

It is clear (by (2.13)) that $(\zeta(\xi)+w) \in W$ for $\xi \in \operatorname{supp} \chi_\ell$ and that

$$\operatorname{Re} \langle\zeta(\xi)+w\rangle \geq 2^{-1/2}\left|\zeta(\xi)+w\right| \geq |\xi|/4. \tag{2.30}$$

Hence $v_\ell \in C_\Delta(\mathbb{R}^{n+1})$ and

$$v := \sum_{l\in\widetilde{L}} v_\ell.$$

converges in $C''(V)$, (compare (2.20) (2.22) and (2.22')). Thus $v \in C_\Delta(V)$.

e) We will show now that λ can be chosen so large that an estimate like (2.12) holds for v: let $(\psi_{k,\nu}) \in \widetilde{B}_{16B_1/(\varepsilon A_3)\ W_{2\varepsilon A_3}}(A_3 I)$ for B_1 from (1.5) and A_3 from b). For $y > 16A_3\varepsilon t$ we get $W_{2\varepsilon A_3}$ x {y} C V. For $s \in \mathbb{R}^n$ and these y we get by (2.29), (2.30), (2.10') and the properties of Φ (see Hörmander [12, 7.3.191) for $g \in B$ if B is bounded in $D(W_{2\varepsilon A_3})$:

$$\begin{aligned}
&\left|\left((\partial_y^{\nu+d} v(\ ,y) * P_t(D)g)\psi_{k,\nu}\right)\widehat{\ }(s)y^\nu/\nu!\right| \\
&\leq (2\pi)^{-n}\textstyle\sum_{l\in\widetilde{L}} \int \chi_\ell(\xi)\langle\zeta(\xi)\rangle^{\nu+d-1}\,\widehat{g}(\zeta(\xi))\,\widehat{\psi}_{k,\nu}(s-\zeta(\xi))/2e^{-y\langle\zeta(\xi)\rangle}d\zeta\ \ |y|^\nu/\nu! \\
&\leq C_2\textstyle\sum_\ell\int\chi_\ell(\xi)\left(2|\xi|y|)|\right)^\nu/\nu!\left(16B_1B_2k/\left(\varepsilon A_3(1+|s-\zeta(\xi)|\widehat{\ })\right)\right)^k\times \\
&\quad \text{x}\,\left(16B_1B_2\nu/\left(\varepsilon A_3(1+|s-\zeta(\xi)|\widehat{\ })\right)\right)^\nu e^{4A_3\varepsilon|\operatorname{Im}(\zeta_n(\xi))|-y|\xi|/4}d\zeta.
\end{aligned} \tag{2.31}$$

We now show that for $\xi \in \operatorname{supp}\chi_\ell, s \in \Gamma_t(e_1)$ and $|x|\widehat{\ }_\infty := \left|(x_1,x'',x_n/(A_3 I))\right|_\infty$

$$|s-\xi|\widehat{\ }_\infty \geq \lambda t|\xi|_\infty/8 \text{ if } t < 1/\left(\lambda(4n^{1/2}L_2A_3I)^{1/\varkappa}\right). \tag{2.32}$$

If (2.32) were not true, then $|s|_\infty \leq 2|\xi|_\infty$ if $t \leq 1/(A_3\lambda I)$. Moreover, if also $\lambda \geq 8$

$$\begin{aligned}
|(\xi'',\xi_n)|_\infty \leq A_3 I\left|\left(\xi'',\xi_n/(A_3I)\right)\right|_\infty &\leq A_3I\left(|\xi-s|\widehat{\ }_\infty + \left|s-|s|e_1\right|\right) \\
< (\lambda t/8+t)A_3I|\xi|_\infty &\leq \lambda t A_3 I|\xi|_\infty/4 \leq |\xi|_\infty/2
\end{aligned}$$

since $s \in \Gamma_t(e_1)$. Hence $|\xi_1| = |\xi|_\infty$ and therefore $\xi_1 = |\xi|_\infty$ since otherwise (2.32) would hold (if $t < 8/\lambda$) since $s_1 > 0$. We thus get for $t < 1/\left(\lambda(4n^{1/2}L_2A_3I)^{1/\varkappa}\right)$

$$\left|\left(\xi_1 - |\xi|_\infty e_1, \xi'', \xi_n 4n^{1/2}(\lambda t)^\varkappa L_2\right)\right|_\infty \leq \left|\left(\xi'',\xi_n/(A_3I)\right)\right|_\infty < \lambda t|\xi_\infty/4. \tag{2.33}$$

This leads to the following contradiction: by the definition of $\widetilde{L}$ there is $\eta \in \operatorname{supp}\chi_\ell\backslash\widehat{\Gamma}_{\lambda t}$. Thus

$$|\eta-\xi|_\infty \leq |\eta-\xi| < 2\gamma t|\xi| < 2tn^{1/2}|\xi|_\infty \leq 4tn^{1/2}|\eta|_\infty \text{ for } \xi \in \operatorname{supp}\chi_\ell$$

if $t < \min\left(1/(2\gamma), 1/(4n^{1/2})\right)$, and

$$\begin{aligned}
&\left|\left(\xi_1-|\xi|_\infty,\xi'',\xi_n 4n^{1/2}(\lambda t)^\varkappa L_2\right)\right|_\infty \\
&\geq \left|\left(\eta_1-|\eta|_\infty,\eta'',\eta_n 4n^{1/2}(\lambda t)^\varkappa L_2\right)\right|_\infty - |\eta-\xi|_\infty - \left||\eta|_\infty-|\xi|_\infty\right| \\
&\geq t(\lambda-8n^{1/2})|\eta|_\infty > \lambda t|\xi|_\infty/2 \text{ if } \lambda > 16n^{1/2} \textit{ and if } t < 1/\left(\lambda(4n^{1/2}L_2)^{1/\varkappa}\right).
\end{aligned}$$

This contradicts (2.33) and proves (2.32). Hence we get for $\xi \in \operatorname{supp} \chi_\ell$ and $s \in \Gamma_t(e_1)$

$$|s - \zeta(\xi)| \geq |s - \xi| \geq |s - \xi|_\infty \geq \lambda t|\xi|/(8n^{1/2}). \tag{2.32'}$$

We now show that for $\xi \in \operatorname{supp} \chi_\ell$ and $s \in \Gamma_t(e_1)$

$$|s - \xi| \geq \lambda t|s|/8. \tag{2.34}$$

To prove this we can assume by (2.32') that $|\xi| < |s|/2$. If (2.34) is not true we get the contradiction

$$\begin{aligned} |s| &= \big||s|e_1\big| \leq \big|s - |s|e_1\big| + |s - \xi| + |\xi| \\ &\leq \big|s - |s|e_1\big| + \lambda t|s|/(16n^{1/2}) + |\xi| \\ &< \Big(t + \lambda t/\big(16n^{1/2} + 1/2\big)\Big)|s| < |s| \text{ if } t \leq \min(1/\lambda, 1/4) \end{aligned}$$

since $s \in \Gamma_t(e_1)$. By (2.31), (2.32') and (2.34) we get for $y = T := 64A_3\varepsilon t$

$$\begin{aligned} &\sum_\nu \big|\big((\partial_y^{\nu+d} v(\,,y) * P_t(D)g)\psi_{k,\nu}\big)\hat{}(s)y^\nu/\nu!\big| \\ &\leq C_3 \int \exp\big(-4A_3\varepsilon t|\xi|\big)d\xi\Big(2^8 eB_1B_2k/\big(\varepsilon\lambda t(1+|s|)\big)\Big)^k \sum_\nu \big(2^{14}eB_1B_2n^{1/2}/\lambda\big)^\nu \\ &\leq C_4\Big(k/\big(T(1+|s|)\big)\Big)^k \text{ for } s \in \Gamma_t(e_1) \text{ and } d = 0,1 \text{ if } \lambda = 2^{15}eB_1B_2n^{1/2}. \end{aligned} \tag{2.35}$$

III) We finally change u such that we obtain an elementary solution for $P_t(D_x)$ and define F, G and E in g):

f) Since $\langle\zeta(\xi) + z\vartheta_j e_n\rangle$ is holomorphic in z for $|z| \leq 1$ and $\xi \in \operatorname{supp} \chi_j$, $j \in J$, (see a)), we get for $j \in J$ by Cauchy's integral formula

$$P_t(D_x)u_j(x,y) = (2\pi)^{-n}/2\int \chi_j(\xi)\exp\big(i\langle x,\zeta(\xi)\rangle - y\langle\zeta(\xi)\rangle\big)/\langle\zeta(\xi)\rangle d\zeta.$$

Similarly, we get for $\ell \in \widetilde{L}$ by the properties of Φ (see Hörmander [12, (7.3.19)])

$$P_t(D_x)v_\ell(x,y) = (2\pi)^{-n}/2\int \chi_\ell(\xi)\exp\big(i\langle x,\zeta(\xi)\rangle - y\langle\zeta(\xi)\rangle\big)/\langle\zeta(\xi)\rangle d\zeta.$$

We now set $\chi(\xi) := \sum_{j \in J \cup \widetilde{L}} \chi_j(\xi)$ and get for $(x,y) \in V_1$ where

$$V_1 := \{(x,y) \mid x_n > 0,\ y > 0\}:$$

$$P_t(D_x)(u+v)(x,y) = (2\pi)^{-n}/2\int \chi(\xi)\exp\big(i\langle x,\zeta(\xi)\rangle - y\langle\zeta(\xi)\rangle\big)/\langle\zeta(\xi)\rangle d\zeta. \tag{2.36}$$

Choose $C_1 \geq 1$ such that $\chi(\xi) = 1$ for $|\xi| \geq C_1$. Since $\langle(\xi', \xi_n)\rangle$ is holomorphic in ξ_n near $S := \{\xi_n \mid \xi' \in \mathbb{R}^{n-1}, |\operatorname{Im} \xi_n| \leq t|\xi|, |\xi| \geq C_1\}$ and satisfies

$$\operatorname{Re}\langle(\xi', \xi_n)\rangle \geq |\operatorname{Re} \xi|/4 \text{ for } \xi \in S$$

(compare (2.20)) we can shift the path $\zeta(\xi)$ in (2.36) by Cauchy's integra1 theorem to $\mathbb{R}$ if $|\xi'| \geq$ Cs. Similarly, for $|\xi'| < C_1$ we can change the path $\zeta(\xi)$ for $|\xi_n| \geq C_1$ such that it is contained in $\mathbb{R}$ for $|\xi_n \geq C_1 + 1$. We can assume that Im $\eta \leq t$ Re η on these new paths, which we denote by γ. For $\eta \in sp(\gamma)$ with Re $\eta \in$ supp χ we have

$$\exp\left(-y\langle\eta\rangle\right)/\left(2\langle\eta\rangle\right) = \int_{\mathbb{R}} e^{iy\tau}/\left(2\pi Q(\eta,\tau)\right)d\tau$$

with $Q(\eta,\tau) = \langle\eta,\eta\rangle + \tau^2$. For $\varphi \in D(V_1)$ we thus get

$$\begin{aligned}\langle P_t(D_x)(u+v),\varphi\rangle &= (2\pi)^{-n}\int_{\mathbb{R}}\int_{\gamma}\chi(\text{ Re }\eta)(\mathfrak{F}_x\varphi)(-\eta,y)\left(e^{-y\langle\eta\rangle}/(2\langle\eta\rangle)\right)d\eta dy\\ &= (2\pi)^{-n-1}\int_{\mathbb{R}}\int_{\gamma}\int_{\mathbb{R}}\chi(\text{ Re }\eta)(\mathfrak{F}_x\varphi)(-\eta,y)e^{iy\tau}/Q(\eta,\tau)d\tau d\eta\, dy\\ &= (2\pi)^{-n-1}\int_{\mathbb{R}}\int_{\gamma}\chi(\text{ Re }\eta)\widehat{\varphi}(-\eta,-\tau)/Q(\eta,\tau)d\eta d\tau\end{aligned} \tag{2.37}$$

by Fubini's theorem since $(\mathfrak{F}_x\varphi)(-\eta,y)/Q(\eta,\tau) \in L_1\left((\mathbb{R}^{n+1}\setminus W_{C_1+1})\text{ x }\mathbb{R}\right)$ (here $\mathfrak{F}_x$ denotes the partial Fourier transform w.r.t. x). By means of (2.37) $P_t(D_x)(u$ + v) can be extended to a distribution H on $\mathbb{R}^{n+1}$. For $\varphi \in D(\mathbb{R}^{n+1})$ we get by the Fourier inversion formula

$$\begin{aligned}\langle\Delta H,\varphi\rangle &= (2\pi)^{-n-1}\int_{\mathbb{R}}\int_{\gamma}\chi(\text{ Re }\eta)\widehat{\varphi}(-\eta,-\tau)d\eta\, d\tau\\ &= (2\pi)^{-n}\int_{\gamma}\chi(\text{ Re }\eta)\mathfrak{F}_x(\varphi)(-\eta,0)d\eta = \langle\delta + h_x\otimes\delta_y,\varphi\rangle\end{aligned} \tag{2.38}$$

with $h \in H(\mathbb{C}^n)$. Thus $H \in C_\Delta\left(\mathbb{R}^n \times]0,\infty[\right)$ and H extends $P_t(D_x)(u+v)$ also from V to $\mathbb{R}^{n+1}$. Let $\widetilde{H}(x,y) := H(x,-y)$. Since $\Delta\widetilde{H} = \Delta H$ on $\mathbb{R}^{n+1}$ by (2.38), we have $\widetilde{H} - H =: g \in C_\Delta(\mathbb{R}^{n+1})$. Set

$$(u+v)\widetilde{\ }(x,y) := u(x,|y|) + v(x,|y|) \text{ for } (x,y) \in V_2 := \left\{(x,y) \quad |y| > 8t|x_n|\right\}.$$

Then

$$P_t(D_x)(u+v)\widetilde{\ }(x,y) = \widetilde{H}(x,y) = H(x,y) + g(x,y) \text{ for } y < -8t|x_n|.$$

Let ψ be the characteristic function of $\mathbb{R}^n$ x$]-\infty,0]$. Then $H + g\psi$ is an extension of $P_t(D_x)(u$ t $v)\widetilde{\ }$ from V_2 to $\mathbb{R}^{n+1}$ such that

$$\Delta\left(H + g\psi\right) = \delta + \left(h - 2\partial_y g(\ ,0)\right)\otimes\delta_y =: \delta - f\otimes\delta_y$$

by partial integration, since g is odd w.r.t. y and thus $g|_{\mathbb{R}^n} = 0$. Since $f \in H(\mathbb{C}^n)$ *we* can solve the equation

$$P_t(D_x)w_1 = f/2 \text{ with } w_1 \in H(\mathbb{C}^n)$$

and then solve the Cauchy problem

$$\text{Aw} = 0 \text{ on } \mathbb{R}^{n+1},\ w(x,0) = 0,\ \partial_y w(x,0) = w_1(x).$$

g) We finally set for $(x,y) \in V_2$:

$$\begin{aligned}F(x,y) := v(x,|y|),\ G(x,y) := u(x,|y|) + w(x,|y|)\\ \text{and } E(x,y) := F(x,y) + G(x,y).\end{aligned}$$

Then $E \in \widetilde{C}_\Delta(V_2)$. G also satisfies b) and c) since w $\in H(\mathbb{C}^{n+1})$. $P_t(D_x)E$ is extended to $\mathbb{R}^{n+1}$ by $H_1 := \left(H + g\psi + P_t(D_x)w(\ ,|\ |)\right)$ and H_1 is an elementary solution for A since

$$\Delta P_t(D_x)w(\ ,|\ |) = 2\partial_y P_t(D_x)w(\ ,0)\otimes\delta_y = 2P_t(D_x)w_1\otimes\delta_y = f\otimes\delta_y.$$

The theorem is proved. □

3 Extension of the regularity set

In this section we will apply the regular fundamental solutions constructed in Theorem 2.3 to extend the regularity set of C^∞– zerosolutions of P(D). As an abbreviation we introduce the following notation:

For $f,g \in D(\mathbb{R}^n \times]a,b[)$ and $a < y_k < b$ let

$$\langle f(\ ,y_1) * g(\ ,y_2)\rangle(x) := \int \big(f(x-\xi,y_1)\partial_y g(\xi,y_2) - \partial_y f(x-\xi,y_1)g(\xi,y_2)\big)d\xi.$$

To apply the regular fundamental solutions constructed in section 2 we use the following simple lemma:

Lemma 3.1 *Let $E \in \widetilde{C}_\Delta(\Omega \times (\mathbb{R} \setminus [-T/2,T/2]))$ be an elementary solution for P(D) and let H be a distributional extension of $P(D)E$ as in Definition 2.1. Let $u \in C_\Delta(W \times [-T,T])$ where $W \subset \mathbb{R}^n$ is open. Then we have for $x \in \omega$ if $\overline{\omega} + W \subset \Omega$ and $|y| < T/2$*

$$u(x,y) = \langle E(\ ,y+T) * P(D)(hu)(\ ,-T)\rangle(x) - \langle E(\ ,y-T) * P(D)(hu)(\ ,T)\rangle(x) + \int_{W\times[-T,T]} H(x-\xi,y-\eta)\Delta(hu)(\xi,\eta)d\xi d\eta$$

if $h \in D(W)$ und $h = 1$ near 0.

Proof. Let χ be the characteristic function of $\mathbb{R}^n \times [-T,T]$ and let $h \in D(W)$. By Leibniz' rule we have

$$\Delta(\chi hu) = \chi\Delta(hu) + 2_x \otimes (\delta_{-T}(y) - \delta_T(y))\partial_y(hu) + 1_x \otimes (\partial_y\delta_{-T}(y) - \partial_y\delta_T(y))\,hu.$$

Choose $\varphi \in C_0^\infty(\Omega \times \mathbb{R})$ such that $\varphi \equiv 1$ near (W-W) x $[-2T,2T]$. We then get for h,x and y as above since $H(\xi,\eta) = P(D)E(\xi,\eta)$ for $|\eta| > T/2$

$$\begin{aligned} u(x,y) &= \chi hu(x,y) = \Delta(\varphi H) * \chi hu(x,y) = H * \Delta(\chi hu)(x,y) \\ &= \int_{W\times[-T,T]} H(x-\xi,y-\eta)\Delta(hu)(\xi,\eta)d\xi d\eta \\ &\quad + \langle H(\ ,y+T) * hu(\ ,-T)\rangle(x) - \langle H(\ ,y-T) * hu(\ ,T)\rangle(x) \\ &= \langle E(\ ,y+T) * P(D)(hu)(\ ,-T)\rangle(x) - \langle E(\ ,y-T) * P(D)(hu)(\ ,T)\rangle(x) \\ &\quad + \int_{W\times[-T,T]} H(x-\xi,y-\eta)\Delta(hu)(\xi,\eta)d\xi d\eta. \end{aligned}$$

□

We also need a more precise version of the fact that harmonic functions are real analytic. Let $V_\delta := \{(x,y) \in \mathbb{R}^{n+1} \ \ |(x,y)| \le \delta\}$.

Lemma 3.2 *There are $B_5 \ge 1$ and $C \ge 1$ such that for any $0 < \delta < \varepsilon \le 1$ and any $u \in C_\Delta(V_\varepsilon)$ which is bounded on V_ε*

$$|\partial_y^\nu \partial_x^\alpha u(0)| \le C(\varepsilon-\delta)^{-n-1}\nu!\delta^{-\nu}\alpha!(B_5/(\varepsilon-\delta))^{|\alpha|}\sup\{|u(w)| \ \ w \in V_\varepsilon\}$$

for any ν and α.

Proof. By the Poisson integra1 formula we have for $(x,y) \in \overline{V}_{\delta_1}$ and $0 < \delta \le \delta_1 < \varepsilon_1 < \varepsilon$

$$u(x,y) = \left(\varepsilon_1^2 \quad |(x,y)|^2\right) / (\omega_{n+1}\varepsilon_1) \int_{\partial V_{\varepsilon_1}} u(\xi,\eta)|(\xi,\eta) - (x,y)|^{-n-1} d\sigma(\xi,\eta). \tag{3.0}$$

For $|(\xi,\eta)| = \varepsilon_1$ and $z \in \mathbb{C}$ with $|z| \le \delta_1$ we bave

$$\sum_{i\le n} \xi_i^2 + (\eta - z)^2 \notin]-\infty, 0].$$

Indeed, if this were not true, then $\eta = \operatorname{Re} z$ and

$$\varepsilon_1^2 = |(\xi,\eta)|^2 = |(\xi, \operatorname{Re} z)|^2 \le |z|^2 = \delta_1^2,$$

a contradiction. The integrand in (3.0) can be extended for x = 0 as a holomorphic function of y to the complex ball with radius ε_1.

$$\begin{aligned} &\inf\left\{ \left| \sum_{i\le n} \xi_i^2 + (\eta - z)^2 \right|^2 \,\middle|\, |(\xi,\eta)| = \varepsilon_1, |z|^2 = \delta_1^2 \right\} \\ &= \inf\left\{ \left(\varepsilon_1^2 - 2\eta x + x^2 - y^2\right)^2 + \left(2xy - 2\eta y\right)^2 \,\middle|\, |\eta| \le \varepsilon_1, x^2 + y^2 = \delta_1^2 \right\} \\ &= \inf\left\{ f_x(\eta) := 4\delta_1^2\eta^2 - 4\eta x(\varepsilon_1^2 + \delta_1^2) + 4x^2\varepsilon_1^2 + \left(\varepsilon_1^2 - \delta_1^2\right)^2 \quad |\eta| \le \varepsilon_1, |x| \le \delta_1 \right\} \\ &\ge (\varepsilon_1 - \delta_1)^4. \end{aligned}$$

To see this we notice that for fixed x the global infinumum of f_x is attained at $\eta_0 = \eta_0(x) :=$ *(x/2)* $\left(\varepsilon_1^2 + \delta_1^2\right)/\delta_1^2$. If $|\eta_0| \ge \varepsilon_1$ we have

$$\inf\{ f_x(\eta) \quad |\eta| \le \varepsilon_1 \} = \min f_x(\pm\varepsilon_1) = \min \left(\varepsilon_1^2 + \delta_1^2 \pm 2x\varepsilon_1\right)^2 \ge (\varepsilon_1 \quad \delta_1)^4$$

since $|x| \le \delta_1$. If $|\eta_0| < \varepsilon_1$, then

$$x^2 < 4\varepsilon_1^2\delta_1^4/(\varepsilon_1^2 + \delta_1^2)^2$$

and therefore

$$\begin{aligned} \inf\{ f_x(\eta) \mid |\eta| \le \varepsilon_1 \} &= 4x^2\varepsilon_1^2 + \left(\varepsilon_1^2 - \delta_1^2\right)^2 - x^2\left(\varepsilon_1^2 + \delta_1^2\right)^2/\delta_1^2 \\ &= \left(\varepsilon_1^2 - \delta_1^2\right)^2\left(1 - x^2/\delta_1^2\right) \ge \left(\varepsilon_1^2 - \delta_1^2\right)^4/\left(\varepsilon_1^2 + \delta_1^2\right)^2 \ge \left(\varepsilon_1 - \delta_1\right)^4. \end{aligned}$$

This shows the above estimate. By Cauchy's estimate with radius δ we get

$$\left|\partial_y^\nu u(0)\right| \le C(\varepsilon_1 - \delta)^{-n-1}\nu!\delta^{-\nu} \sup\{ |u(w)| \quad w \in V_{\varepsilon_1} \} \text{ for any } \nu.$$

This is applied to $\partial_x^\alpha u$ and the claim follows from the well-known fact that there is $B_0 \ge 1$ such that for any $\gamma > 0$

$$|D^\beta v(0)| \le B_0(B_0/\gamma)^{|\beta|} \sup\{ |v(\eta)| \quad \eta \in V_\gamma \} \text{ if } v \in C_\Delta(V_\gamma)$$

(take $\varepsilon_1 := (\delta + \varepsilon)/2$ and $\gamma := (\varepsilon - \delta)/4$)). □

The basic result on extension of the uniform regularity set is contained in the following theorem. For $\Omega \subset \mathbb{R}^n$ let

$$\Omega_+ := \{x \in \Omega \mid x_n > 0\}$$

and let

$$\widetilde{W}_\varepsilon(\xi) := \{x \in \mathbb{R}^n \quad |\xi' - x'| < \varepsilon, |\xi_n - x_n| < \varepsilon/A_3\} \text{ and } \widetilde{W}_\varepsilon := \widetilde{W}_\varepsilon(0)$$

with A_3 from Theorem 2.3.

Theorem 3.3 *Let $P_{m,e_1}(e,) \neq 0$. There are $A_k \geq 1$ such that the following holds for any $0 < \varepsilon \leq 1/A_0$ und $L_1 \geq A_0$: let $\Omega \subset \mathbb{R}^n$ be open and let $\omega \subset \mathbb{R}^{n-1}$ satisfy $(\omega + \widetilde{W}_\varepsilon) \subset \Omega$. If $u \in BC_\Delta(\Omega \times (\mathbb{R} \setminus \{0\})), \Omega_+ \times \{e_1\} \subset \mathrm{UReg}_L(u)$ und $\Omega \times \{e_1\} \subset \mathrm{UReg}_L(P(D_x)u)$, then $\widetilde{\Omega} \times \{e_1\} \subset \mathrm{UReg}_{\widetilde{L}}(u)$ where $\widetilde{\Omega} := (\Omega_+ \cup (\omega + \widetilde{W}_{\varepsilon/A_1}))$ and $\widetilde{L} = A_1((L_0 + L_1/\varepsilon), (L_0\varepsilon + L_1))$.*

Proof. We can assume that the conditions hold for 4ε instead of ε. When proving Theorem 3.3 we will consider only y > 0 (the case when y < 0 is treated similarly).

I) There is $A \geq 1$ and $C_1 \geq 1$ such that for any $0 < y < 1/(2L_1)$ and any $\xi \in \overline{\omega + \widetilde{W}_{2\varepsilon}}$ there are $u_{k,y} \in D(\widetilde{W}_\varepsilon(\xi))$ such that $\{u_{k,y} \mid k \in \mathbb{N}, 0 < y < 1/(2L_1)\}$ is bounded in $D(\widetilde{W}_\varepsilon(\xi)), u_{k,y}|_{\widetilde{W}_{\varepsilon/16}(\xi)} = u(\,,y)$ and

$$\left|\widehat{u}_{k,y}(s)\right| \leq C_1\left(A(L_0 + L_1/\varepsilon)k/(1+|s|)\right)^k$$

for any $s \in \Gamma_{1/(A(L_0\varepsilon + L_1))}(e_1)$ and any $0 < y < 1/(2L_1)$.

Proof. a) Let $0 < y \leq 1/(2L_1)$ and $0 < T < 1/(4L_1)$. Further bounds on T will be given in the proof below. We can assume that $\xi = 0$ and get for $T < \varepsilon/(2A_3)$ and $x \in \widetilde{W}_\varepsilon$

$$u(x,y) = \sum_\nu \partial_y^\nu u(x, y+T)(-T)^\nu/\nu! \tag{3.1}$$

where

$$\sup_{0<y\leq 1/2} \sum_\nu \sup_{x \in \widetilde{W}_\varepsilon} |\partial_x^a \partial_y^\nu u(x, y+T)| T^\nu/\nu! < \infty \text{ for any } a \in \mathbb{N}_0^n. \tag{3.2}$$

(3.2) is seen as follows: since $u \in BC_\Delta(\Omega \times (\mathbb{R} \setminus \{0\}))$ for any $a \in \mathbb{N}^n$ there is $C \geq 1$ by Lemma 3.2 (used fora = 0) such that for $d = 0, 1$

$$\sum_\nu \sup_{x\in\widetilde{W}_\varepsilon} \left|\partial_x^a \partial_y^{\nu+d} u(x,y+T)\right| \frac{T^\nu}{\nu!} \leq C y^{-n-2} \text{ if } 0 < y \leq 1/2 \text{ and} 0 < T < \varepsilon/(2A_3).$$

Since $\partial_y^2 u_f = -\Delta_x u_f$, this implies that for these y and T and $0 \leq j \leq n+3$

$$\sum_\nu \sup_{x\in\widetilde{W}_\varepsilon} \left|\partial_x^a \partial_y^{\nu+j} u(x, y+T)\right| T^\nu/\nu! \leq C y^{-n-2}.$$

By Taylors formula with Lagrange remainder term we get for these y and T

$$\begin{aligned} &\mathrm{CV} \sup_{x\in\widetilde{W}_\varepsilon} \left|\partial_x^a \partial_y^\nu u(x, y+T)\right| T^\nu/\nu! \\ &\leq \mathrm{CV} \; \Sigma_{0\leq j\leq n+1} \sup_{x\in\Omega} \partial_x^a \partial_y^{\nu+j} u\left(x, \tfrac{1}{2} + T\right)\left|T^\nu\right|y - \tfrac{1}{2}\Big|^j/(\nu!j!) \\ &+ \Sigma_\nu \sup_{x\in\Omega} \int_{y-\frac{1}{2}}^0 \left|\left(y - \tfrac{1}{2} - t\right)^{n+2} \partial_x^a \partial_y^{\nu+n+2} u\left(x, T + \tfrac{1}{2} + t\right)\right| T^\nu/\left(\nu!(n+2)!\right) dt \\ &\leq eC + C\int_{y-\frac{1}{2}}^0 \left(\left(\tfrac{1}{2} + t - y\right)\Big/\left(\tfrac{1}{2} + t\right)\right)^{n+2}/(n+2)!\,dt \leq C(e+1). \end{aligned}$$

b) Choose $(\psi_{k,\nu})$ in the following way: applying (1.5) to the variables x' and x_n separately we can choose $(\psi_k) \in D(\widetilde{W}_{\varepsilon/4})$ such that $\psi_k = 1$ on $\widetilde{W}_{\varepsilon/8}$ and $(\widetilde{\psi}_k) \in D(\widetilde{W}_{\varepsilon/16})$ such that $\int \widetilde{\psi}_k(x)dx = 1$ and such that

$$||\psi_k^{(\alpha+\beta)}||_\infty + ||\widetilde{\psi}_k^{(\alpha+\beta)}||_\infty \leq C_d(16B_1k/\varepsilon)^{|\alpha|} A_3^{\alpha_n} \text{ if } |\alpha| \leq k \text{ and } |\beta| \leq d.$$

Set $\psi_{k,\nu} := \psi_k * \widetilde{\psi}_\nu$. Then

$$(\psi_{k,\nu}) \in \widetilde{B}_{16B_1/\varepsilon,\widetilde{W}_{5\varepsilon/16}}(A_3) \text{ and } \psi_{k,\nu} = 1 \text{ on } \widetilde{W}_{\varepsilon/16}. \tag{3.3}$$

Set

$$u_{k,y} := \sum_\nu \psi_{k,\nu}\, \partial_y^\nu u(\ ,y+T)(-T)^\nu/\nu!.$$

Then $u_{k,y}(x) = u(x,y)$ for $x \in \widetilde{W}_{\varepsilon/16}$ by (3.1) and (3.3), and $\{u_{k,y}\ \ k \in \mathbb{N}, 0 < y < 1/(2L_1)\}$ is bounded in $D(\widetilde{W}_{5\varepsilon/16})$ by (3.2) since $\{\psi_{k,\nu}\ \ k,\nu \in \mathbb{N}\}$ is bounded in $D(\widetilde{W}_{5\varepsilon/16})$ by (3.3).

With $B_6 := 128B_1$ we now choose $(\varphi_{k,\nu}) \in B_{B_6A_3/\varepsilon,\widetilde{W}_{2\varepsilon}}$ (again by (1.5) and convolution as above) such that

$$\begin{aligned} &\operatorname{supp}(\varphi_{k,\nu}) \subset \{x\ \ |x'| \le 25\varepsilon/16,\ |x_n| \le 9\varepsilon/(16A_3)\} =: W \\ &\text{and } \varphi_{k,\nu}(x) = 1 \text{ if } |x'| \le 3\varepsilon/2 \text{ and } |x_n| \le \varepsilon/(2A_3). \end{aligned} \tag{3.4}$$

The assumption (2.11) of Theorem 2.3 is satisfied for $P_t \equiv P$ *and* $\varkappa = 1$ by *Lemma 2.2.* If $2L_1 \ge \rho$ and $0 < t \le 1/A_2$ we can apply Lemma 3.1 (with $h = \varphi_{k,\nu}$, $\omega = \widetilde{W}_{5\varepsilon/16}$ and $\Omega = W_{2\varepsilon}$) to $\widetilde{u}(x,\eta) := u(x, T + y + \eta)$ and an elementary solution $E = E_{\varepsilon/A_3,t,2L_1}$ chosen by Theorem 2.3. Taking derivatives w.r.t. η and setting $\eta = 0$ this implies (since E is even w.r.t. y) for $s \in \mathbb{R}^n$

$$\begin{aligned} |\widehat{u}_{k,y}(s)| &= \left|\textstyle\sum_\nu \int \psi_{k,\nu}(x)\partial_y^\nu u(x,y+T)e^{-i\langle x,s\rangle}dx(-T)^\nu/\nu!\right| \\ &\le \textstyle\sum_\nu \left(\psi_{k,\nu}\langle\partial_y^\nu E(\ ,T)*\left(P(D)(\varphi_{k,\nu}u)(\ ,y) - P(D)(\varphi_{k,\nu}u)(\ ,y+2T)\right)\rangle\right)\widehat{\ }(s)\left|\frac{T^\nu}{\nu!}\right. \\ &+ \textstyle\sum_\nu \left|\int\int\int_{|\eta|\le T} \psi_{k,\nu}(x)\partial_y^\nu H(x-\xi,-\eta)\Delta\left(\varphi_{k,\nu}u\right)(\xi,y+T+\eta)e^{-i\langle x,s\rangle}d\eta\, d\xi\, dx\right|\frac{T^\nu}{\nu!} \\ &\le \textstyle\sum_\nu \left(\psi_{k,\nu}\langle\partial_y^\nu F(\ ,T)*\left(P(D)(\varphi_{k,\nu}u)(\ ,y) - P(D)(\varphi_{k,\nu}u)(\ ,y+2T)\right)\rangle\right)\widehat{\ }(s)\left|\frac{T^\nu}{\nu!}\right. \\ &+ \textstyle\sum_\nu \left(\psi_{k,\nu}\langle\partial_y^\nu G(\ ,T) * \varphi_{k,\nu}\left(P(D)u(\ ,y) - P(D)u(\ ,y+2T)\right)\rangle\right)\widehat{\ }(s)\left|\frac{T^\nu}{\nu!}\right. \\ &+ \textstyle\sum_{a\ne 0,\nu}\left|\left(\psi_{k,\nu}\langle\partial_y^\nu G(\ ,T)*\partial_x^a\varphi_{k,\nu}\left(P^{(a)}(D)u(\ ,y) - P^{(a)}(D)u(\ ,y+2T)\right)\rangle\right)\widehat{\ }(s)\right|\frac{T^\nu}{a!\nu!} \\ &+ \textstyle\sum_\nu \left|\int\int\int_{|\eta|\le T} \psi_{k,\nu}(x)\partial_y^\nu H(x-\xi,-\eta)\Delta\left(\varphi_{k,\nu}u\right)(\xi,y+T+\eta)e^{-i\langle x,s\rangle}d\eta\, d\xi\, dx\right|\frac{T^\nu}{\nu!} \end{aligned} \tag{3.5}$$

where $T = 64\varepsilon t$.

c) The four terms in (3.5) are now estimated uniformly for $0 < y \le 1/(2L_1)$, where in i) – iii) only $u(\ , y)$ is considered for shortness since $u(\ ,y + 2T)$ can be treated in exactly the same way.

i) $\{\varphi_{k,\nu}\partial_y^d u(\cdot,y)\ \ d = 0,1;\ k,\nu \in \mathbb{N},\ 0 < y \le 1\}$ is bounded in $D(\overline{W}_{2\varepsilon})$ since $u \in BC_\Delta(\Omega \times (\mathbb{R}\setminus\{0\}))$. We thus we get by Theorem 2.3ii)

$$\begin{aligned} &\textstyle\sum_\nu \left(\psi_{k,\nu}\langle\partial_y^\nu F(\ ,T) * \left(P(D)(\varphi_{k,\nu}u(\ ,y)\right)\rangle\right)\widehat{\ }(s)\left|T^\nu/\nu!\right. \\ &\quad\le C\left(k/\left(T(1+|s|)\right)\right)^k \text{ if } s \in \Gamma_t(e_1), \end{aligned} \tag{3.6}$$

and if $t \le 1/(2A_1L_1A_3)$ (set $I := 1$).

ii) Since $\widetilde{W}_{2\varepsilon} \times \{e_1\} \subset \mathrm{UReg}_{2L}(P(D)u)$ and $(\psi_{k,\nu}), (\varphi_{k,\nu}) \in B_{B_6A_3/\varepsilon, \widetilde{W}_{2\varepsilon}}$, we get by Theorem 2.3i) for $s \in \mathbb{R}^n$ and $0 < y \le 1/(2L_1)$

$$\sum_\nu \left|\left(\psi_{k,\nu}\langle \partial_y^\nu G(\,,T) * \varphi_{k,\nu} P(D)u(\,,y)\rangle\right)^\frown(s)\right| T^\nu/\nu! \le C\big(k/\big(T(1+|s|)\big)\big)^k$$

if $t < 1/\big(2A_1(L_0\varepsilon + B_6L_1A_3)\big)$.

iii) To estimate the third term in (3.5) we choose $f_{k,\nu} = f_{k,\nu}(x_n)$ such that $(f_{k,\nu}) \in B_{B_6A_3/\varepsilon,]0,\infty[}$ and such that $f_{k,\nu} = 1$ near $[\varepsilon/(16A_3), 1]$. Then we have for $a \ne 0$

$$\begin{aligned}
&\sum_\nu \left(\psi_{k,\nu}\langle \partial_y^\nu G(\,,T) * (\partial_x^a \varphi_{k,\nu}) P^{(a)}(D)u(\,,y)\rangle\right)^\frown(s)\Big| T^\nu/\nu! \\
&\le \sum_\nu \left|\left(\psi_{k,\nu}\langle \partial_y^\nu G(\,,T) * (f_{k,\nu}\partial_x^a \varphi_{k,\nu}) P^{(a)}(D)u(\,,y)\rangle\right)^\frown(s)\right| T^\nu/\nu! \\
&+ \sum_\nu \left|\left(\psi_{k,\nu}\langle \partial_y^\nu G(\,,T) * ((1-f_{k,\nu})\partial_x^a \varphi_{k,\nu}) P^{(a)}(D)u(\,,y)\rangle\right)^\frown(s)\right| T^\nu/\nu!
\end{aligned} \tag{3.7}$$

By assumption and (1.11) we get $\widetilde{W}_{2\varepsilon,+} \times \{e_1\} \subset \mathrm{UReg}_{2L}(P^{(a)}(D)u)$. Since $(f_{k,\nu}\partial_x^a\varphi_{k,\nu}) \in B_{2B_6A_3/\varepsilon, \widetilde{W}_{2\varepsilon,+}}$ (compare (1.4)), we get by Theorem 2.3i) for $s \in \mathbb{R}^n$ and $0 < y \le 1/(2L_1)$

$$\begin{aligned}
&\sum_\nu \left(\psi_{k,\nu}\langle \partial_y^\nu G(\,,T) * (f_{k,\nu}\partial_x^a \varphi_{k,\nu}) P^{(a)}(D)u(\,,y)\rangle\right)^\frown(s)\Big| T^\nu/\nu! \\
&\le C\big(k/\big(T(1+|s|)\big)\big)^k
\end{aligned}$$

if $t < 1/\big(2A_1(L_0\varepsilon + 2B_6L_1A_3)\big)$. To estimate the second term in (3.7) we use the harmonic extension of G+ (see Theorem 2.3) and Lemma 3.2 and get for $x \in \operatorname{supp} \psi_{k,\nu}$ and $\xi \in \operatorname{supp}\big((1-f_{k,\nu}) \operatorname{grad} \varphi_{k,\nu}\big)$

$$\left|\partial_x^a \partial_y^{\nu+d} G(x-\xi T)\right| T^\nu/\nu! \le C_1 \big((1+1/(32A_3A_4)\big)^{-\nu} (A_6|a|/T)^{|a|}$$

if $T \le \varepsilon/(64A_3)$ (with $A_6 := 64B_5A_3A_4$ and $d = 0,1$).

So $\langle \partial_y^\nu G(\,,T) * ((1-f_{k,\nu})\partial_x^a\varphi_{k,\nu}) P^{(a)}(D)u(\,,y)\rangle$ satisfies these Cauchy estimates on $\operatorname{supp} \psi_{k,\nu}$. Since the functions in $\widetilde{B}_{C,\Omega}$ satisfy estimates in k and ν simultaneously, we can use (1.8) (for $(\psi_{k,\nu})_k$ uniformly in ν) and thus get for any $s \in \mathbb{R}^n$ and $0 < y \le 1/(2L_1)$

$$\begin{aligned}
&\sum_\nu \left|\left(\psi_{k,\nu}\langle \partial_y^\nu G(\,,T) * ((1-f_{k,\nu})\partial_x^a \psi_{k,\nu}) P^{(a)}(D)u(\,,y)\rangle\right)^\frown(s)\right| T^\nu/\nu! \\
&\le C_2\big(B_3k(A_6/T + 16B_1A_3/\varepsilon)/(1+|s|)\big)^k
\end{aligned}$$

if $T < \varepsilon/(64A_3)$.

iv) Since

$$\operatorname{dist}\big(\{x-\xi \mid x \in \operatorname{supp}\psi_{k,\nu}, \xi \in \operatorname{supp} \operatorname{grad}\varphi_{k,\nu}\}, \{0\} \cup \partial(W_{2\varepsilon} \times \mathbb{R})\big) \ge \varepsilon/(8A_3)$$

we get for these x and ξ by Lemma 3.2 if $T < \varepsilon/(32A_3)$ and $d = 0,1$

$$\left|\partial_y^\nu \partial_x^a H(x-\xi, \eta)\right| T^\nu/\nu! \le C_3(B_5|a|/T)^{|a|} 2^{-\nu}.$$

We now use (1.8) to estimate the last term in (3.5) as in the second part of iii) and get for $s \in \mathbb{R}^n$ and $0 < y \leq 1/(2L_1)$

$$\begin{aligned}
&\sum_\nu \Big| \int \psi_{k,\nu}(x) \int \int_{|\eta|\leq T} \partial_y^\nu H(x-\xi,-\eta)\Delta(\varphi_{k,\nu}u)(\xi,y+T+\eta)e^{-i\langle x,s\rangle}d\eta\, d\xi\, dx \Big| \frac{T^\nu}{\nu!} \\
&\leq \sum_\nu \int_{|\eta|\leq T} \Big| \Big(\psi_{k,\nu} \, (\partial_y^\nu H(\ ,-\eta) * \Delta(\varphi_{k,\nu}u)(\ ,y+T+\eta))\Big)^\frown(s) \Big| \frac{T^\nu}{\nu!} \\
&\leq C_4 \sup\Big\{ \big|\Delta(\varphi_{k,\nu}u)(x,\eta)\big| \,\Big|\, \eta \in [y,y+2T], x \in \mathbb{R}^n \Big\} \times \\
&\quad \times \big(B_3(B_5/T + 16A_3B_1/\varepsilon)k/(1+|s|)\big)^k \\
&\leq C_5\big(B_3(B_5/T + 16A_3B_1/\varepsilon)k/(1+|s|)\big)^k.
\end{aligned}$$

We now set $t := 1/\big(2^{12}(A_1+A_2)(L_0\varepsilon + B_6L_1A_3)\big)$. Then $T = 64\varepsilon t = 1/\big(64(A_1+A_2)(L_0 + A_3B_6L_1/\varepsilon)\big)$ and t and T satisfy the restrictions needed above. This proves claim 1).

11) From 1) the theorem follows by means of a resolution of the identity chosen as follows: choose $\xi_j \in \omega + \widetilde{W}_{3\varepsilon/2}$ and $\chi_j \in D\big(W_{\varepsilon/(32A_3)}(\xi_j)\big)$ such that $\sum \chi_j = 1$ on $\omega + \widetilde{W}_{5\varepsilon/4}$. Choose $(g_k) \in A_{32B_1A_3/\varepsilon, W_{\varepsilon/(32A_3)}}$ such that $\int g_k = 1$ and set $\psi_{k,j} := \chi_j * g_k$. Then $(\psi_{k,j})_k \in A_{32B_1A_3/\varepsilon, \widetilde{W}_{\varepsilon/16}(\xi_j)}$ and

$$\psi_k := \sum \psi_{k,j} = 1 \text{ on } \omega + \widetilde{W}_\varepsilon.$$

Choose $u_{k,y,j}$ for $\widetilde{W}_\varepsilon(\xi_j)$ by I). For $(\varphi_k) \in A_{C,\widetilde{\Omega}}$, $\widetilde{\Omega} := \Omega_+ \cup (\omega + \widetilde{W}_\varepsilon)$, we then have

$$u(\ ,y)\varphi_k = \sum u_{k,y,j}(\psi_{k,j}\varphi_k) + u(\ ,y)(1-\psi_k)\varphi_k.$$

Since $(\psi_{k,j}\varphi_k) \in A_{32B_1A_3/\varepsilon + C, \widetilde{W}_{\varepsilon/16}(\xi_j)}$ by (1.4) and since $u_{k,y,j}$ satisfies 1), there is $\widetilde{A} \geq 1$ such that (by Remark 1.2)

$$\big|\big(u_{k,y,j}(\psi_{k,j}\varphi_k)\big)^\frown(s)\big| \leq C_1\Big(\big(\widetilde{A}(L_0 + L_1/\varepsilon) + C\widetilde{A}(L_0\varepsilon + L_1)\big)k/(1+|s|)\Big)^k$$

if $s \in \Gamma_{1/\left(2A(L_0\varepsilon+L_1)\right)}(e_1)$. Since $\big((1-\psi_k)\varphi_k\big) \in A_{32B_1A_3/\varepsilon + C, \Omega_+}$ and $\Omega_+ \times \{e_1\} \subset \operatorname{UReg}_L(u)$ we can estimate also $\big(u(\ ,y)(1-\psi_k)\varphi_k\big)^\frown(s)$ uniformly for $0 < y \leq 1/(2L_1)$ by Definition 1.3 (obtaining better bounds). Since $-\partial_y^2 u = \Delta_x u$, also $\partial_y u$ satisfies the assumptions of the theorem (use also (1.11)). By the proof above we thus have the same estimates for $\partial_y u$. The theorem is proved. □

Repeated application of Theorem 3.3 yields the following quantitative result on the extension of the regularity set in certain cones up to the edge (with polynomial bounds on the index L measuring regularity). It is the main result of this paper and it will also be a central tool in the paper Langenbruch [18] on partial differential operators which are surjective on real analytic functions. Let always $P_m(\Theta) = 0$.

Theorem 3.4 *a) Let* $P_{m,\Theta}(N) \neq 0$. *There are* $B \geq 1$ *and open cones* $K_1 \subset K_2 \subset \{x \in \mathbb{R}^n \mid \langle x, N\rangle > 0\}$ *such that* $\overline{K}_2 \cap \{x \in \mathbb{R}^n \mid \langle x,N\rangle \leq 0\} = \{0\}$ *and such that the following holds for the truncated cones* S_j *and* $\sum_\tau$ *defined by*

$$S_1 := \{x \in K_2 \mid t_1 < \langle x,N\rangle < t_2\},\ S_2 := \{x \in K_2 \mid \langle x,N\rangle < t_2\}$$
$$\text{and } \Sigma_\tau := \{x \in K_1 \mid \tau < \langle x,N\rangle < (t_1+t_2)/2\}:$$

for any $0 < t_1 < t_2 < 2t_1 \leq 1$ *there is* $B_0 \geq 1$ *such that for any* $L \geq B$ *and* $0 < \tau \leq t_1$*: if* $f \in C^\infty(S_2)$, $S_1 \times \{\Theta\} \subset \operatorname{reg}_{(L,L)}(f)$ *and* $S_2 \times \{\Theta\} \subset \operatorname{reg}_{(L,L)}(P(D_x)f)$, *then* $\Sigma_\tau \times \{\Theta\} \subset \operatorname{reg}_{h(\tau)(L,L)}(f)$ *with* $h(\tau) := B_0\tau^{-B}$.
b) If there are $C \geq 1$ *and* $0 < c$ *such that*

$$(P_m)^\sim(x,t) \leq C(P_m)^\sim_{\langle N\rangle}(x,t) \text{ if } t \in]0,1] \text{ and } |x - \widehat{\Theta}| \leq c \tag{3.8}$$

then a) holds for any Θ *with* $|\Theta - \widehat{\Theta}| \leq c/2$ *with the cones* K_j *and the constant* B *and* B_0 *independent of* Θ.

Proof. a) i) N and Θ are not collinear since $P_{m,\Theta}(N) \neq 0 = P_{m,\Theta}(\Theta)$ since $P_m(\Theta) = 0$. We can thus choose an invertible real $n \times n$-matrix M such that ${}^tMe_n = N$ and ${}^tMe_1 = \Theta$. Now consider $\widetilde{K}_j := MK_j$, $\widetilde{S}_j := MS_j$, $\widetilde{\Sigma}_\tau := M\Sigma_\tau$, e_1, e_n, $Q := P \circ {}^tM$ and $\widetilde{f} := f \circ M^{-1}$ instead of $K_j, S_j, \Sigma_\tau, \Theta, N, P(D)$ and f. Then $f \in C^\infty(\widetilde{S}_2)$ and there is $B_1 \geq 1$ such that $\widetilde{S}_1 \times \{e_1\} \subset \operatorname{reg}_{B_1(L,L)}(\widetilde{f})$, $\widetilde{S}_2 \times \{e_1\} \subset \operatorname{reg}_{B_1(L,L)}(Q(D)\widetilde{f})$ and $Q_{m,e_1}(e_n) = P_{m,\Theta}(N) \neq 0$. If the claim is proved for $\widetilde{f}$, then it directly follows for f. We can thus assume that $\Theta = e_1$ and $N = e_n$, and we will show that the claim holds for the truncated cones S_j and Σ_τ defined by

$$S_1 := \{x \mid \max(t_1, |x'|/(2B_2)) < x_n < t_2\},\ S_2 := \{x \mid |x'|/(2B_2) < x_n < t_2\}$$
$$\text{and } \Sigma_\tau := \{x \mid \max(\tau, 4|x'|/B_2) < x_n < (t_2+t_1)/2\},$$

where $B_2 := 2A$ with $A := A_1A_3$ for A_1 from Theorem 3.3 and A_3 from Theorem 2.3.
ii) We first show by induction how the regularity of a defining function u_f for f extends through a union Q_k of layers defined as follows:
Fix $0 < t_1 < t_2$ and $\delta := A_3/2$ and set $\tau_{-1} := \widetilde{t}_2 := t_2 - (t_2 - t_1)/4$, $\widetilde{t}_1 := t_1 + (t_2 - t_1)/4$ and

$$\tau_k := \widetilde{t}_1(1 - \delta/A)^k,\ d_k := A\tau_k \text{ and}$$
$$Q_k := \{x \in \mathbb{R}^n \mid \exists\, 0 \leq j \leq k : \tau_j < x_n \leq \tau_{j-1}, |x'| < d_j\} \text{ for } k \geq 0.$$

We then have for large $C \geq 1$ (independent of t_1, t_2) and $C_1 = C_1(t_1, t_2) \geq 1$:

$$Q_k \times \{e_1\} \subset \operatorname{UReg}_{LC_1C^k(1,\varepsilon_k)}(u_f) \text{ for any } k \geq 1 \text{ with } \varepsilon_k := \delta\tau_{k-1}.$$

Proof. We want to apply Theorem 3.3 to $\Omega_{k,+} := Q_{k-1}$, $\Omega_k := Q_{k-1} \cup (\omega_k + \widetilde{W}_{\varepsilon_k})$, $k \geq 1$, where

$$\omega_k := \{(x', \tau_{k-1}) \mid |x'| < d_k\}.$$

First notice that there is $C \geq 1$ such that

$$\Omega_k \times \{e_1\} \subset \operatorname{UReg}_{LC_1C^{k-1}(1,\varepsilon_{k-1})}(P(D_x)u_f) \text{ for } k \geq 1\ (\varepsilon_0 := 1). \tag{3.9}$$

Indeed, if $|x'| < d_j$ and $\tau_j < x_n \leq \tau_{j-1}$ for some $j \geq 0$, then

$$|x'|/B_2 < d_j/B_2 = \tau_j/2 < x_n$$

and therefore

$$Q_{k-1} \subset L_k := \left\{x \in \mathbb{R}^n \;\middle|\; |x'|/B_2 \leq x_n, \tau_{k-1}/2 < x_n \leq \widetilde{t}_2\right\} \subset S_2.$$

Also,

$$\omega_k + \widetilde{W}_{\varepsilon_k} \subset L_k \text{ for } k \geq 1$$

since we get for $\xi \in \omega_k + \widetilde{W}_{\varepsilon_k}$

$$|\xi'|/B_2 < (d_k + \varepsilon_k)/B_2 = \tau_{k-1}/2 = \tau_{k-1} - \varepsilon_k/A_3 < \xi_n.$$

Since

$$\text{dist } (L_k, \partial S_2) \geq \delta_k := \min\left((t_2 - t_1)/2, \tau_{k-1}/6\right)$$

we get by Proposition 1.4

$$\Omega_k \times \{e_1\} \subset \text{UReg }_{\widetilde{L}}\left(P(D_x)u_f\right)$$

with $\widetilde{L} = B_5 L(1 + 1/\delta_k, 1) \leq C_0 C^{k-1} L(1, \varepsilon_{k-1})$ for $k \geq 1, C \geq 1/(1 - \delta/A)$ and sufficiently large $C_0 = C_0(t_1, t_2)$.

Let $k = 1$. Since $t_2 \leq 2t_1$, *we* get

$$\text{dist } (\Omega_{1,+}, \partial S_1) = \text{dist } (Q_0, \partial S_1) \geq (t_2 - t_1)/4$$

and Proposition 1.4 implies that for sufficiently large $C \geq 1$

$$\Omega_{1,+} \times \{e_1\} \subset \text{UReg }_{C(L,L)}(u_f).$$

Using also (3.9) we thus have by Theorem 3.3

$$\left(Q_0 + (\omega_1 + \widetilde{W}_{\varepsilon_1/A_1})\right) \times \{e_1\} \subset \text{UReg }_{A_1 CL(1+1/\varepsilon_1, 1+\varepsilon_1)}(u_f)$$

and thus if $C_0 \geq A_1(1 + 1/\varepsilon_1)$ and $C_1 := C_0^2$

$$Q_1 \times \{e_1\} \subset \text{UReg }_{LC_1(1,\varepsilon_1)}(u_f)$$

since $t_1 - \varepsilon_1/(A_1 A_3) = \tau_1$. This proves the claim for $k = 1$.

If $k > 1$, then

$$\Omega_{k,+} \times \{e_1\} = Q_{k-1} \times \{e_1\} \subset \text{UReg }_{LC_1 C^{k-1}(1,\varepsilon_{k-1})}(u_f)$$

by the induction hypothesis. Using also (3.9) we get by Theorem 3.3

$$\left(Q_{k-1} + \left(\omega_k + \widetilde{W}_{\varepsilon_k/A_1}\right)\right) \times \{e_1\} \subset \text{UReg }_{A_1 LC_1 C^{k-1}(1+\varepsilon_{k-1}/\varepsilon_k, \varepsilon_k+\varepsilon_{k-1})}(u_f)$$

and thus

$$Q_k \times \{e_1\} \subset \text{UReg }_{LC_1 C^k(1,\varepsilon_k)}(u_f)$$

since $\tau_{k-1} - \varepsilon_k/(A_1A_3) = \tau_k$ (if $C \geq 4A_1 \geq 1 + \varepsilon_{k-1}/\varepsilon_k$). Claim ii) is proved.

iii)

$$\Sigma_\tau \subset Q_k \text{ if } \tau_k \leq \tau \leq \tau_{k-1}, k \geq 1. \tag{3.10}$$

Indeed, let $x \in \Sigma_\tau$. If $\tilde{t}_1 = \tau_0 \leq x_n \leq \tau_{-1} = \tilde{t}_2$ we have

$$|x'| < B_2 x_n/4 \leq A\tilde{t}_2/2 \leq A\tilde{t}_1 = d_0$$

since $\tilde{t}_2 \leq 2\tilde{t}_1$, and thus $x \in Q_0$. If $k \geq j > 0$ and $\tau_j \leq x_n \leq \tau_{j-1}$, we have

$$|x'| < B_2 x_n/4 \leq A\tau_{j-1}/2 \leq A\tau_j = d_j$$

since $(1 - \delta/A) \geq 1/2$. This shows (3.10). Set $h(\tau) := \delta C_1 (tC_2/\tau)^{\ell n(C)/\ell n(C_2)}$ with $C_2 := 1/(1-\delta/A)$ and C from ii). Let $k \geq 1$ and $\tau_k \leq \tau \leq \tau_{k-1}$. Then

$$\Sigma_\tau \times \{e_1\} \subset Q_k \times \{e_1\} \subset \text{UReg}_{\delta LC_1C^k(1,1)}(u_f) \subset \text{UReg}_{h(\tau)(L,L)}(u_f)$$

by (3.10) and ii). Hence

$$\Sigma_\tau \times \{e_1\} \subset \text{reg}_{h(\tau)(L,L)}(f)$$

by Proposition 1.4

This proves the theorem in case a) since $\tau_0 = \tilde{t}_1$.

b) As in ii) of the proof of Lemma 2.2 one proves that (3.8) implies that there exist $0 < \delta \leq 1$, $b_1 \geq 1$ such that for all $0 < t < 6$:

$$\tilde{P}(\xi, t|\xi|) \leq b_1 \tilde{P}_{\langle N\rangle}(\xi, t|\xi|) \text{ if } \xi \in \Gamma_{\frac{c}{2}}(\Theta),\ |\hat{\Theta} - \Theta| < c/4 \text{ and } |\xi| \geq C(t). \tag{3.11}$$

(Compare (2.5)). For $\Theta \in \Gamma_{c/4}(\hat{\Theta}) \cap S^n$ we can now make the normalization from i) with matrices M_Θ such that ${}^tM_\Theta e_n = N$ and ${}^tM_\Theta e_1 = \Theta$ and such that

$$\left\{ ({}^tM_\Theta)^{-1}, {}^tM_\Theta \ \ \Theta \in \Gamma_{c/4}(\hat{\Theta}) \cap S^n \right\} \text{ is bounded.} \tag{3.12}$$

For $Q_\Theta := P o\, {}^tM_\Theta$ we get: there are $b_2 \geq 1$ and $\rho \geq 1$ such that for any $\Theta \in \Gamma_{c/4}(\hat{\Theta}) \cap S^n$, any $\lambda \geq 1$, $0 < t \leq 1/(\rho\lambda)$ and any $\xi \in \tilde{\Gamma}_{\lambda t}(\rho, 1)$

$$\tilde{Q}_\Theta(\xi, t|\xi|) \leq b_2 (Q_\Theta)\tilde{}_{\langle e_n\rangle}(\xi, t|\xi|) \text{ if } |\xi| \geq C(t). \tag{3.13}$$

Indeed, for $|\xi|_\infty < 1/\rho < 1$ we have by (3.12)

$$\begin{aligned} \left|{}^tM_\Theta\xi - \Theta|{}^tM_\Theta\xi|\right| &\leq \left|\Theta(|\xi|_\infty - |{}^tM_\Theta\xi|)\right| + \left|{}^tM_\Theta(0,\xi'',\xi_n)\right| \\ &\leq 2\left|{}^tM_\Theta(0,\xi'',\xi_n)\right| \leq 2B_1|\xi|/\rho < c|\xi|/2 \end{aligned} \tag{3.14}$$

if $\rho > 4B_1/c$. Hence ${}^tM_\Theta\xi \in \Gamma_{c/2}(\Theta)$. Also by (3.12) we get

$$\tilde{Q}_\Theta(\xi, t|\xi|) \leq B_2 \tilde{P}({}^tM_\Theta\xi, t|{}^tM_\Theta\xi|)$$

and

$$\tilde{P}_{\langle N\rangle}({}^tM_\Theta\xi, t|{}^tM_\Theta\xi|) \leq B_2 (Q_\Theta)\tilde{}_{\langle e_n\rangle}(\xi, t|\xi|) \tag{3.15}$$

(use also (2.3)). (3.13) now easily follows from (3.11), (3.14) and (3.15). Since the constants in (3.13) are uniform w.r.t. $\Theta \in \Gamma_{c/4}(\hat{\Theta}) \cap S^n$, also the constants A_k in Theorem 2.3 and hence the constants A_k in Theorem 3.3 can be chosen uniformly for these Θ. Since these constants (and the uniform bound from (3.12)) are the only data for the proof of Theorem 3.4a) this proof shows the claim in b). □

Though we will only use the sets reg $_{(L,L)}(f)$ in Langenbruch [18], we had to consider the more complicated sets reg $_{(L_o,L_1)}$ (f) in this paper to obtain polynomial bounds on the regularity in Theorem 3.4

4 Extension of the complement of the wave front set

In this fina1 section the results of section 3 will be applied to get bounds for the wave front set of hyperfunctions. These are direct consequences of Theorem 3.4. Let always $N \in S^n$ and $\Theta \in S^n$ with $P_m(\Theta) = 0$.

Theorem 4.1 *Let $\Omega \subset \mathbb{R}^n$ be open* and $x_0 \in \Omega$. *Let* $\psi \in C^1(\Omega)$ *with* $N := \operatorname{grad} \psi(x_0) \neq 0$ *und set* $\Omega_+ := \{x \in \Omega \quad \psi(x) > \psi(x_0)\}$. *Let $P_{m,\Theta}(N) \neq 0$. Then there is a neighbourhood U of x_0 such that the following holds for any $[u] \in \mathfrak{B}(\Omega)$: $(U \times \{\Theta\}) \cap WF_A([u]) = 0$ if $(\Omega_+ \times \{\Theta\}) \cap WF_A([u]) = 0$ und if $(\Omega \times \{\Theta\}) \cap WF_A(P(D)[u]) = 0$.*

Proof. By Kaneko [13, Corollary 1.121 we can choose an elliptic loca1 operator J(D) and $f \in C^\infty(\Omega)$ such that $[u] = J(D)f$. Since J(D) is elliptic, we have

$$WF_A(f) = WF_A([u]) \text{ and } WF_A(P(D)f) = WF_A(P(D)[u]).$$

(by Kawai [15, Theorem 4.1.8] (since the support of the microfunction image of a hyperfunction $[u]$ coincides with $WF_A([u])$) and Hörmander [12, Theorem 9.3.3 and 9.3.4]). f thus satisfies the assumptions of the theorem and we only have to prove the claim for f.

b) We can assume that $x_0 = 0$. The second assumption implies by Hörmander [12, Lemma 8.4.4] that there is $L \geq 1$ such that $U_{1/L} \times \{\Theta\} \subset \text{reg}_{(L,L)}(P(D)f)$. With the cones $K_1 \subset K_2$ chosen for N by Theorem 3.4 we can choose $t > 0$ and $0 < t_1 < t_2 < 2t_1 \leq 1$ and define the truncated cones S_j as in Theorem 3.4 such that $tN + \overline{S}_1 \subset\subset \Omega_+$ and $tN + S_2 \subset U_{1/L}$. Hence also $(tN + S_1) \times \{\Theta\} \subset \text{reg}_{(L,L)}(f)$ for sufficiently large L by the first assumption and [12, Lemma 8.4.4] again. By Theorem 3.4 we thus get $(tN + \Sigma_\tau) \times \{\Theta\} \subset \text{reg}_{h(\tau)(L,L)}(f)$ and hence $(tN + \Sigma_\tau) \times \{\Theta\} \subset WF_A(f)$ for any $0 < \tau \leq t_1$. This proves the claim since $0 \in tN + \Sigma_\tau$ for $0 < \tau < t$. □

Theorem 4.1 essentially is a special case of a result of Sjostrand [24, Theorem 5.11. Holmgren type theorems for the analytic wave front set (usually for operators with variable coefficients) have been obtained by many authors (see J.M. Bony [3, 4], J.M. Bony, P. Schapira [5], A. Grigis, P. Schapira, J. Sjöstrand [6], N. Hanges [7], N. Hanges, J. Sjöstrand [8], L. Hormander [10], M. Kashiwara, T. Kawai [14], P. Laubin [20, 21], 0. Liess [22, 23], J. Sjöstrand [24], the reader is also referred to the literature cited in these papers).

We will now state global versions of Theorem 4.1.

Corollary4.2 *Let $P_{m,\Theta}(N) \neq 0$. Let $[u] \in \mathfrak{B}(\mathbb{R}^n)$ und $(x,\Theta) \notin WF_A(P(D)[u])$ for any $x \in \mathbb{R}^n$. If there is $\tau \in \mathbb{R}$ such that $(x,\Theta) \notin WF_A([u])$ if $\langle x,N\rangle < \tau$, then $(x,\Theta) \notin WF_A([u])$ for any XE $\mathbb{R}^n$.*

Proof. Application of Theorem 4.1 to any x_0 with $\langle x_0,N\rangle = \tau$ shows that there is $\delta > 0$ such that (x, Θ) $\notin WF_A([u])$ if $\langle x,N\rangle < \tau + \delta$. This implies the claim. □

Corollary 4.2 can be generalized to a global version of Theorem 4.1 stated for convex sets:

Theorem 4.3 *Let $\Theta \in S^n$ und let Ω_1 c Ω_2 c $\mathbb{R}^n$ be open und convex. Assume thut every hyperplane $\xi + N^\perp$ with $P_{m,\Theta}(N) = 0$ intersects Ω_2 if it intersects Ω_1. Then the following holds for $[u] \in \mathfrak{B}(\Omega_2)$:*
(x, Θ)$\notin WF_A([u])$ for uny $x \in \Omega_2$ if (x, Θ) $\notin WF_A([u])$ for uny $x \in \Omega_1$ und if (x, Θ)$\notin$ $WF_A(P(D)[u])$ for uny $x \in \Omega_2$.

Proof. This is proved exactly as the corresponding corollary of Holmgren's theorem (see Hormander [9, Theorem 5.3.3], with reference to [9, Theorem 5.3.1] substituted by the reference to Theorem 4.1). cl

The convex sets in Theorem 4.3 can be chosen as columns if the vectors N with $P_{m,\Theta}(N)$ = 0 are contained in a hyperplane. We are then in the extreme case where singularities trave1 along lines:

Theorem 4.4 *Fix $\Theta \in S^n$. Assume that there is $N \in S^n$ such thut*

$$\langle N,M\rangle = 0 \text{ if } P_{m,\Theta}(M) = 0. \tag{4.1}$$

*Let $[u] \in \mathfrak{B}(\Omega)$ and $(x,\Theta) \in WF_A([u])$. Then I x $\{\Theta\} \in WF_A([u])$ if I c $\Omega \cap (x + N\mathbb{R})$ is a line segment contuining x such thut$(I$ x $\{\Theta\})$ **n** $WF_A(P(D)[u])$ = 0.*

Proof. Assume that there is $x_0 \in I$ such that $x_0 \notin WF_A([u])$. We can assume that x_0 = x + aN for some $a > 0$. We can choose $\Omega_1 := U_\varepsilon(x_0)$ and $\Omega_2 := [0,a]N + U_\varepsilon(0)$ such that $(\Omega_1$ x $\{\Theta\}) \cap WF_A([u]) = \emptyset$ and $(\Omega_2 \times \{\Theta\}) \cap WF_A(P(D)[u]) = \emptyset$. By (4.1) the assumptions of Theorem 4.3 then hold for Ω_1 and Ω_2, and therefore (x, Θ)$\notin WF_A([u])$ by that theorem, a contradiction. □

(4.1) is clearly satisfied for P_m if Θ is a root of first order: then $P_{m,\Theta}(x) = \langle \text{grad} P_m(\Theta), x\rangle$ and (4.1) holds for N $\in$ span $\{$ Re grad $P_m(\Theta)$, Im grad P,(O)}. Thus Theorem 4.4 extends the corresponding result for operators of rea1 principal type (Hormander [12, Theorem 8.6.13]), i.e. where any root of P_m is of first order and P_m is real.

Theorem 4.4 also contains the following result of Liess [23, Theorem 1.8] who proved the conclusion of Theorem 4.4 under the following assumption (for $\Theta = e_n$ and N = $e_1, q := P_{m,\Theta}$ and P,(D) involving only deratives w.r.t. $x_1, \ldots, x_{n'}, x_n$ for some $n' < n$ and $\eta = (\tau,\vartheta) \in \mathbb{C}$ x $\mathbb{C}^{n'-1}$):
there is $\beta > 1$ such that for any $0 \neq \eta^0 = (\tau^0,\vartheta^0) \in \mathbb{R}^{n'}$ with $P_{m,\Theta}(\eta^0, 0) = 0$ there are $c_k > 0$ such that

$$|\operatorname{Re}\tau| \le c_1\left(|\operatorname{Im}\eta| + \left|\vartheta^0/|\vartheta^0| - \operatorname{Re}\vartheta/|\operatorname{Re}\vartheta|\right|^\beta |\operatorname{Re}\vartheta|\right)$$

if $P_{m,\Theta}(\eta^0,0) = 0, \left| \eta^0/|\eta^0| - \operatorname{Re}\eta/|\operatorname{Re}\eta| \right| < c_2$ and $|\operatorname{Im}\eta| < c_2|\operatorname{Re}\eta|$.

Since we can take $\eta = \eta^0$ in this condition, we get $\langle N, \eta^0 \rangle = \tau^0 = 0$ if $P_{m,\Theta}(\eta^0,0) = 0$, $\eta^0 \in \mathbb{R}^n$. Since $P_{m,\Theta}$ only depends on the variables in $\mathbb{R}^{n'}$, (4.1) holds for $P_{m,\Theta}$.

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