Note di Matematica Vol. 18 - n. 1, 27-37 (1998)

COMPLETE LIFTS OF TENSOR FIELDS ON A PURE CROSS-SECTION IN THE TENSOR BUNDLE $T_q^1(M_n)$

A.A. SALIMOV, A. MAĞDEN

1. INTRODUCTION

Let M_n be a differentiable manifold of class C^∞ and finite dimension n, and let $T_q^1(M_n)$ be the tensor bundle of M_n: that is, the bundle of all tensors of type $(1,q)$ in M_n. Then $T_q^1(M_n)$, $q \geq 0$ is also a differentiable manifold of class C^∞; the dimension of $T_q^1(M_n)$ is $n(1+n^q)$.

The main purpose of the present paper is to study the behaviour on the pure cross-section of the complete lifts of tensor fields in a manifold M_n to its tensor bundle $T_q^1(M_n)$, $q>0$. Our main interest focuses on complete lifts of tensor fields of type $(1,1)$ and tensor fields of type $(1,2)$. The results obtained are to some extent similar to results previously established for tangent bundles (see [1], [2]). However there are various important differences and it appears that the problem of lifting tensor fields to the tensor bundle $T_q^1\ (M_n), q>0$ on the pure cross-section presents difficulties which are not encountered in the case of the tangent bundle. Our method related with applications of the Tachibana and Yano-Ako operators is a new method.

Throughout we use the following notations and conventions:

i. $\pi : T_q^1(M_n) \mapsto M_n$ is the projection $T_q^1(M_n)$ onto M_n.

ii. The indices $I=(i,\bar{i}), J=(j,\bar{j}), K=(k,\bar{k}), \ldots$ run from 1 to $n(1+n^q)$, the indices $i,j,k,\cdots$ from 1 to n and the indices $\bar{i},\bar{j},\bar{k},\cdots$ from $n+1$ to $n(1+n^q)$. The so-called Einsteins summation convention is used.

iii. $\mathfrak{F}(M)$ is the ring of real-valued C^∞ functions on M_n. $\mathfrak{T}_q^p(M_n)$ denotes the module over $\mathfrak{F}(M)$ of C^∞ tensor field of type (p,q).

iv. Vector fields in M_n are denote by $V, W, \cdots$. The Lie derivative with respect to V is denoted by $\mathcal{L}_V$. Tensor field of type $(1,1)$ is denoted by φ and tensor field of type $(1,2)$ by S.

2. COMPLETE LIFTS OF VECTOR FIELDS ON A CROSS-SECTION

Let x^i be local coordinates in a neighborhood U of a point $X \in M_n$. Then a tensor t of type $(1,q)$ at X which is an element of $T_q^1(M_n)$ is expressible in the form $(x^i, t^i_{i_1\cdots i_q})$, where $t^j_{i_1\cdots i_q}$ are components of t with respect to the natural frame ∂_i. We may consider $(x^i, t^j_{i_1\cdots i_q}) = (x^i, x^{\bar{i}})$ as local coordinates in a neighborhood $\pi^{-1}(U)$ of $T_q^1(M_n)$. To a transformation of local coordinates of $M_n : x^{i'} = x^{i'}(x^i)$, there corresponds in $T_q^1(M_n)$ the coordinates transformations

$$\begin{cases} x^{i'} &= x^{i'}(x^i) \\ t^{j'}_{(i')} &= A^{(i)}_{(i')} A^{j'}_j t^j_{(i)} = A^{(i)}_{i'} A^{j'}_j x^{(\bar{i})} \end{cases} \tag{2.1}$$

where $t^{j'}_{(i')} = t^{j'}_{i'_1\cdots i'_q}, t^{j}_{(i)} = t^{j}_{i_1\cdots i_q}, A^{(i)}_{(i')} = A^{i_1}_{i'_1}\cdots A^{i_q}_{i'_q}, A^{i}_{i'} = \frac{\partial x^i}{\partial x^{i'}}, A^{j'}_{j} = \frac{\partial x^{j'}}{\partial x^j}$.
If we put $x^{\bar{i}\,'} = t^{j'}_{i'_j\cdots i'_q}$, then we may write (2. 1) as

$$x^{I'} = x^{I'}(x^1,\cdots,x^n,x^{n+1},\cdots,x^{n(1+n^q)}).$$

Let V be a vector field on M_n and v^i be its components with respect to a coordinate neighborhood $U(x^i)\subset M_n$. Making use of the Jacobian matrix $(A^{I'}_I) = (\frac{\partial x^{I'}}{\partial x^I})$, i.e.

$$\begin{pmatrix} \frac{\partial x^{i'}}{\partial x^i} & \frac{\partial x^{i'}}{\partial x^{\bar{i}}} \\ \frac{\partial x^{\bar{i}\,'}}{\partial x^i} & \frac{\partial x^{\bar{i}\,'}}{\partial x^{\bar{i}}} \end{pmatrix} = \begin{pmatrix} A^{i'}_i & 0 \\ t^{j}_{(k)}\partial_i(A^{(k)}_{(i')}A^{j'}_j) & A^{(i)}_{(i')}A^{j'}_j \end{pmatrix}, \tag{2.2}$$

we have a vector field ${}^cV$ on $T^1_q(M_n)$ whose components are ${}^cV^H$:

$$\begin{cases} {}^cV^h = v^h, \\ {}^cV^{\bar{h}} = t^m_{(h)}\partial_m v^k - \sum_{\mu=1}^{q} t^k_{h_1\cdots m\cdots h_q}\partial_{h_\mu} v^m, \end{cases} \tag{2.3}$$

with respect to the coordinate neighborhood $\pi^{-1}(U)(x^h,x^{\bar{h}})\subset T^1_q(M_n)$, where $x^{\bar{h}} = t^k_{h_1\cdots h_q}$.

If $\alpha\in\mathfrak{T}^q_1(M_n)$, it is regarded, in a natural way (by contraction), as a function in $T^1_q(M_n)$, which we denote by $\iota\alpha$. If α has the local components $\alpha^{i_1\cdots i_q}_j$ in a coordinate neighborhood $U(x^i)\subset M_n$, then $\iota\alpha$ has the local expression

$$\iota\alpha = \alpha(t) = a^{i_1\cdots i_q}_j t^j_{i_1\cdots i_q},$$

with respect to the coordinates $(x^i,x^{\bar{i}})$ in $\pi^{-1}(U)$. Using (2.3), we can easily verify that

$${}^cV(\iota\alpha) = \iota(\mathcal{L}_V\alpha), \quad \text{for any } \alpha\in\mathfrak{T}^q_1(M_n).$$

Therefore, ${}^cV$ is the complete lift of V to $T^1_q(M_n)$ [5] (when $q=0$, see [2, p. 27-29]). If we put $q=0$ then ${}^cV^I$ are the components of the complete lift of V from a manifold M_n to its tangent bundle $T(M)$ [3].

Suppose that there is given a tensor field $\xi\in\mathfrak{T}^1_q(M_n)$ in M_n. Then the correspondence $X\mapsto\xi_X$, ξ_X being the value of ξ at $X\in M_n$, determines a mapping $\sigma_\xi : M_n\mapsto T^1_q(M_n)$, such that $\pi\circ\sigma_\xi = id_{M_n}$, and the n dimensional submanifold $\sigma_\xi(M_n)$ of $T^1_q(M_n)$ is called the cross-section determined by ξ. If the tensor field ξ has the local components $\xi^k_{h_1\cdots h_q}(x^h)$, the cross-section $\sigma_\xi(M_n)$ is locally expressed by

$$\begin{cases} x^h = x^h \\ x^{\bar{h}} = \xi^k_{h_1\cdots h_q}(x^h) \end{cases} \tag{2.4}$$

with respect to the coordinates $(x^h, x^{\bar h})$ in $T_q^1(M_n)$. Differentiating (2.4) by x^i, we see that the n tangent vector fields B_i to $\sigma_\xi(M_n)$ have components

$$(B_i^H) = \left(\frac{\partial x^H}{\partial x^i}\right) = \begin{pmatrix} \delta_i^h \\ \partial_i \xi^k_{h_1\cdots h_q} \end{pmatrix}, \tag{2.5}$$

with respect to the natural frame $\{\partial_h, \partial_{\bar h}\}$ in $T_q^1(M_n)$.

On the other hand, the fibre is locally expressed by

$$\begin{cases} x^h & = const, \\ t^k_{h_1\cdots h_q} & = t^k_{h_1\cdots h_q}, \end{cases}$$

$t^k_{h_1\cdots h_q}$ being considered as parameters. Thus, on differentiating with respect to $x^{\bar i} = t^j_{i_1\cdots i_q}$, we see that the n^{1+q} tangent vector fields $C_{\bar i}$ to the fibre have components

$$(C_{\bar i}^H) = \left(\frac{\partial x^H}{\partial x^{\bar i}}\right) = \begin{pmatrix} 0 \\ \delta_j^k \delta_{h_1}^{i_1} \cdots \delta_{h_q}^{i_q} \end{pmatrix} \tag{2.6}$$

with respect to the natural frame $\{\partial_h, \partial_{\bar h}\}$ in $T_q^1(M_n)$.

We consider in $\pi^{-1}(U) \subset T_q^1(M_n)$, $n(1+n^q)$ local vector fields B_i and $C_{\bar i}$ along $\sigma_\xi(M_n)$. They form a local family of frames $\{B_i, C_{\bar i}\}$ along $\sigma_\xi(M_n)$, which is called the adapted (B,C) -frame of $\sigma_\xi(M_n)$ in $\pi^{-1}(U)$. From ${}^cV = {}^cV^h\partial_h + {}^cV^{\bar h}\partial_{\bar h}$ and ${}^cV = \tilde V^i B_i + \tilde V^{\bar i} C_{\bar i}$, we easily obtain ${}^cV^h = \tilde V^i B_i^h + \tilde V^{\bar i} C_{\bar i}^h$, ${}^cV^{\bar h} = \tilde V^i B_i^{\bar h} + +\tilde V^{\bar i} C_{\bar i}^{\bar h}$. Now, taking account of (2.3) on the cross-section ξ, and also (2.5) and (2.6), we have $\tilde V^i = {}^cV^i$, $\tilde V^{\bar i} = -\mathcal{L}_V \xi^j_{i_1\cdots i_q}$.

Thus, ${}^cV$ has along $\sigma_\xi(M_n)$ components of the form

$${}^cV = \begin{pmatrix} v^i \\ -\mathcal{L}_V \xi^j_{i_1\cdots i_q} \end{pmatrix} \tag{2.7}$$

with respect to the adapted (B,C)-frame.

3. COMPLETE LIFTS OF TENSOR FIELD OF TYPE $(1,1)$ ON A PURE CROSS-SECTION

A tensor field $\xi \in \mathfrak{T}_q^1(M_n)$ is called pure with respect to $\varphi \in \mathfrak{T}_1^1(M_n)$, if

$$\varphi_r^i \xi^r_{j_1\cdots j_q} = \varphi^r_{j_1} \xi^i_{r\cdots j_q} = \cdots = \varphi^r_{j_q} \xi^i_{j_1\cdots r}.$$

In particular, vector fields will be considered to be pure.

Let $\overset{*}{\mathfrak{T}}{}_q^1(M_n)$ denotes a module of all the tensor fields $\xi \in \mathfrak{T}_q^1(M_n)$ which are pure with respect to φ. We consider the Tachibana operator on the module $\overset{*}{\mathfrak{T}}{}_q^1(M_n)$ (see [4]):

$$\begin{aligned}(\Phi_\varphi \xi)^i_{kj_1\ldots j_q} &= \varphi_k^m \partial_m \xi^i_{j_1\ldots j_q} - \varphi_r^i \partial_k \xi^r_{j_1\ldots j_q} - \\ &\quad - (\partial_r \varphi_k^i)\xi^r_{j_1\cdots j_q} + \sum_{a=1}^q (\partial_{j_a} \varphi_k^r)\xi^i_{j_1\cdots r\cdots j_q}.\end{aligned} \tag{3.1}$$

After some calculations we have

$$\gamma^k(\Phi_\varphi \xi)^i_{kj_1\cdots j_q} = \mathcal{L}_{\varphi V}\xi^{i_1}_{j_1\cdots j_q} - \varphi^{i_1}_m \mathcal{L}_V \xi^m_{j_1\cdots j_q}, \tag{3.2}$$

for any $V \in \mathfrak{T}^1_0(M_n)$.

Suppose that $A \in \mathfrak{T}^1_q(M_n)$ with local components $A^j_{i_1\cdots i_q}$ in $U(x^i) \subset M_n$. Making use o (2.8), we have a vertical vector field

$$^VA = \begin{pmatrix} ^VA^i \\ ^VA^{\bar{i}} \end{pmatrix} = \begin{pmatrix} 0 \\ A^j_{i_1\cdots i_q} \end{pmatrix} \tag{3.3}$$

with respect to the coordinates $(x^i, x^{\bar{i}})$ in $\pi^{-1}(U) \subset T^1_q(M_n)$.

Using (3.3), we can easily verify that $^VA(\iota\alpha) = \alpha(A) \circ \pi = {}^V(\alpha(A))$, i.e. $^VA \in \mathfrak{T}^1_0(T^1_q(M_n))$ is the vertical lift of $A \in \mathfrak{T}^1_q(M_n)$ [5] (when $q = 0$, see [2, p. 6]). From (2.5), (2.6), (3.3) and $^VA = {}^V\tilde{A}^iB_i + {}^V\tilde{A}^{\bar{i}}C_{\bar{i}}$, we easily obtain $^V\tilde{A}^i = 0$, $^V\tilde{A}^{\bar{i}} = VA^{\bar{i}} = A^j_{i_1\cdots i_q}$. Thus the vertical lift ^VA also has components of the form (3.3) with respect to the adapted (B, C)-frame of $\sigma_\xi(M_n)$.

Now, we consider a pure cross-section $\sigma^\varphi_\xi(M_n)$ determined by $\xi \in \overset{*}{\mathfrak{T}}{}^1_q(M_n)$. We define a tensor field $^c\varphi \in \mathfrak{T}^1_1(T^1_1(M_n))$ along the pure cross-section $\sigma^\varphi_\xi(M_n)$ by

$$\begin{cases} ^c\varphi(^cV) = & ^c(\varphi(V)), \forall V \in \mathfrak{T}^1_0(M_n), \\ ^c\varphi(^VA) = & ^V(\varphi(A)), \forall A \in \mathfrak{T}^1_q(M_n), \end{cases} \tag{3.4}$$

where $\varphi(A) \in \mathfrak{T}^1_q(M_n)$ and call $^c\varphi$ the complete lift of $\varphi \in \mathfrak{T}^1_1(M_n)$ to $T^1_q(M_n)$ along $\sigma^\varphi_\xi(M_n)$.

Let $^c\varphi^K_L$ components of $^c\varphi$ with respect to the adapted (B, C)-frame of the pure cross-section $\sigma^\varphi_\xi(M_n)$. Then, from (3.4) we have

$$\begin{cases} ^c\varphi^{K\,c}_L V^L = & ^c(\varphi(V))^K, (i) \\ ^c\varphi^{K\,V}_L A^L = & ^V(\varphi(A))^K, (ii) \end{cases} \tag{3.5}$$

where $^V(\varphi(A)) = \begin{pmatrix} 0 \\ ^V(\varphi(A))^{\bar{k}} \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi^{l_1}_m A^m_{k_1\cdots k_q} \end{pmatrix}$.

First, consider the case where $K = k$. In this case, (i) of (3.5) reduces to

$$^c\varphi^{k\,c}_l V^l + {}^c\varphi^{k\,c}_{\bar{l}} V^{\bar{l}} = {}^c(\varphi(V))^k = (\varphi(V))^k = \varphi^k_l v^l. \tag{3.6}$$

Since the right-hand side of (3.6) are functions depending only on the base coordinates x^i, the left-hand side of (3.6) are too. Then, since $^cV^{\bar{l}}$ depend on fibre coordinates, from (3.6) we obtain

$$^c\varphi^k_{\bar{l}} = 0. \tag{3.7}$$

From (3.6) and (3.7), we have $^c\varphi^{k\,c}_l V^l = {}^c\varphi^k_l v^l = \varphi^k_l v^l$, v^i being arbitrary, which implies

$$^c\varphi^k_l = \varphi^k_l. \tag{3.8}$$

Thus, from (3.7) and (3.8) we see that ${}^c\varphi$ is projectable with projection φ [6].

When $K = k$, (*ii*) of (3.5) can be rewritten, by virtue of (3.3), (3.7) and (3.8), as $0 = 0$.

When $K = \overline{k}$, (*ii*) of (3.5) reduces to

$$ {}^c\varphi_l^{\overline{k}}\, {}^V A^l + {}^c\varphi_{\overline{l}}^{\overline{k}}\, {}^V A^{\overline{l}} = {}^V(\varphi(A))^{\overline{k}} $$

or

$$ {}^c\varphi_{\overline{l}}^{\overline{k}} A_{r_1\cdots r_q}^{s_1} = \varphi_m^{l_1} A_{k_1\cdots k_q}^m = \varphi_{s_1}^{l_1} \delta_{k_1}^{r_1} \cdots \delta_{k_q}^{r_q} A_{r_1\cdots r_q}^{s_1}, $$

for all $A \in \mathfrak{T}_q^1(M_n)$, which implies

$$ {}^c\varphi_{\overline{l}}^{\overline{k}} = \varphi_{s_1}^{l_1} \delta_{k_1}^{r_1} \cdots \delta_{k_q}^{r_q}, \tag{3.9} $$

where $x^{\overline{l}} = t_{r_1\cdots r_q}^{s_1}, x^{\overline{k}} = t_{k_1\cdots k_q}^{l_1}$.

When $K = \overline{k}$, (*i*) of (3.5) reduces to

$$ {}^c\varphi_l^{\overline{k}}\, {}^c V^l + {}^c\varphi_{\overline{l}}^{\overline{k}}\, {}^c V^{\overline{l}} = {}^c(\varphi(V))^{\overline{k}}. \tag{3.10} $$

We shall investigate components ${}^c\varphi_l^{\overline{k}}$. From (3.2) we have

$$ v^k(\Phi_\varphi \xi)_{lk_1\cdots k_q}^{l_1} + \varphi_l^{l_1} \mathcal{L}_V \xi_{k_1\cdots k_q}^l = \mathcal{L}_{\varphi(V)} \xi_{k_1\cdots k_q}^{l_1} \tag{3.11} $$

Using (2.7), from (3.11) we have

$$ \begin{aligned} & v^k(\Phi_\varphi \xi)_{lk_1\cdots k_q}^{l_1} + \varphi_l^{l_1} \mathcal{L}_V \xi_{k_1\cdots k_q}^l = \\ & = v^k \Phi_l \xi_{k_1\cdots k_q}^{l_1} + \varphi_{s_1}^{l_1} \delta_{k_1}^{r_1} \cdots \delta_{k_q}^{r_q} \varphi_l^{l_1} \mathcal{L}_V \xi_{r_1\cdots r_q}^{s_1} \\ & = {}^c V^l \Phi_l \xi_{k_1\cdots k_q}^{l_1} - \delta_{k_1}^{r_1} \cdots \delta_{k_q}^{r_q} \varphi_{s_1}^{l_1}\, {}^c V^{\overline{l}} = -{}^c(\varphi(V))^{\overline{k}} \end{aligned} $$

or

$$ (\Phi_\varphi \xi)_{k_1\cdots k_q}^{l_1}\, {}^c V^l - \delta_{k_1}^{r_1} \cdots \delta_{k_q}^{r_q} \varphi_{s_1}^{l_1}\, {}^c V^{\overline{l}} = -{}^c(\varphi(V))^{\overline{k}} \tag{3.12} $$

From (3.10), (3.9) and (3.12) we write

$$ ({}^c\varphi_l^{\overline{k}} + (\Phi_\varphi \xi)_{lk_1\cdots k_q}^{l_1}) v^l = 0 $$

or

$$ {}^c\varphi_l^{\overline{k}} = -(\Phi_\varphi \xi)_{lk_1\cdots k_q}^{l_1}. $$

Thus the complete lift ${}^c\varphi$ of φ has along the pure cross-section $\sigma_\xi^\varphi(M_n)$ components

$$ \begin{cases} {}^c\varphi_l^k = \varphi_l^k, & {}^c\varphi_{\overline{l}}^k = 0 \\ {}^c\varphi_l^{\overline{k}} = -(\Phi_\varphi \xi)_{lk_1\cdots k_q}^{l_1}, & {}^c\varphi_{\overline{l}}^{\overline{k}} = \varphi_{s_1}^{l_1} \delta_{k_1}^{r_1} \cdots \delta_{k_q}^{r_q} \end{cases} \tag{3.13} $$

with respect to the adapted (B, C)-frame of $\sigma_\xi^\varphi(M_n)$, where $\Phi_\varphi \xi$ is the Tachibana operator.

Remark 1. The formula (3.2) is valid if and only if $\Phi_\varphi\xi$ is the Tachibana operator, i.e. in the form (3.13) is unique solution of (3.4). Therefore, if $\tilde{\varphi}$ is element of $\mathfrak{T}^1_1(T^1_q(M_n))$, su that $\tilde{\varphi}({}^cV) = {}^c\varphi({}^cV) = {}^c(\varphi(V))$, $\tilde{\varphi}({}^VA) = {}^c\varphi({}^VA) = {}^V(\varphi(A))$, then $\tilde{\varphi} = {}^c\varphi$.

If we write $q = 0$, then (3.13) is the formula of the complete lift to the tangent bundle alo the cross-section $\sigma_\xi(M_n)$ (see [2, p. 126]).

Remark 2. On putting $B_{\bar{i}} = C_{\bar{i}}$, we write the adapted (B,C)-frame of $\sigma^\varphi_\xi(M_n)$ as B_I $\{B_i, B_{\bar{i}}\}$. We define a coframe $\tilde{B}^I$ of $\sigma^\varphi_\xi(M_n)$ by $\tilde{B}^J(B_I) = \delta^J_I$. If $B_I = B^H_I\partial_H$, then we have

$$B^H_I\tilde{B}^J_H = \delta^J_I,$$

where $\tilde{B}^J_H = \tilde{B}^J(\partial_H)$. From (2.5), (2.6), (3.13) and (*), we see that covector fields $\tilde{B}^J$ ha components

$$\tilde{B}^i = (\tilde{B}^i_H) = (\delta^i_h, 0)$$
$$\tilde{B}^{\bar{i}} = (\tilde{B}^{\bar{i}}_H) = (-\partial_h\xi^j_{i_1\cdots i_q}, \delta^{h_1}_{i_1}\cdots\delta^{h_q}_{i_q}\delta^j_k)$$

with respect to the natural frame $\{dx^h, dx^{\bar{h}}\}$.

Taking account of representation

$${}^c\varphi = {}^c\varphi^I_J B_I \otimes \tilde{B}^J$$

and

$$\begin{aligned}{}^c\tilde{\varphi}^K_L &= {}^c\varphi(dx^K, \partial_L) = {}^c\varphi^I_J B_I \otimes \tilde{B}^J(dx^K, \partial_L)\\ &= {}^c\varphi^I_J dx^K(B_I)\tilde{B}^J(\partial_L) = {}^c\varphi^I_J dx^K(B^H_J\partial_H)\tilde{B}^J_L\\ &= {}^c\varphi^I_J B^H_I\delta^K_H\tilde{B}^J_L = {}^c\varphi^I_J B^K_I\tilde{B}^J_L,\end{aligned}$$

and also (2.5), (2.6), (3.13) and (**), we have along $\sigma^\varphi_\xi(M_n)$ the formulas

$$\begin{aligned}{}^c\tilde{\varphi}^k_l &= \varphi^k_l, {}^c\tilde{\varphi}^k_{\bar{l}} = 0,\\ {}^c\tilde{\varphi}^{\bar{k}}_{\bar{l}} &= \varphi^{l_1}_{s_1}\delta^{r_1}_{k_1}\cdots\delta^{r_q}_{k_q},\\ {}^c\tilde{\varphi}^{\bar{k}}_l &= -(\Phi_\varphi\xi)^{l_1}_{lk_1\cdots k_q} + \varphi^m_l\partial_m\xi^{l_1}_{k_1\cdots k_q} - \varphi^{l_1}_m\partial_l\xi^m_{k_1\cdots k_q}.\end{aligned}$$

Thus, ${}^c\varphi$ has along the pure cross-section $\sigma^\varphi_\xi(M_n)$ components of the form (see [7])

$$\begin{aligned}{}^c\tilde{\varphi}^k_l &= \varphi^k_l, {}^c\tilde{\varphi}^k_{\bar{l}} = 0,\\ {}^c\tilde{\varphi}^{\bar{k}}_{\bar{l}} &= \varphi^{l_1}_{s_1}\delta^{r_1}_{k_1}\cdots\delta^{r_q}_{k_q},\\ {}^c\tilde{\varphi}^{\bar{k}}_l &= \xi^m_{k_1\cdots k_q}\partial_m\varphi^{l_1}_l - \sum_{a=1}^{q}(\partial_{k_a}\varphi^m_l)\xi^{l_1}_{k_1\cdots m\cdots k_q}\end{aligned}$$

with respect to the natural frame $\{\partial_h, \partial_{\bar{h}}\}$ of $\sigma^\varphi_\xi(M_n)$ in $\pi^{-1}(U)$.

Theorem 3.1. The complete lift ^{c} : *End* $M_n \mapsto$ *End* $T^1_q(M_n)$ along the pure cross-section $\sigma^{\varphi}_{\xi}(M_n)$ is a monomorphism.

Proof. The peculiarity

$$(\Phi_{a\varphi_1+b\varphi_2}\xi)^{l_1}_{lk_1\cdots k_q} = a(\Phi_{\varphi_1}\xi)^{l_1}_{lk_1\cdots k_q} + b(\Phi_{\varphi_2}\xi)^{l_1}_{lk_1\cdots k_q}, \forall a, b \in \mathbb{R}$$

of the Tachibana operator and from (3.13), we find that, ^{c} : *End* $M_n \mapsto$ *End* $T^1_q(M_n)$ is a linear. From (3.4), we write

$$\begin{aligned}
{}^c(\varphi \circ \psi)({}^cV) &= {}^c((\varphi \circ \psi)(V)) = {}^c(\varphi(\psi(V))) \\
&= {}^c\varphi({}^c\psi(V)) = {}^c\varphi({}^c\psi({}^cV)) = ({}^c\varphi \circ {}^c\psi)({}^cV), \\
{}^c(\varphi \circ \psi)({}^VA) &= {}^V((\varphi \circ \psi)(A)) = {}^V(\varphi(\psi(A))) \\
&= {}^c\varphi({}^V(\psi(A))) = {}^c\varphi({}^c\psi({}^VA)) = ({}^c\varphi \circ {}^c\psi)({}^VA).
\end{aligned}$$

With respect to the above Remark, we find

$${}^c(\varphi \circ \psi) = {}^c\varphi \circ {}^c\psi, \tag{3.14}$$

that is, ^{c} is a homeomorphism. However, ${}^c\varphi = 0$ if and only if $\varphi = 0$, i.e. ^{c} is a monomorphism.

Theorem 3.2. Let I be the field of identity transformation of *End* M_n. Then, ${}^cI$ is the field of identity automorphism of *End* T^1_q (M_n).

Proof. By (3.4), we write

$$\begin{aligned}
{}^cI({}^cV) &= {}^c(I(V)) = {}^cV = id_{T^1_q(M_n)}({}^cV), \\
{}^cI({}^VA) &= {}^V(I(A)) = {}^VA = id_{T^1_q(M_n)}({}^VA)
\end{aligned}$$

and hence, we have ${}^cI = id_{T^1_q(M_n)}$. From (3.14) and Theorem 3.2., we have

Theorem 3.3. If $\varphi \in \mathfrak{T}^1_1(M_n)$ defines an almost complex structures on M_n, so does ${}^c\varphi$ on $T^1_q(M_n)$ along the pure cross-section $\sigma^{\varphi}_{\xi}(M_n)$.

4. COMPLETE LIFTS OF TENSOR FIELDS OF TYPE (1, 2) ON A PURE CROSS-SECTION

A tensor field $\xi \in \mathfrak{T}^1_q(M_n)$ is called pure with respect to $S \in \mathfrak{T}^1_2(M_n)$, if

$$\begin{aligned}
S^i_{rk}\xi^r_{j_1\cdots j_q} &= S^r_{j_1k}\xi^i_{r\cdots j_q} = \cdots = S^r_{j_qk}\xi^i_{j_1\cdots r}, \\
S^i_{lr}\xi^r_{j_1\cdots j_q} &= S^r_{lj_1}\xi^i_{r\cdots j_q} = \cdots = S^r_{lj_q}\xi^i_{j_1\cdots r}.
\end{aligned}$$

Let $\overset{*}{\mathfrak{T}}{}^1_q(M_n)$ denotes a module of all the tensor fields $\xi \in \mathfrak{T}^1_q(M_n)$ which are pure with respect to the S. We consider the Yano-Ako operator on the module $\overset{*}{\mathfrak{T}}{}^1_q(M_n)$ (see [8]):

$$\begin{aligned}
(\Phi_S\xi)^h_{i_1i_2j_1\cdots j_q} &= S^m_{i_1i_2}\partial_m\xi^h_{j_1\cdots j_q} - \xi^m_{j_1\cdots j_q}\partial_m S^h_{i_1i_2} + \\
&+ \sum_{b=1}^{q} \xi^h_{j_1\cdots m\cdots j_q}\partial_{j_b}S^m_{i_1i_2} - S^h_{mi_2}\partial_{i_1}\xi^m_{j_1\cdots j_q} - S^h_{i_1m}\partial_{i_2}\xi^m_{j_1\cdots j_q}.
\end{aligned} \tag{4.1}$$

After some calculations we find

$$v^i(\Phi_S\xi)^h_{i_1 i_2 j_1\cdots j_q} = (\Phi_{S_V}\xi)^h_{i_2 j_1\cdots j_q} - S^h_{mi_2}\mathcal{L}_V\xi^m_{j_1\cdots j_q}. \tag{4.2}$$

for any $V \in \mathfrak{T}^1_0(M_n)$, where, S_V is the tensor field of type (1,1) in M_n defined by $S_V(W) = S(V, W)$ and $\Phi_{S_V}\xi$ is the Tachibana operator.

We now consider a pure cross-section $\sigma^S_\xi(M_n)$ determined by the pure tensor field $\xi \in \overset{*}{\mathfrak{T}}{}^1_q(M_n)$ with respect to the S. We define a tensor field ${}^cS \in \mathfrak{T}^1_2\ (T^1_q(M_n))$ along the pure cross-section $\sigma^S_\xi(M_n)$ by

$$\begin{cases} {}^cS({}^cV_1, {}^cV_2) = {}^c(S(V_1, V_2)), & \forall V_1, V_2 \in \mathfrak{T}^1_0(M_n), \\ {}^cS({}^cA, {}^cV_2) = {}^V(S_{V_2}(A)), & \forall A \in \mathfrak{T}^1_q(M_n), \\ {}^cS({}^cV_1, {}^VB) = {}^c(S_{V_1},(B)), & \forall B \in \mathfrak{T}^1_q(M_n), \\ {}^cS({}^VA, {}^VB) = 0 \end{cases} \tag{4.3}$$

and called ${}^cS$ the complete lift of $S \in \mathfrak{T}^1_2(M_n)$ to $T^1_q(M_n)$ along $\sigma^S_\xi(M_n)$.

Let ${}^cS^K_{L_1L_2}$ be components of ${}^cS$ with respect to the adapted (B, C)-frame of $\sigma^S_\xi(M_n)$. Then, from (4.3) we write

$$\begin{cases} {}^cS^K_{L_1L_2}\,{}^cV^{L_1}_1\,{}^cV^{L_2}_2 = {}^c(S(V_1, V_2))^K, & (i) \\ {}^cS^K_{L_1L_2}\,{}^VA^{L_1}\,{}^cV^{L_2}_2 = {}^V(S_{V_2}(A))^K, & (ii) \\ {}^cS^K_{L_1L_2}\,{}^cV^{L_1}_1\,{}^VB^{L_2} = {}^V(S_{V_1}(B))^K, & (iii) \\ {}^cS^K_{L_1L_2}\,{}^VA^{L_1}\,{}^VB^{L_2} = 0, & (iv) \end{cases} \tag{4.4}$$

where

$${}^V(S_{V_2}(A)) = \begin{pmatrix} 0 \\ {}^V(S_{V_2}(A))^{\bar{k}} \end{pmatrix} = \begin{pmatrix} 0 \\ S^h_{mj}v^j_2 A^m_{k_1\cdots k_q} \end{pmatrix},$$

$${}^V(S_{V_1}(B)) = \begin{pmatrix} 0 \\ {}^V(S_{V_1}(B))^{\bar{k}} \end{pmatrix} = \begin{pmatrix} 0 \\ S^h_{jm}v^j_1 B^m_{k_1\cdots k_q} \end{pmatrix}, x^{\bar{k}} = t^h_{k_1\cdots k_q}.$$

Substituting $K = k$ in (i) of (4.4) and calculating in the same way as in §3, we obtain

$${}^cS^k_{\bar{l}_1 l_2} = {}^cS^k_{l_1\bar{l}_2} = {}^cS^k_{\bar{l}_1\bar{l}_2} = 0, \quad {}^cS^k_{l_1l_2} = S^k_{l_1l_2}. \tag{4.5}$$

When $K = \bar{k}$, (i) of (4.4) reduces to

$$\begin{aligned} &{}^cS^{\bar{k}}_{l_1l_2}\,{}^cV^{l_1}_1\,{}^cV^{l_2}_2 + {}^cS^{\bar{k}}_{\bar{l}_1l_2}\,{}^cV^{\bar{l}_1}_1\,{}^cV^{l_2}_2 + {}^cS^{\bar{k}}_{l_1\bar{l}_2}\,{}^cV^{l_1}_1\,{}^cV^{\bar{l}_2}_2 + \\ &+ {}^cS^{\bar{k}}_{\bar{l}_1\bar{l}_2}\,{}^cV^{\bar{l}_1}_1\,{}^cV^{\bar{l}_2}_2 = {}^c(S(V_1, V_2))^{\bar{k}}. \end{aligned} \tag{4.6}$$

We will study components ${}^cS^{\bar{k}}_{l_1l_2}$, ${}^cS^{\bar{k}}_{\bar{l}_1l_2}$, ${}^cS^{\bar{k}}_{l_1\bar{l}_2}$ and ${}^cS^{\bar{k}}_{\bar{l}_1\bar{l}_2}$ of the complete lift ${}^cS$. When $K = k$ (ii) and (iv) of (4.4) can be rewritten by virtue of (3.3) and (4.5) as $0 = 0$. For a case where $K = \bar{k}$, (iv) of (4.4) we have ${}^cS^{\bar{k}}_{\bar{l}_1\bar{l}_2} = 0$. When $K = \bar{k}$, from (ii) of (4.4) we write

$$\begin{aligned} &{}^cS^{\bar{k}}_{l_1l_2}\,{}^VA^{l_1}\,{}^cV^{l_2}_2 + {}^cS^{\bar{k}}_{\bar{l}_1l_2}\,{}^VA^{\bar{l}}_1\,{}^cV^{l_2}_2 + {}^cS^{\bar{k}}_{l_1\bar{l}_2}\,{}^VA^{l_1}\,{}^cV^{\bar{l}_2}_2 + \\ &+ {}^cS^{\bar{k}}_{\bar{l}_1\bar{l}_2}\,{}^VA^{\bar{l}_1}\,{}^cV^{\bar{l}_2}_2 = {}^V(S_{V_2}(A))^{\bar{k}} \end{aligned}$$

or

$$
\begin{aligned}
&{}^cS^{\bar{k}}_{\bar{l}_1 l_2}\,{}^VA^{\bar{l}_1}\,{}^cV^{l_2}_2 = {}^V(S_{V_2}(A))^{\bar{k}}\\
&{}^cS^{\bar{k}}_{\bar{l}_1 l_2}A^{s_1}_{r_1\cdots r_q}v^{l_2}_2 = S^h_{mj}v^j_2A^m_{k_1\cdots k_q} = S^h_{s_1 l_2}\delta^{r_1}_{k_1}\cdots\delta^{r_q}_{k_q}A^{s_1}_{r_1\cdots r_q}v^{l_2}_2
\end{aligned}
$$

$\forall A \in \mathfrak{T}^1_q(M_n)$, $\forall V_2 \in \mathfrak{T}^1_0(M_n)$, which implies

$$
{}^cS^{\bar{k}}_{\bar{l}_1 l_2} = S^h_{s_1 l_2}\delta^{r_1}_{k_1}\cdots\delta^{r_q}_{k_q}, \tag{4.7}
$$

where $x^{\bar{l}_1} = t^{s_1}_{r_1\cdots r_q}$. We also have by (*iii*) of (4.4)

$$
{}^cS^{\bar{k}}_{l_1\bar{l}_2} = S^h_{l_1 s_1}\delta^{r_1}_{k_1}\cdots\delta^{r_q}_{k_q}, \tag{4.8}
$$

where $x^{\bar{l}_2} = t^{s_1}_{r_1\cdots r_q}$.
Thus, by virtue of (4.7) and (4.8), (4.6) reduces to

$$
\begin{aligned}
&{}^cS^{\bar{k}}_{l_1 l_2}\,{}^cV^{l_1}_1\,{}^cV^{l_2}_2 + S^h_{s_1 l_2}\delta^{r_1}_{k_1}\cdots\delta^{r_q}_{k_q}\,{}^cV^{\bar{l}_1}_1\,{}^cV^{l_2}_2 +\\
&+ S^h_{l_1 s_1}\delta^{r_1}_{k_1}\cdots\delta^{r_q}_{k_q}\,{}^cV^{l_1}_1\,{}^cV^{\bar{l}_2}_2 = {}^c(S(V_1,V_2))^{\bar{k}}.
\end{aligned} \tag{4.9}
$$

From (4.2) and (3.2), we obtain

$$
\begin{aligned}
v^{l_2}_2 v^{l_1}_1(\Phi_{S(V_1,V_2)}\xi)^h_{l_1 l_2 k_1\cdots k_q} &= v^{l_2}_2(\Phi_{S_{V_1}(V_2)}\xi)^h_{l_2 k_1\cdots k_q} - v^{l_2}_2 S^h_{ml_2}\mathcal{L}_{V_1}\xi^m_{k_1\cdots k_q}\\
= \mathcal{L}_{S(V_1,V_2)}\xi^h_{k_1\cdots k_q} &- v^{l_1}_1 S^h_{l_1 m}\mathcal{L}_{V_2}\xi^m_{k_1\cdots k_q} - v^{l_2}_2 S^h_{ml_2}h_{V_1}\xi^m_{k_1\cdots k_q}
\end{aligned}
$$

or

$$
v^{l_2}_2 v^{l_1}_1(\Phi_{S(V_1,V_2)}\xi)^h_{l_1 l_2 k_1\cdots k_q} + v^{l_1}_1 S^h_{l_1 m}\mathcal{L}_{V_2}\xi^m_{k_1\cdots k_q} + v^{l_2}_2 S^h_{ml_2}\mathcal{L}_{V_1}\xi^m_{k_1\cdots k_q} = \mathcal{L}_{S(V_1,V_2)}\xi^h_{k_1\cdots k_q}. \tag{4.10}
$$

By virtue of (3.9), (4.10) reduces to

$$
\begin{aligned}
&v^{l_2}_2 v^{l_1}_1(\Phi_{S(V_1,V_2)}\xi)^h_{l_1 l_2 k_1\cdots k_q} + v^{l_1}_1 S^h_{l_1 m}\mathcal{L}_{V_2}\xi^m_{k_1\cdots k_q} + v^{l_2}_2 S^h_{ml_2}\mathcal{L}_{V_1}\xi^m_{k_1\cdots k_q} =\\
&= v^{l_2}_2 v^{l_1}_1(\Phi_{S(V_1,V_2)}\xi)^h_{l_1 l_2 k_1\cdots k_q} + v^{l_1}_1 S^h_{l_1 s_1}\delta^{r_1}_{k_1}\cdots\delta^{r_q}_{k_q}\mathcal{L}_{V_2}\xi^{s_1}_{r_1\cdots r_q} +\\
&+ v^{l_2}_2 S^h_{s_1 l_2}\delta^{r_1}_{k_1}\cdots\delta^{r_q}_{k_q}\mathcal{L}_{V_1}\xi^{s_1}_{r_1\cdots r_q} = ((\Phi_{S(V_1,V_2)}\xi)^h_{l_1 l_2 k_1\cdots k_q})^c\,V^{l_1}_1\,{}^cV^{l_2}_2 -\\
&- S^h_{l_1 s_1}\delta^{r_1}_{k_1}\cdots\delta^{r_q}_{k_q}\,{}^cV^{l_1}_1\,{}^cV^{\bar{l}_2}_2 - S^h_{s_1 l_2}\delta^{r_1}_{k_1}\cdots\delta^{r_1}_{k_q}\,{}^cV^{\bar{l}_1}_1\,{}^cV^{l_2}_2 = -{}^c(S(V_1,V_2))^{\bar{k}}
\end{aligned}
$$

or

$$
\begin{aligned}
&((\Phi_{S(V_1,V_2)}\xi)^h_{l_1 l_2 k_1\cdots k_q})^c\,V^{l_1}_1\,{}^cV^{l_2}_2 - S^h_{l_1 s_1}\delta^{r_1}_{k_1}\cdots\delta^{r_q}_{k_q}\,{}^cV^{l_1}_1\,{}^cV^{\bar{l}_2}_2 -\\
&- S^h_{s_1 l_2}\delta^{r_1}_{k_1}\cdots\delta^{r_q}_{k_q}\,{}^cV^{\bar{l}_1}_1\,{}^cV^{l_2}_2 = -{}^c(S(V_1,V_2))^{\bar{k}}, \forall V_1, V_2 \in \mathfrak{T}^1_0(M_n).
\end{aligned} \tag{4.11}
$$

Comparing (4.9) and (4.11) we get

$$
{}^cS^{\bar{k}}_{l_1 l_2} = -(\Phi_S\xi)^h_{l_1 l_2 k_1\cdots k_q}.
$$

Thus, the complete lift ${}^{c}S \in \mathfrak{T}_2^1(T_q^1(M_n))$ of $S \in \mathfrak{T}_2^1(M_n)$ has along the pure cross-section $\sigma_\xi^S(M_n)$ components:

$$\begin{cases} {}^{c}S_{l_1 l_2}^{k} = S_{l_1 l_2}^{k}, \ {}^{c}S_{\bar{l}_1 l_2}^{k} = {}^{c}S_{l_1 \bar{l}_2}^{k} = {}^{c}S_{\bar{l}_1 \bar{l}_2}^{k} = {}^{c}S_{\bar{l}_1 \bar{l}_2}^{\bar{k}} = 0, \\ {}^{c}S_{\bar{l}_1 l_2}^{\bar{k}} = S_{s_1 l_2}^{h} \delta_{k_1}^{r_1} \cdots \delta_{k_q}^{r_q}, \ {}^{c}S_{l_1 \bar{l}_2}^{\bar{k}} = S_{l_1 s_1}^{h} \delta_{k_1}^{r_1} \cdots \delta_{k_q}^{r_q}, \\ {}^{c}S_{l_1 l_2}^{\bar{k}} = -(\Phi_S \xi)_{l_1 l_2 k_1 \cdots k_q}^{h}, \end{cases} \qquad (4.12)$$

with respect to the adapted (B, C)-frame of $\sigma_\xi^S(M_n)$, where $\Phi_S \xi$ is the Yano-Ako operator.

If we write $q = 0$, then (4.12) is the formula of the complete lift to the tangent bundle along the cross-section $\sigma_\xi(M_n)$ ([2, p. 126]). By similar devices (see §3) from (4.12) we have components of the complete lift ${}^{c}S$ along the pure cross-section with respect to the natural frame $\{\partial_i, \partial_{\bar{i}}\}$ ([9], [10]).

REFERENCES

[1] Yano K., *Tensor fields and connections on cross-section in the tangent bundle of a differentiable manifold*, Proc. Royal Soc. Edinburg, Sect., ALXVII (1967), 277-288.

[2] Yano K., Ishihara S., *Tangent and Cotangent Bundles*, Marcel Dekker, Inc., New York (1973).

[3] Yano K., Kobayashi N., *Prolongations of tensor fields and connections to tangent bundles*, I, General Theory, Jour. Math. Soc. Japan, 18 (1966).

[4] Tachibana S., *Analytic tensor and its generalization*, Tohoku Math. J. 12 (1960), 208-221.

[5] Ledger A., Yano K., *Almost complex structures on tensor bundles*, J. Dif. Geom. 1 (1967), 355-368.

[6] Shapukov B.N., *Connections on differentiable fiber bundles*, Itogi nauki i tekhn., Viniti, Problemy geometry 15 (1983), 61-91, (Russian).

[7] Sato I., *Almost analytic tensor field in almost complex manifolds*, Tensor., 17 (1966), 105-119.

[8] Yano K., Ako M., *On certain operators associated with tensor fields*, Kodai Math. Sem. Rep., 20 (1968), 414-436.

[9] Salimov A.A., *The Generalized Yano-Ako operator and complete lift of tensor fields*, Tensor N.S. 55 (1994), 142-146.

[10] Salimov A.A., *A new method in theory of lifts of tensor fields to a tensor bundle*, Iz. VUZ. Mat., 3 (1994), 69-75, (Russian).

Received January 12, 1998
A.A. Salimov and A. Magden
Atatürk Üniversitesi
Fen Edebiyat Fakültesi
Matematik Bölümü
25240 ERZURUM-TURKEY
e-mail: amagden@hotmail.com