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ON THE CAUCHY PROBLEM FOR NONLINEAR EVOLUTION EQUATIONS AND REGULARITY OF SOLUTIONS[1]

MARKUS POPPENBERG

Abstract. *In some previous works a generalized implicit function theorem of Nash–Moser type has been applied to prove the local well posedness for the Cauchy problem for several types of nonlinear evolution equations. For instance, applications of this new method have been given to ordinary differential equations in Fréchet spaces, to nonlinear parabolic partial differential equations and to some specific nonlinear Schrödinger type equations. All these results take* $C^1([a,b],E)$ *as the basic function space where* E *is a general Fréchet space or* $E = H^\infty(\mathbb{R}^n)$*, respectively. The purpose of this note is to show that a similar approach based on Nash–Moser techniques works with the function space* $C^\infty([a,b],E)$ *as well providing the existence of* C^∞*–solutions smoothly depending on the initial value. In particular, sufficient conditions on* E *are given such that* $C^\infty([a,b],E)$ *satisfies a required smoothing property.*

1 Introduction

In [26] the generalized implicit function theorem of Nash–Moser type [25] has been applied to prove an existence and uniqueness result for ordinary differential equations in Fréchet spaces. Subsequently, similar techniques have been used to prove the local well posedness for the Cauchy problem for nonlinear evolution equations of different type, for instance for a class of fully nonlinear parabolic problems in [27] and for certain nonlinear Schrödinger type equations in [10]. All the above mentioned results are concerned with a nonlinear Cauchy problem of the form

$$\begin{cases} u_t & = & \mathcal{F}(t,u)\ ,\ t \in [0,a] \\ u(0) & = & \phi \end{cases}$$

and provide the existence of a local solution $u \in C^1([0,a],E)$ for some small $a > 0$ under suitable assumptions. The abstract framework for the Nash–Moser technique is set up with the basic function space $C^1([a,b],E)$ where E is a general Fréchet space in [26] and $E = H^\infty(\mathbb{R}^n)$, the intersection of all Sobolev spaces, in [27], [10].

The purpose of this note is to show that a similar approach works with the function space $C^\infty([a,b],E)$ as well. As a consequence, the solution map $\phi \to u$ is shown to be C^∞ as a map $E \to C^\infty([0,a],E)$ in several cases improving results of [26], [27], [10]. A motivation for taking e.g. $C^\infty([a,b],H^\infty)$ rather than $C^1([a,b],H^\infty)$ comes from the fact that in the theory

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of evolutionary partial differential equations time regularity and space regularity are always coupled.

The results [26], [27], [10] are based on an application of the implicit function theorem of Nash–Moser type [25] which generalizes classical implicit function theorems (cf. [5], [7], [13], [16], [17]). While these classical results require that the involved spaces admit smoothing operators as introduced by Nash [17] (cf. [5], [13]), it is in [25] enough that the spaces satisfy the much weaker smoothing property $(\mathrm{S}_\Omega)_\mathrm{t}$ introduced in [21] and property (DN) of D. Vogt [31].

In section 3 we are concerned with the proof of the smoothing property $(\mathrm{S}_\Omega)_\mathrm{t}$ for the space $C^\infty([-1,1],E)$ equipped with some natural fundamental system of seminorms (which does not coincide with the usual 'tensor product grading'). We first show that with E also the space $C^\infty([-1,1],E)$ has property $(\mathrm{S}_\Omega)_\mathrm{t}$ if E is tamely nuclear. It is then proved that for any Fréchet–Hilbert space E which is an (Ω)–space in standard form both E and $C^\infty([-1,1],E)$ have property $(\mathrm{S}_\Omega)_\mathrm{t}$.

In section 5 a general existence and uniqueness result is proved for ordinary differential equations in Fréchet spaces; the proof of the main result Theorem 15 is based on the method of [26] replacing the space $C^1([a,b],E)$ by $C^\infty([a,b],E)$. Different from [26] we have to estimate higher order time derivatives of the linearized problem; some technical tools on linear equations are given in section 4. The corresponding theorem of Picard–Lindelöf for Banach spaces is well known (cf. [35]). A lot of negative results show that a straightforward generalization to Fréchet spaces fails (cf. [2], [4], [6], [12]), and positive results in Fréchet spaces can only be obtained under rather restrictive assumptions (cf. [3], [5], [6], [11], [12], [26]).

Finally, two applications on nonlinear parabolic and Schrödinger type equations are given. Theorems 17, 18 are obvious modifications of [27], [10] taking $C^\infty([a,b],H^\infty)$ in place of $C^1([a,b],H^\infty)$. The additional part of the proof requires estimates for the higher order time derivatives of the solutions of the linearized problem. Since this is quite similar to the proof of Theorem 15 it will be omitted.

2 Preliminaries

In this preliminary section we state some notations and formulate the implicit function theorem which is proved in [25].

A Fréchet space E equipped with a fixed sequence $|\ |_0 \leq |\ |_1 \leq |\ |_2 \leq \dots$ of seminorms defining the topology is called a graded Fréchet space (cf. [5]). Graded subspaces and graded quotient spaces are equipped with the induced seminorms, the product $E \times F$ is graded by $|(x,y)|_k = \max\{|x|_k, |y|_k\}, x \in E, y \in F$. A linear map $A : E \to F$ between graded Fréchet spaces is called tame (cf. [5]) if there exist a fixed integer $b \geq 0$ and constants $c_n > 0$ so that $|Ax|_n \leq c_n |x|_{n+b}$ for all n and all $x \in E$; if $b = 0$ then A is called normwisely tame. E is called a tame direct summand of F if there exist tame linear maps $T : E \to F$ and $S : F \to E$ so that $S \circ T = \mathrm{id}_E$; if in addition $T \circ S = \mathrm{id}_F$ then we say that $E \cong F$ tamely isomorphic, and T is called a tame isomorphism. A short exact sequence $0 \to F \xrightarrow{i} G \xrightarrow{q} E \to 0$ of graded Fréchet spaces with tame linear maps is called tamely exact if the induced linear isomorphisms $\bar{i} : F \to iF$ and $\bar{q} : G/iF \to E$ are tame isomorphisms.

In [21] a smoothing property $(S_\Omega)_t$ is introduced for graded Fréchet spaces generalizing the classical concept of smoothing operators ([5], [7], [13], [16], [17], [33]).

Definition 1 *Let E be a graded Fréchet space.*

(i) E has property $(S)_t$ *(smoothing operators) if there are* $b, p \geq 0$ *and* $c_n > 0$ *so that for any* $\theta \geq 1$ *there is a (not necessarily linear) map* $S_\theta : E \to E$ *so that*

$$\begin{array}{rclclcl} |S_\theta x|_n & \leq & c_n \theta^{n+p-k} |x|_k & , & b \leq k \leq n+p & , & x \in E. \\ |x - S_\theta x|_n & \leq & c_k \theta^{n+p-k} |x|_k & , & k \geq n+p & , & x \in E. \end{array}$$

(ii) E has property (Ω_{DZ}) *(cf. [19]) if there exist* $p \geq 0$ *and constants* $c_n > 0$ *so that for* $U_n = \{x \in E : |x|_n \leq 1\}$ *and all* $n \geq p$ *and all* $r > 0$ *we have*

$$U_n \subset c_n(\cap_{i=p}^{n} r^{i-p} U_{n-i}) + (\cap_{k=-p}^{\infty} c_{n+k} r^{-k-p} U_{n+k}) .$$

(iii) E has property (Ω_{DS}) *(cf. [20]) if there exist constants* $c_n > 0$ *so that*

$$U_n \subset c_n(((rU_{n-1}) \cap U_n) + (((1/r)U_{n+1}) \cap U_n)) \ , \ 0 < r < 1 \ , \ n \geq 1.$$

(iv) E has smoothing property $(S_\Omega)_t$ *(cf. [21]) if there exists a tamely exact sequence* $0 \longrightarrow F \longrightarrow G \longrightarrow E \times H \longrightarrow 0$ *of graded Fréchet spaces so that F has property* (Ω_{DZ}) *and G has property* $(S)_t$ *where H is arbitrary.*

(v) E has property (DN) *(cf. [31]) if there exist* $b \geq 0$ *and for every n a constant* $c_n > 0$ *and k so that* $|x|_n^2 \leq c_n |x|_b |x|_{n+k}$ *for all* $x \in E$.

Remark 2 *(i) The properties* $(S)_t$, (Ω_{DZ}) *and* $(S_\Omega)_t$ *are preserved by tame isomorhpisms, (DN) by topological isomorphisms. If both E,F have property* $(S_\Omega)_t$ *(or* $(S)_t$, (Ω_{DZ})*, respectively) then* $E \times F$ *has this property as well.*

(ii) Property $(S)_t$ *implies* $(S_\Omega)_t$*, property* $(S_\Omega)_t$ *implies* (Ω_{DZ}).

(iii) If the compact $K \subset \mathbb{R}^n$ *is the closure of its interior and is subanalytic in the sense of Bierstone [1] then the space* $C^\infty(K)$ *satisfies both properties* $(S_\Omega)_t$ *and (DN) (cf. [21], 5.3., 5.4. for* $(S_\Omega)_t$ *and [1] for (DN)). If K has singularites like cusps then* $C^\infty(K)$ *does not admit classical smoothing operators (cf. [34]).*

(iv) Any graded Fréchet–Hilbert space which is an (Ω) *space in standard form has property* $(S_\Omega)_t$ *([21], 4.7.). Here Fréchet–Hilbert means that the seminorms are defined by semiscalar products, and* (Ω) *in standard form (cf. [33]) means estimates of the form* $|\ |_n^{*2} \leq c_n |\ |_{n-1}^* |\ |_{n+1}^*$ *for all n where the extended real valued dual norm is* $|\phi|_n^* := \sup\{|\phi(x)| : |x|_n \leq 1\} \in [0,\infty], \phi \in E'$.

The assumptions of the implicit function theorem [25], 4.3. are formulated in terms of the expressions $[\ ]_{m,k}$ forming part of a general symbolic calculus (cf. [25]).

Definition 3 ([25], 1.1.) *Let $p,q \geq 0$ be integers and $E_1,\ldots,E_p,F_1,\ldots,F_q$ be graded Fréchet spaces, $p+q \geq 1$. For $m,k \geq 0, x_i \in E_i, y_j \in F_j$ we put*

$$[x_1,\ldots,x_p;y_1,\ldots,y_q]_{m,k} = \sup\{1 \cdot |x_{k_1}|_{m+i_1} \ldots |x_{k_r}|_{m+i_r} |y_1|_{m+j_1} \ldots |y_q|_{m+j_q}\}$$

where the 'sup' is running over all $i_1,\ldots,i_r,j_1,\ldots,j_q \geq 0$ and $1 \leq k_1,\ldots,k_r \leq p$ with $0 \leq r \leq k$ and $i_1+\ldots+i_r+j_1+\ldots+j_q \leq k$ (for $r=0$ the $|x|$–terms are omitted). For $q=0$ we write $[x_1,\ldots,x_p]_{m,k}$ (where the $|y|$–terms are omitted), and for $p=0$ we write $[;y_1,\ldots,y_q]_{m,k}$. For $m=0$ we always put $[\ldots]_k = [\ldots]_{0,k}$.

In addition, for the vectors $x=(x_1,\ldots,x_p), y=(y_1,\ldots,y_q)$ we shall use the abbreviation $[x;y]_{m,k} = [x_1,\ldots,x_p;y_1,\ldots,y_q]_{m,k}$ and $[x;y]_k = [x;y]_{0,k}$.

Observe that the expression $[x_1,\ldots,x_p;y_1,\ldots,y_q]_{m,k}$ is a seminorm seperately in each y_j–component while it is 'completely nonlinear' in the x_j–components. The expressions $[\ ]_{m,k}$ are increasing in m and in k. For the purely nonlinear expressions we have $[x_1,\ldots,x_p]_{m,0} = 1$ and $[x_1,\ldots,x_p]_{m,k} \geq 1$ for all m,k.

The following theorem on implicit functions is proved in [25]. For the notion of differentiability in Fréchet spaces we refer to [5] or [25].

Theorem 4 (**on implicit functions** [25], 4.3.). *Let E,F,G be graded Fréchet spaces so that $E,F \in (\mathrm{S}_\Omega)_\mathrm{t}$ and $E,F,G \in$ (DN). Let $U \subset E$ and $V \subset F$ be open sets and $x_0 \in U, y_0 \in V$. Let $f: U \times V \to G$ be a C^∞–mapping, $f = f(x,y)$, and $f(x_0,y_0) = 0$. Assume that for any $w \in U \times V$ the partial derivative $f_y(w): F \to G$ is bijective so that for some fixed $d \geq 0$ and suitable constants $c_n > 0$ for all $n \geq 0$ the following estimates hold:*

$$\begin{array}{llll} (1) & |f'(w)x|_n \leq c_n[w;x]_{d,n} & , \ w \in U \times V & , \ x \in E \times F. \\ (2) & |f_y(w)^{-1}z|_n \leq c_n[w;z]_{d,n} & , \ w \in U \times V & , \ z \in G. \\ (3) & |f''(w)\{x,x\}|_n \leq c_n[w;x,x]_{d,n} & , \ w \in U \times V & , \ x \in U \times V. \end{array}$$

Then there exist open neighbourhoods $U_0 \subset U$ of x_0, $V_0 \subset V$ of y_0 and a C^∞–map $h: U_0 \to V_0$ so that $f(x,y) = 0$ is uniquely solved by $y = h(x)$ for all $(x,y) \in U_0 \times V_0$.

3 The Smoothing Property for some Function Spaces

For a graded Fréchet space E let E_k denote the Banach space obtained by completion of $(E/\ker|\ |_k, |\ |_k)$. We call E tamely nuclear (cf. [18], 3.2.) if there is a fixed $b \geq 0$ so that the canonical maps $E_{k+b} \to E_k$ are nuclear for all k; this definition coincides with other tame variants (cf. [18], 3.2.) of nuclearity. The spaces $C^\infty(K)$ in Remark (iii) are tamely nuclear by means of Whitney's extension theorem (cf. [21], 5., [22], 4.2., 4.12.) since tame nuclearity is inherited by graded quotient spaces. Any nuclear (DN)–space in standard form (i.e. $|\ |_k^2 \leq c_k |\ |_{k-1} |\ |_{k+1}$ for all k) and any nuclear space admitting smoothing operators is tamely nuclear (cf. [18], 3.3.). We call E locally l_1 if $E_k \cong l_1(J_k)$ for all k and suitable sets J_k.

For $0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \nearrow +\infty$ we consider the E–valued power series space

$$\Lambda_\infty^\infty(\alpha)\{E\} = \{x = (x_j)_{j=1}^\infty \subset E : |x|_k = \sup_{i=0,\ldots,k} \sup_{j \in \mathbb{N}} |x_j|_{k-i} e^{i\alpha_j} < +\infty \text{ for all } k\}.$$

Observe that this grading is not the 'usual tensor product grading' (indicated by the curved brackets). If $E = \mathbb{R}$ or $E = \mathbb{C}$ then we write $\Lambda_\infty^\infty(\alpha)$, and for $\varepsilon > 0$ we put $s_\varepsilon = \Lambda_\infty^\infty(\varepsilon \log j)$ and $s = s_1$. The space $s\{E\}$ is defined in the same way. Likewise the space $C^\infty\{[-1,1],E\}$ of all E–valued C^∞–functions in $[-1,1]$ is equipped with

$$|f|_k = \sup_{i=0,\dots,k} \sup_{-1\le x\le 1} |f^{(i)}(x)|_{k-i} \, , \; f \in C^\infty\{[-1,1],E\}.$$

Analogously we can consider the graded Fréchet space $C_{2\pi}^\infty\{\mathbb{R},E\}$ of all 2π–periodic C^∞–functions $f : \mathbb{R} \to E$ equipped with the grading of $C^\infty\{[-\pi,\pi],E\}$.

For tamely nuclear spaces properties (Ω_{DZ}) and $(\mathrm{S}_\Omega)_{\mathrm{t}}$ are equivalent by means of the following theorem which is proved in [19], 7.1. We say that a linear map $q : G \to E$ between graded Fréchet spaces has property $(*)_{\mathrm{t}}$ (cf. [19], 5.1.) if there are a fixed $b \ge 0$ and constants $C_k > 0$ so that for $U_k = \{x \in G : |x|_k \le 1\}$ we have

$$(*)_{\mathrm{t}} \qquad \cap_{i=0}^k r^i q(U_{k-i}) \subset C_k q(\cap_{i=b}^k r^{i-b} U_{k-i}) \; , \; k \ge b \, , \, 0 < r \le 1 \, .$$

Theorem 5 ([19], 7.1.) *For a graded Fréchet space E the following are equivalent:*

(i) *E is tamely nuclear and has property* (Ω_{DZ}).

(ii) *E is tamely isomorphic to a graded quotient space of some nuclear* $\Lambda_\infty^\infty(\alpha)$.

(iii) *There exist* $\varepsilon > 0$ *and a tamely exact sequence* $0 \to s_\varepsilon \to s_\varepsilon \xrightarrow{q} E \to 0$ *so that q has property* $(*)_{\mathrm{t}}$.

(iv) *E is tamely nuclear and has property* $(\mathrm{S}_\Omega)_{\mathrm{t}}$.

The proof of Theorem 5 given in [19], 7.1. relies on the existence of tamely exact sequences of the form $0 \to s_\varepsilon \to s_\varepsilon \to (s_\varepsilon)^{\mathbb{N}} \to 0$ satisfying the lifting property $(*)_{\mathrm{t}}$ (cf. [19], 6.11.); for a different construction of such sequences we refer to [29], 4.1.

Lemma 6 *Let* $0 \to F \xrightarrow{d} G \xrightarrow{q} E \to 0$ *be tamely exact where q has property* $(*)_{\mathrm{t}}$. *Then* $D(x_j)_j = (dx_j)_j$ *and* $Q(x_j)_j = (qx_j)_j$ *give rise to a tamely exact sequence*

$$0 \to s\{F\} \xrightarrow{D} s\{G\} \xrightarrow{Q} s\{E\} \to 0 \, .$$

Proof. We only have to show the tame surjectivity of Q. We consider D,Q also as maps $D : F^{\mathbb{N}} \to G^{\mathbb{N}}$ and $Q : G^{\mathbb{N}} \to E^{\mathbb{N}}$. Let $U_k = \{x \in G : |x|_k \le 1\}$ and choose b so that condition $(*)_{\mathrm{t}}$ above holds for q. We choose $a,p \ge 0$ and constants $1 \le D_k \le D_{k+1}$ so that $\inf\{|\xi|_k : q\xi = z\} \le \frac{1}{2} D_k |z|_{k+a}$ and $|dx|_k \le D_k |x|_{k+p}, |x|_k \le D_k |dx|_{k+p}$ for all $x \in F, z \in E$ and all k.

Now let $n \ge 0$ and $z = (z_j)_j \in s\{E\}$ such that $|z|_{n+a+b} \le 1$. For $0 \le i \le n+b$ we choose $x_{j,i} \in G$ with $qx_{j,i} = z_j$ and $|x_{j,i}|_{n+b-i} \le D_{n+b} |z_j|_{n+a+b-i} \le D_{n+b} j^{-i}$; hence $(*)_{\mathrm{t}}$ implies that $z_j \in D_{n+b} \cap_{i=0}^{n+b} j^{-i} q(U_{n+b-i}) \subset C_{n+b} D_{n+b} q(\cap_{i=0}^n j^{-i} U_{n-i})$.

We fix $z \in s\{E\}, k \ge 2p+2, \varepsilon > 0$. For every n there is $x^n = (x_j^n)_j \in G^{\mathbb{N}}$ so that $Qx^n = z$ and $|x^n|_n \le C_{n+b} D_{n+b} |z|_{n+a+b}$. For fixed n the finite vectors are dense in $\{x = (x_j)_j \in F^{\mathbb{N}} : \|x\|_n := \sup_{0\le i \le n} \sum_j |x_j|_{n-i} j^i < +\infty\}$ with respect to $\| \;\; \|_n$, in particular $s\{F\}$ is dense. Thus for any

$n \geq p+2$ we can choose $h^n \in s\{F\}$ so that $\|D^{-1}(x^{n+1}-x^n)-h^n\|_{n-p-2} \leq 2^{-n}D_n^{-1}\varepsilon$. Here we consider D^{-1} as a map $D(F^{\mathbb{N}}) \to F^{\mathbb{N}}$ and observe that $x^{n+1}-x^n \in D(F^{\mathbb{N}})$ and $\|D^{-1}(x^{n+1}-x^n)\|_{n-p-2} \leq 2D_n|x^{n+1}-x^n|_n < +\infty$. Putting $H^n = Dh^n$ we get $|x^{n+1}-x^n-H^n|_{n-2p-2} \leq 2^{-n}\varepsilon$ for $n \geq 2p+2$. For $x = x^k + \sum_{n\geq k}(x^{n+1}-x^n-H^n)$ we have $x \in s\{G\}$ and $Qx = z$. The result follows since $|x|_{k-2p-2} \leq C_{k+b}D_{k+b}|z|_{k+a+b}+\varepsilon$. □

Remark 7 *(i) Let $0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \nearrow +\infty$ and $0 \leq \beta_1 \leq \beta_2 \leq \ldots \nearrow +\infty$ be two sequences. Then there is a sequence $0 \leq \gamma_1 \leq \gamma_2 \leq \ldots \nearrow +\infty$ so that*

$$\Lambda^\infty_\infty(\alpha)\{\Lambda^\infty_\infty(\beta)\} \cong \Lambda^\infty_\infty(\gamma) \ \text{normwisely tamely isomorphic}$$

since for $x = (x^n)_n \in \Lambda^\infty_\infty(\alpha)\{\Lambda^\infty_\infty(\beta)\}, x^n = (x^n_j)_j \in \Lambda^\infty_\infty(\beta)$ we have

$$|x|_k = \sup_{0\leq i\leq k} \sup_{n,j} |x^n_j| e^{i\alpha_n} e^{(k-i)\beta_j} = \sup_{n,j} |x^n_j| \max\{e^{\alpha_n}, e^{\beta_j}\}^k$$

and an increasing rearrangement of $(\max\{e^{\alpha_n}, e^{\beta_j}\})_{n,j}$ gives the assertion.

(ii) For every graded Fréchet space E we have $C^\infty_{2\pi}\{\mathbb{R}, E\} \cong s\{E\}$ tamely isomorphic. This follows by Fourier expansion (cf. [22], 3.1.).

(iii) For a graded Fréchet space E the space $C^\infty\{[-1,1],E\}$ is a tame direct summand of $C^\infty_{2\pi}\{\mathbb{R},E\}$. For this it is enough to show the existence of a (normwisely) tame linear extension operator $C^\infty\{[-1,1],E\} \to C^\infty\{[-4,4],E\}$ because then cutting off yields the result. This extension operator can be obtained by applying Seeley's construction [30] (for the estimates see [22], 4.6.).

(iv) If E has property (DN) then $C^\infty\{[-1,1],E\}$ has (DN) as well (cf. [22], 4.11.).

Theorem 8 *Let the tamely nuclear graded Fréchet space E have property (Ω_{DZ}). Then the spaces E and $C^\infty\{[-1,1],E\}$ have the smoothing property $(\mathrm{S}_\Omega)_{\mathrm{t}}$.*

Proof. By Theorem 5 there is a tamely exact sequence $0 \to s_\varepsilon \to s_\varepsilon \xrightarrow{q} E \to 0$ where q has property $(*)_{\mathrm{t}}$. Therefore Theorem 6 gives a tamely exact sequence $0 \to s\{s_\varepsilon\} \to s\{s_\varepsilon\} \to s\{E\} \to 0$. Remark (i) shows that $s\{s_\varepsilon\}$ has properties $(\mathrm{S})_{\mathrm{t}}$ and (Ω_{DZ}); thus $s\{E\}$ enjoys smoothing property $(\mathrm{S}_\Omega)_{\mathrm{t}}$. By definition, condition $(\mathrm{S}_\Omega)_{\mathrm{t}}$ is inherited by tame direct summands; hence $C^\infty\{[-1,1],E\}$ satisfies this property as a tame direct summand of $s\{E\}$ by means of Remarks (ii), (iii). □

Lemma 9 *Let E be a graded Fréchet–Hilbert space which is an (Ω)–space in standard form. Then there exist a Hilbert space H and a tamely exact sequence*

$$0 \to s\{H\} \to s\{H\} \xrightarrow{q} E \times s\{H\} \to 0$$

where q has property $(*)_{\mathrm{t}}$.

Proof. The existence of a tamely exact sequence of this form is proved in [21], 4.7. It remains to show property $(*)_t$ for the map q constructed there. There exist a Hilbert space H and a tamely exact sequence $0 \to E \to H^{\mathbb{N}} \to H^{\mathbb{N}} \to 0$ (cf. [21], 4.6.). For H there is a tamely exact sequence $0 \to s\{H\} \to s\{H\} \xrightarrow{p} H^{\mathbb{N}} \to 0$ where p has property $(*)_t$; this is proved in the case $H = \mathbb{K}$ in [19], 6.11.; the same proof gives the result for a Hilbert space H. A well known construction of D. Vogt [32], 3.3. gives a commutative diagram with tamely exact columns and rows.

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & E & \longrightarrow & H^{\mathbb{N}} & \longrightarrow & H^{\mathbb{N}} & \longrightarrow & 0 \\ & & & & \uparrow P & & \uparrow p & & \\ 0 & \longrightarrow & E & \longrightarrow & G & \longrightarrow & s\{H\} & \longrightarrow & 0 \\ & & & & \uparrow & & \uparrow & & \\ & & & & s\{H\} & & s\{H\} & & \\ & & & & \uparrow & & \uparrow & & \\ & & & & 0 & & 0 & & \end{array}$$

It is easy to see that the map P in the first column inherits property $(*)_t$ from the map p in the second column (it is not necessary that the surjective map in the first row has property $(*)_t$). The second row splits tamely by means of the tame splitting theorem [28], 6.1. The same procedure applied with the first row $0 \to s\{H\} \to E \times s\{H\} \to H^{\mathbb{N}} \to 0$ and the same second column as above gives the desired tamely exact sequence (cf. [21], 4.4.) where the map q appearing in the first column inherits property $(*)_t$ from the map p occuring in the second column. □

Theorem 10 *Let the graded Fréchet–Hilbert space E have* (Ω) *in standard form. Then the spaces E and* $C^\infty\{[-1,1],E\}$ *have the smoothing property* $(\mathrm{S}_\Omega)_t$.

Proof. This holds for E by Lemma 9. As in the proof of Theorem 8 it remains to show that $s\{E\}$ has $(\mathrm{S}_\Omega)_t$. Lemmas 6 and 9 give a tamely exact sequence

$$0 \to s\{s\{H\}\} \to s\{s\{H\}\} \to s\{E\} \times s\{s\{H\}\} .$$

This shows the assertion in view of the previous remark (i). □

Examples 1 *The following spaces satisfy the assumptions of Theorem 10.*

(i) $E = H^\infty(\mathbb{R}^n) = \cap_{k=0}^\infty H^k(\mathbb{R}^n)$ *with its canonical norms (cf. [14], 14.12.). We notice that* $H^\infty(\mathbb{R}^n)$ *is by no means nuclear.*

(ii) $E = H^\infty(\Omega) = \cap_{k=0}^\infty H^k(\Omega)$ *for a bounded open* $\Omega \subset \mathbb{R}^n$ *with* C^∞*–boundary.*

(iii) *Let m be a positive integer. With E then* $(E, \|\ \|_{mk})$ *satisfies the assumptions of Theorem 10 and thus* $C^\infty\{[-1,1],(E,|\ \ |_{mk})\}$ *has property* $(\mathrm{S}_\Omega)_t$. *In particular, this applies to* $E = H^\infty(\mathbb{R}^n)$. *Hence* $C^\infty\{[-1,1],H^\infty(\mathbb{R}^n)\}$ *has* $(\mathrm{S}_\Omega)_t$ *for the grading (which naturally appears in the theory of evolution equations where m denotes the highest order of the space derivatives involved)*

$$|u|_k = \sum_{0 \le i \le k} \sup_{-1 \le t \le 1} \|u^{(i)}(t)\|_{H^{m(k-i)}(\mathbb{R}^n)}, \ u \in C^\infty\{[-1,1],H^\infty(\mathbb{R}^n)\}.$$

Theorem 11 *Let the graded Fréchet space E be locally l_1 and have property* (Ω_{DS}). *Then both the spaces E and $C^\infty\{[-1,1],E\}$ have the smoothing property* $(\mathrm{S}_\Omega)_{\mathrm{t}}$.

Proof. By [26], 2.4. (cf. Theorem 10) there exist a Banach space B and a tamely exact sequence $0 \to s\{B\} \to s\{B\} \stackrel{q}{\to} E \times s\{B\} \to 0$ where q has property $(*)_{\mathrm{t}}$. □

For instance, the convolution algebra $(\lambda^1(a),*)$ (cf. [26], 5.4.) satisfies the assumptions of Theorem 11 if the Köthe space $\lambda^1(a)$ has (Ω) in standard form.

4 Linear Equations

In this section some results on linear ordinary differential equations in Fréchet spaces are stated which are important for the application of the Nash–Moser implicit function theorem in the next section. For a graded Fréchet space E the space

$$LNT(E) \;=\; \{A : E \to E \text{ linear } : |A|_n := \sup_{|x|_n \le 1} |Ax|_n < +\infty \text{ for all } n\}$$

of all normwisely tame endomorphisms of E is a graded Fréchet space for the grading $||A||_k = \sup_{0 \le n \le k} |A|_n$. The following generalization of the theorem of Picard–Lindelöf for linear equations in Banach spaces (cf. [35], 3.3.) is proved in [26], 3.4.

Lemma 12 *Let E be a graded Fréchet space, $I \subset \mathbb{R}$ an open interval, $t_0 \in I$, let $A : I \to LNT(E), b : I \to E$ be continuous. Then the linear initial value problem*

$$\text{(IVP)} \qquad y'(t) \;=\; A(t)y(t) \;+\; b(t)\;,\; y(t_0) \;=\; y_0$$

has for any $y_0 \in E$ a unique solution $y \in C^1(I,E)$.

The following Lemma generalizing Gronwall's Lemma is proved in [26], 3.6., 3.7.

Lemma 13 *Let E be a graded Fréchet space. Let $A : J \to LNT(E)$ and $b : J \to E$ be continuous where $J = [0,d], d > 0$ and $t_0 = 0$. Assume that*

$$|A(t)x|_k \le D_k \sum_{i=0}^{k} B_{k-i}|x|_i \;,\; |b(t)|_k \le b_k \textit{ for all } t \in J, x \in E, k = 0,1,\ldots$$

where $0 \le b_k \le b_{k+1}, 0 \le D_k \le D_{k+1}$ and $B_iB_j \le D_{i,j}B_{i+j}$ for all i,j and $D_{i,j} > 0$. Then for $y_0 \in E$ the unique solution $y \in C^1(J,E)$ of (IVP) *satisfies for all k with constants $C_k = C(k,d,B_0,D_k,(D_{i,j})_{1 \le i+j \le k})$ the inequalities*

$$|y(t)|_k \le C_k \sum_{i=0}^{k} (1 + B_{k-i})(b_i + |y_0|_i).$$

Lemma 14 *Let $I \subset \mathbb{R}$ be an interval, $b \in C^\infty(I,E)$ and $A \in C^\infty(I,LNT(E))$. Let $y \in C^1(I,E)$ satisfy $y'(t) = A(t)y(t) + b(t)$ for $t \in I$. Then it follows that $y \in C^\infty(I,E)$, and for any $k \ge 1$ there are constants $c_{i_1,\ldots,i_r}$ and $c_{i_1,\ldots,i_r;j}$ so that*

$$y^{(k)}(t) = b^{(k-1)}(t) + \sum_{\substack{0 \le i_1+\ldots+i_r \le k-1 \\ 1 \le r \le k, i_1,\ldots,i_r \ge 0}} c_{i_1,\ldots,i_r} A^{(i_1)}(t) \circ \ldots \circ A^{(i_r)}(t) y(t)$$

$$+ \sum_{\substack{0 \le i_1+\ldots+i_r+j \le k-2 \\ 1 \le r \le k-1, i_1,\ldots,i_r \ge 0, j \ge 0}} c_{i_1,\ldots,i_r;j} A^{(i_1)}(t) \circ \ldots \circ A^{(i_r)}(t) b^{(j)}(t).$$

Proof. For $A, B \in C^1(I, LNT(E))$ and $y \in C^1(I,E), z(t) = A(t)y(t)$ we have

$$z'(t) = A'(t)y(t) + A(t)y'(t) \quad , \quad (A(t)B(t))' = A'(t)B(t) + A(t)B'(t).$$

Applying these rules the assertion follows by induction on k. □

5 Nonlinear Equations

If E, F are graded Fréchet spaces and $U \subset E$ is an open set then a continuous nonlinear map $f : (U \subset E) \to F$ is called a C^∞–map if all directional derivatives $f^{(n)} : U \times E^n \to F$, $(u; e_1, \ldots, e_n) \mapsto f^{(n)}(u)\{e_1, \ldots, e_n\}$ (cf. [5], I. 3.) exist and are continuous; in this case $f^{(n)}(u)\{e_1, \ldots, e_n\}$ is completely symmetric and linear seperately in $e_1, \ldots, e_n$ (cf. [5], I. 3.6.2.).

Let E, P be graded Fréchet spaces, let $U \subset \mathbb{R} \times E \times P$ be an open set, and let $f : U \to E$ be a C^∞–map, $f = f(t,x,p)$. We then can consider the partial derivative $\frac{\partial^i}{\partial t^i} \frac{\partial^j}{\partial x^j} \frac{\partial^k}{\partial p^k} f$ as a map $\partial_t^i \partial_x^j \partial_p^k f : U \times \mathbb{R}^i \times E^j \times P^k \to E$ (or as a map $U \times E^j \times P^k \to E$), cf. [5], I. 3.4. Using multiindices we can write $\partial^\alpha f = \partial_t^{\alpha_1} \partial_x^{\alpha_2} \partial_p^{\alpha_3} f$ for $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3$; note that $\partial^\alpha \partial^\beta f = \partial^\beta \partial^\alpha f$ for any α, β (cf. [5], I. 3.5.3.).

We now formulate and prove the main result of this paper. Several arguments are quite similar to the proof in [26]. The additional part of the proof are the estimates for the higher order time derivatives for the solutions of the linearized problem. To preserve self–containedness we here give the full proof.

Theorem 15 *Let E, P be graded Fréchet spaces satisfying* (DN) *and* $(S_\Omega)_t$ *such that $C^\infty\{[-1,1], E\}$ has* $(S_\Omega)_t$. *Let $U \subset \mathbb{R} \times E \times P$ be open and $(t_0, x_0, p_0) \in U$. Let $f : U \to E$ be a C^∞–map, $f = f(t,x,p)$. We consider the initial value problem*

$$\text{(P)} \qquad \begin{cases} x'(t) &= f(t, x(t), p) \quad , \quad t \in [t_0 - a, t_0 + a] \\ x(t_0) &= y \end{cases}$$

for $a > 0, y \in E, p \in P$. We assume that for any $n \ge 0$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3$ there is a constant $c_{n,\alpha} > 0$ so that for all $u \in U$ and $z \in \mathbb{R}^{\alpha_1} \times E^{\alpha_2} \times P^{\alpha_3}$ we have

$$(*) \qquad |\partial^\alpha f(u)(z)|_n \le c_{n,\alpha}[u; z]_n .$$

(i) *Then there exist $a > 0$ and open neighbourhoods $U(x_0)$ of x_0 and $U(p_0)$ of p_0 such that problem* (P) *has for any $(y,p) \in U(x_0) \times U(p_0)$ a unique solution $x \in C^\infty([t_0 - a, t_0 + a], E)$.*

(ii) *The solution map $(y,p) \mapsto \Psi(y,p)$ is C^∞ considered as a map*

$$\Psi : U(x_0) \times U(p_0) \to C^\infty([t_0 - a, t_0 + a], E).$$

(iii) *If x_1, x_2 are C^1–solutions of* (P) *for t in an interval J then $x_1 = x_2$ in J.*

(iv) *The solution map $(t,y,p) \mapsto x(t;y,p)$ is C^∞ on $(t_0 - a, t_0 + a) \times U(x_0) \times U(p_0)$. There is a maximal open neighbourhood U_1 of (t_0, x_0, p_0) in U where the solution map $(t,y,p) \mapsto x(t;y,p)$ exists and is C^∞.*

Remark 16 *For $u = (t,x,p)$ and $z = (t_1, \ldots, t_{\alpha_1}, x_1, \ldots, x_{\alpha_2}, p_1, \ldots, p_{\alpha_3})$ the term $[u;z]_n$ in $(*)$ is defined by $[u;z]_n = [t,x,p;t_1, \ldots, t_{\alpha_1}, x_1, \ldots, x_{\alpha_2}, p_1, \ldots, p_{\alpha_3}]_n$.*

Proof. We write $J = [-1,1]$ and consider the transformation in the time variable $t = t_0 + as, z(s) = x(t_0 + as) - y, s \in J$. Now problem (P) is equivalent to

$$\text{(P1)} \qquad \left\{ \begin{array}{rcl} z'(s) & = & af(t_0 + as, z(s) + y, p) \quad , \qquad s \in J \\ z(0) & = & 0 \end{array} \right. .$$

The map $\Phi : (W \subset C^\infty\{J,E\} \times \mathbb{R} \times E \times P)) \to C^\infty\{J,E\} \times E$ defined by

$$\Phi(z,a,y,p) \ = \ (z'(s) - af(t_0 + as, z(s) + y, p), z(0)) \ , \ s \in J$$

is a C^∞–map defined in the open subset $W \subset C^\infty\{J,E\} \times \mathbb{R} \times E \times P$ where

$$W \ = \ \{(z,a,y,p) : (t_0 + as, z(s) + y, p) \in U \text{ for all } s \in J\}.$$

We observe that $(0,0,x_0,p_0) \in W, \Phi(0,0,x_0,p_0) = (0,0)$. Further $(\mathrm{P1})$ is equivalent to $\Phi(z,a, y, p) = (0,0)$. The partial derivative Φ_z is given by

$$\Phi_z(z,a,y,p)(w) = (w'(s) - af_x(t_0 + as, z(s) + y, p)w(s), w(0)) \ , \ s \in J$$

for $(z,a,y,p) \in W, w \in C^\infty\{J,E\}$. In order to apply Theorem 4 we have to solve the equation $\Phi_z(z,a,y,p)(w) = (u,u_0)$ for $u \in C^\infty\{J,E\}, u_0 \in E$. We hence consider

$$\text{(LP)} \qquad \left\{ \begin{array}{rcl} w'(s) & = & A(s)w(s) + u(s) \quad , \qquad s \in J \\ w(0) & = & u_0 \end{array} \right.$$

where $A(s) := af_x(t_0 + as, z(s) + y, p)$. Problem (LP) admits a unique solution $w \in C^\infty\{J,E\}$ since $A \in C^\infty(J, LNT(E))$ in view of $(*)$. Hence $\Phi_z(z,a,y,p)$ is bijective as a map $C^\infty\{J,E\} \to C^\infty\{J,E\} \times E$ for all $(z,a,y,p) \in W$. Next we verify the estimates (1), (2), (3) in Theorem 4. Shrinking U if necessary we may assume that $|(z,a,y,p)|_0 \le C$ for some $C > 0$ and all $(z,a,y,p) \in W$. We have

$$\Phi'(u)\{w,b,x,q\} = \Phi_z(u)w + \Phi_a(u)b + \Phi_y(u)x + \Phi_p(u)q$$

for all $u \in W$ and $w \in C^\infty\{J,E\}, b \in \mathbb{R}, x \in E, q \in P$. Here $\Phi_z(u)w$ is as above and

$$\begin{array}{rcl} \Phi_a(z,a,y,p)b & = & (-bf(t_0 + as, z(s) + y, p) - absf_t(t_0 + as, z(s) + y, p), 0). \\ \Phi_y(z,a,y,p)x & = & (-af_x(t_0 + as, z(s) + y, p)x, 0). \\ \Phi_p(z,a,y,p)q & = & (-af_p(t_0 + as, z(s) + y, p)q, 0). \end{array}$$

For any $n \geq 0$ there are constants $c_* = c_{i;j;i_1,\ldots,i_{j-1}}$ so that

$$A^{(n)}(s)x = \sum c_* a^{i+1} \partial_t^i \partial_x^j f(t_0 + as, z(s) + y, p)\{z^{(i_1)}(s), \ldots, z^{(i_{j-1})}(s), x\}$$

the sum taken over $1 \leq i + j \leq n + 1, i \geq 0, j \geq 1$ with $1 \leq i_1 + \ldots + i_{j-1} \leq n, i_t \geq 1$ where $x \in E, s \in J$; this can be seen by induction on n. We have

$$|\Phi_z(z,a,y,p)w|_k \leq |w|_{k+1} + |A(s)w(s)|_k \ .$$

Inserting the above formula and writing *sup* for the supremum running over the set

$$\{0 \leq n \leq l \leq k, i + j \leq n + 1, i \geq 0, j \geq 1, 1 \leq i_1 + \ldots + i_{j-1} \leq n, i_t \geq 1, s \in J\}$$

we get for $u := (z,a,y,p)$ with constants $c_k > 0$ (which may vary from line to line)

$$\begin{aligned}
&|A(s)w(s)|_k \\
&\quad \leq c_k sup|\partial_t^i \partial_x^j f(t_0 + as, z(s) + y, p)\{z^{(i_1)}(s), \ldots, z^{(i_{j-1})}(s), w^{(l-n)}(s)\}|_{k-l} \\
&\quad \leq c_k sup[t_0 + as, z(s) + y, p; z^{(i_1)}(s), \ldots, z^{(i_{j-1})}(s), w^{(l-n)}(s)]_{k-l} \\
&\quad \leq c_k sup \sup_{m_0 + \ldots + m_j \leq k-l} [u]_{m_0} |z^{(i_1)}(s)|_{m_1} \cdot \ldots \cdot |z^{(i_{j-1})}(s)|_{m_{j-1}} |w^{(l-n)}(s)|_{m_j} \\
&\quad \leq c_k sup \sup_{m_0 + \ldots + m_j \leq k-l} [u]_{m_0} |z|_{i_1 + m_1} \cdot \ldots \cdot |z|_{i_{j-1} + m_{j-1}} |w|_{m_j + l - n} \\
&\quad \leq c_k \ [(z,a,y,p); w]_k.
\end{aligned}$$

We proved that $|\Phi_z(u)w|_k \leq |w|_{k+1} + c_k[u;w]_k \leq c_k[u;w]_{1,k}$ for all $u \in W$ and $w \in C^\infty\{J,E\}$. Analogous calculations show that $|\Phi_a(u)b|_k \leq c_k[u;b]_k$ and $|\Phi_y(u)x|_k \leq c_k[u;x]_k$ and $|\Phi_p(u)q|_k \leq c_k[u;q]_k$. This gives $|\Phi'(u)v|_k \leq c_k[u;v]_{1,k}$ for $u \in W, v \in C^\infty\{J,E\} \times \mathbb{R} \times E \times P$ proving inequality (1) in Theorem 4.

To show inequality (3) in Theorem 4 we have to estimate $|\Phi''(u)\{v,v\}|_k$ for $u, v \in W$. The second derivative $\Phi''(u)\{v,v\}$ is given as a finite sum involving terms $\Phi_{zz}, \Phi_{aa}, \Phi_{yy}, \Phi_{pp}, \Phi_{za}, \Phi_{zy}, \Phi_{zp}, \Phi_{ay}, \Phi_{ap}, \Phi_{yp}$. We here consider the term Φ_{zz}; the other second order partial derivatives are estimated analogously. We have

$$\Phi_{zz}(z,a,y,p)\{w,w\} \ = \ (-af_{xx}(t_0 + as, z(s) + y, p)\{w(s), w(s)\}, 0) \ , \ s \in J \ .$$

For derivatives $B^{(n)}(s)$ of the bilinear mapping $B(s) = af_{xx}(t_0 + as, z(s) + y, p)$ we get the same formula as for $A^{(n)}(s)$ with the modifications $j \geq 2, 2 \leq i + j \leq n + 2$, $1 \leq i_1 + \ldots + i_{j-2} \leq n$ and replacing 'x' by '(x,x)'. Applying the formula

$$\left(\tfrac{d}{ds}\right)^l (B(s)\{w(s), w(s)\}) = \sum_{i+j+n=l} c_{i,j,n} B^{(n)}(s)\{w^{(i)}(s), w^{(j)}(s)\}$$

the same arguments as above give the estimates $|\Phi_{zz}(u)\{w,w\}|_k \leq c_k[u;w,w]_k$ for $u \in W, w \in C^\infty\{J,E\}$. Analogous estimates for the other second partial derivatives yield $|\Phi''(u)\{v,v\}|_k \leq c_k[u;v,v]_k$ for $u, v \in W$ proving inequality (3). It remains to show inequality (2). For that we fix $u \in C^\infty\{J,E\}$ and $u_0 \in E$ and assume that $w \in C^\infty\{J,E\}$ is a solution of (LP). We have to estimate $|w|_k$ in terms of u, u_0 and (z,a,y,p) where $|(z,a,y,p)|_0 \leq C$. We have already proved that

$$\sup_{s \in J} |A(s)x|_k \ \leq \ c_k[(z,a,y,p);x]_k \ \leq \ c_k \sup_{0 \leq i \leq k} [z,a,y,p]_{k-i}|x|_i \ , \ x \in E.$$

We apply Lemma 13 with $B_i = [z,a,y,p]_i$ observing that $[u]_i[u]_j \leq [u]_{i+j}$ and get

$$\sup_{s\in J}|w(s)|_k \leq c_k \sup_{i=0}^{k}[z,a,y,p]_{k-i}(\sup_{s\in J}|u(s)|_i + |u_0|_i) \leq c_k[(z,a,y,p);(u,u_0)]_k.$$

In order to estimate the derivatives $w^{(k)}(s)$ we first observe that the above estimates for $|A(s)w(s)|_k$ also show that $|A^{(i)}(s)x|_j \leq c_{i+j}[(z,a,y,p);x]_{i+j}, i,j \geq 0, x \in E$. This implies $[(z,a,y,p);A^{(i)}(s)x]_j \leq c_{i+j}[(z,a,y,p);x]_{i+j}$. Lemma 14 and the above estimate for $|w(s)|_k$ imply for $1 \leq i \leq k$ and $s \in J$ the inequalities

$$\begin{aligned} |w^{(i)}(s)|_{k-i} \leq \quad & |u^{(i-1)}(s)|_{k-i} + c_k[(z,a,y,p);w(s)]_k \\ & + c_k \sum_{j=0}^{i}[(z,a,y,p);u^{(j)}(s)]_{k-j} \leq c_k[(z,a,y,p);(u,u_0)]_k. \end{aligned}$$

This shows $|w|_k = \sup_{0\leq i\leq k}\sup_{s\in J}|w^{(i)}(s)|_{k-i} \leq c_k[(z,a,y,p);(u,u_0)]_k$ and thus (2).

We apply Theorem 4 to Φ and obtain open neighbourhoods $U_0 \subset U$ of $(0,x_0,p_0)$ in $\mathbb{R} \times E \times P$ and V_0 of 0 in $C^\infty\{J,E\}$ and a C^∞–map $\Psi_1 : U_0 \to V_0$ so that equation $\Phi(z,a,y,p) = (0,0)$ is uniquely solved by $z = \Psi_1(a,y,p)$ for all $(a,y,p) \in U_0$ and $z \in V_0$. We hence can find $\delta > 0$ and open neighbourhoods $U(x_0)$ of x_0 and $U(p_0)$ of p_0 such that $\Psi_1 : (-\delta,\delta) \times U(x_0) \times U(p_0) \to V_0$ is a C^∞–map and $z = \Psi_1(a,y,p)$ is a solution of (P1) which is unique V_0. Using the above transformation $z(s) = x(t_0 + as) - y$ we thus can choose a fixed small number $a > 0$ and a C^∞–map $\Psi : U(x_0) \times U(p_0) \to C^\infty\{[t_0 - a, t_0 + a], E\}$ so that $x = \Psi(y,p)$ is a solution of (P). This proves (ii) and the existence part of (i). Now the proof of uniqueness is standard and will be omitted (cf. [26]). A straightforward continuation argument gives (iv), and the theorem is proved.

A lot of examples of spaces E,P and mappings f satisfying the assumptions of Theorem 15 are given in [26], for instance, applications are given there to the spaces $C^\infty(K), \mathcal{S}(\mathbb{R}^n)$, $\mathcal{B}(\mathbb{R}^n), \lambda^1(a), \mathcal{D}_{L_1}(\mathbb{R}^n)$ and the general class of subbinomic Fréchet algebras (cf. [26]), showing that the hypotheses of Theorem 15 are natural. Instead of repeating these examples we prefer giving here some applications of the above Nash–Moser technique to the proof of well posedness results for nonlinear evolutionary partial differential equations. Due to the limited space we here only mention two particular examples without carrying out the details. The first example in the parabolic case can be obtained with the arguments used in [27]; the second example on nonlinear Schrödinger type equations follows from the results in [10].

Theorem 17 *(cf. [27]). Let $\mathcal{F}$ be a nonlinear partial differential operator of even order m given by $\mathcal{F}(u)(x) = F(\{\partial^\alpha u(x)\}_{|\alpha|\leq m})$ for some real C^∞–map F with $F(0) = 0$. Let $\phi \in H^\infty(\mathbb{R}^n)$ and assume that $\mathcal{F}'(\phi)$ is elliptic, i.e.,*

$$(-1)^{m/2}\sum_{|\alpha|=m} F_{\partial^\alpha u}(\{\partial^\beta\phi(x)\}_{|\beta|\leq m})\xi^\alpha \geq \mu|\xi|^m$$

for all $x,\xi \in \mathbb{R}^n$ and some $\mu > 0$. Then the initial value problem

$$\begin{cases} u_t &= \mathcal{F}(u) \;,\; t \in [0,a] \\ u(0) &= \phi \end{cases}$$

has a unique solution $u \in C^\infty\{[0,a], H^\infty(\mathbb{R}^n)\}$ *for some* $a > 0$. *The solution map* $(U \subset H^\infty(\mathbb{R}^n)) \to C^\infty\{[0,a], H^\infty(\mathbb{R}^n)\}$ *is* C^∞ *in some neighbourhhood* U *of* ϕ.

In [27] the more general case of nonlinear smooth differential operators of the form $\mathcal{F}(t,u)(x) = F(t,x,\{\partial^\alpha u(x)\}_{|\alpha|\leq m})$ is treated under some mild technical assumptions on the map F; obviously, this case could be considered here as well. Theorem 17 improves the statement on the smooth dependence of the solutions from the initial value proved in [27], 6.10. The proof of Theorem 17 is based on the arguments used in [27]. In addition, the proof requires estimates for the higher order time derivatives for the solutions of the linearized problem; since this is similar to the proof of Theorem 15 it will be omitted.

Theorem 18 *Let* $\kappa \in \mathbb{R}$ *and* $\phi_0 \in H^\infty(\mathbb{R})$ *such that* $2\kappa|\phi_0(x)|^2 < 1$ *for all* $x \in \mathbb{R}$. *Then there exist* $a > 0$ *and an open neighbourhood* U *of* ϕ_0 *in* $H^\infty(\mathbb{R})$ *such that*

$$\begin{cases} iu_t &= -\Delta u + \kappa(\Delta|u|^2)\cdot u\ ,\ t \in [-a,a] \\ u(0) &= \phi \end{cases}$$

admits for any $\phi \in U$ *a unique solution* $u \in C^\infty\{[-a,a], H^\infty(\mathbb{R})\}$. *The induced solution map* $U \to C^\infty\{[-a,a], H^\infty(\mathbb{R})\}$ *is a* C^∞*–map between Fréchet spaces.*

Theorem 18 can be proved following the arguments used in [10] supplemented by suitable estimates for the higher order time derivatives. A survey on the literature and an exhaustive list on references concerning Theorems 17, 18 are given in [27], [10]. The nonlinear Schrödinger type equation in Theorem 18 has been considered in the theory of superfluids in plasma physics, we refer to [8], [9] and to the references given in [10]. The various difficulties which arise when trying to solve this equation by means of more conventional methods are discussed in [10]. A main problem is caused by the fact that the nonlinearity appears in the highest order space derivatives, and the linearized equation fails to be dissipative. Using the Nash–Moser technique the problem of loss of derivatives can be overcome.

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M. Poppenberg
Fachbereich Mathematik
Universität Dortmund
D-44221 Dortmund
GERMANY
E-mail address: poppenberg@math.uni-dortmund.de