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A CHARACTERIZATION OF A CERTAIN REAL HYPERSURFACE OF THE COMPLEX PROJECTIVE SPACE

TAKAHITO KIKUKAWA

Abstract. *Let M be a geodesic hypersphere or a tube of radius r over a totally geodesic CP^k $(1 \leq k \leq n-2, 0 < r < \pi/2)$ in a complex projective space $CP^n(4)$. We characterize M by using specific properties of the tensor field T of type (1,2) defined by*

$$T_X Y = \eta(Y)\phi AX - \eta(X)\phi AY - g(\phi AX, Y)\xi.$$

1 Introduction

Let $\bar{M} = CP^n(4)$ be an n-dimensional complex projective space with *Fubini-Study* metric g of constant holomorphic sectional curvature 4 , and let M be a connected real hypersurface of $\bar{M}$. For real hypersurfaces in $\bar{M}$ it is well-known that there exists no locally symmetric Riemannian spaces. But homogeneous real hypersurfaces of $\bar{M}$ exist and are classified by R.Takagi [9] by means of six model spaces of type $(A_1), (A_2), (B), (C), (D)$ and (E). Some characterizations of these model spaces are investigated by R.Takagi, [10] T.E.Cecil and P.J.Ryan [3]. Particularly, M.Kimura [4] proved the following:

Theorem 1 ([4]) *Let M be a connected real hypersurface of $\bar{M}$. Then M has constant principal curvatures and the structure vector ξ is principal with principal curvature $\alpha = 2\cot(2r)$ if and only if M is locally congruent to one of the following spaces:*

(A_1) *a geodesic hypersphere (that is, a tube of radius r over a hyperplane CP^{n-1}, where $0 < r < \pi/2$);*

(A_2) *a tube of radius r over a totally geodesic CP^k $(1 \leq k \leq n-2)$, where $0 < r < \pi/2$;*

(B) *a tube of radius r over a complex quardric Q^{n-1}, where $0 < r < \pi/4$;*

(C) *a tube of radius r over $CP^1 \times CP^{\frac{n-1}{2}}$, where $0 < r < \pi/4$ and n (≥ 5) is odd;*

(D) *a tube of radius r over a complex Grassmann $G_{2,5}(c)$, where $0 < r < \pi/4$ and $n = 9$;*

(E) *a tube of radius r over a Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \pi/4$ and $n = 15$.*

In the following we call real hypersurfaces of type (A_1) and type (A_2) "real hypersurfaces of type (A)" without distinguishing.

W.Ambrose and I.M.Singer[1] gave a characterization of homogeneous Riemannian manifolds:

Theorem 2 ([1]) *A connected, complete and simply connected Riemannian manifold M is homogeneous if and only if there exists a tensor field T of type$(1,2)$ on M such that*

(1) $g(T_X Y, Z) + g(Y, T_X Z) = 0$

(2) $(\nabla_X R)(Y,Z) = [T_X, R(Y,Z)] - R(T_X Y, Z) - R(Y, T_X Z)$

(3) $(\nabla_X T)_Y = [T_X, T_Y] - T_{T_X Y}$

for $X,Y,Z \in \chi(M)$. Here ∇ denotes the Levi Civita connection, R is the Riemannian curvature tensor of M and $\chi(M)$ is the Lie algebra of all C^∞ vector fields over M.

If T satisfies in addition

(4) $T_X X = 0$,

then M is the naturally reductive homogeneous space [11].

In this case, we call T a naturally reductive homogeneous structure on M.

The examples of naturally reductive homogeneous real hypersurfaces of the complex space form $\bar{M}_n(c)$ are given at first by J.Berndt and L.Vanhecke [2]. They proved that η-umbilical real hypersurfaces of $\bar{M}_n(c)$ are naturally reductive homogeneous spaces. Further, S.Nagai [7] generalized their result:

Theorem 3 ([7]) *Let M be a real hypersurface satisfying the commutativity condition $A\phi = \phi A$ in a non-flat complex space form $\bar{M}_n(c)$. Then*

$$T_X Y = \eta(Y)\phi AX - \eta(X)\phi AY - g(\phi AX, Y)\xi \tag{1.1}$$

defines a naturally reductive homogeneous structure on M.

Here (ϕ, ξ, η, g) denotes the almost contact metric structure of M naturally induced from the complex structure of $\bar{M}_n(c)$, and A is the shape operator of M in $\bar{M}_n(c)$. When $\bar{M}_n(c)$ is the complex projective space $\bar{M}$, the real hypersurface satisfying $A\phi = \phi A$ is of type(A) (see Theorem 4 in §2).

We put $\tilde{\nabla} := \nabla - T$, where T is the tensor field of type$(1,2)$ defined by (1.1). Then, the conditions (1), (2), and (3) in Theorem 2 are equivalent to $\tilde{\nabla} g \equiv 0, \tilde{\nabla} R \equiv 0$, and $\tilde{\nabla} T \equiv 0$, respectively. Moreover, in the paper [7], it is shown that a real hypersurface of type(A) in $\bar{M}$ satisfies $\tilde{\nabla} A \equiv 0$ and $\tilde{\nabla}\phi \equiv 0$. From these facts we know that the real hypersurface of type(A) in $\bar{M}$ satisfies $\tilde{\nabla} g \equiv 0, \tilde{\nabla} R \equiv 0, \tilde{\nabla} T \equiv 0, \tilde{\nabla} A \equiv 0$ and $\tilde{\nabla}\phi \equiv 0$. In this paper, we investigate the converse problem of the above result and prove the following:

Main Theorem *Let M be a connected real hypersurface of $\bar{M}$ and $\tilde{\nabla}$ a connection defined by $\tilde{\nabla} := \nabla - T$. Then the following statements are equivalent:*

(1) *M is locally congruent to the real hypersurface of type (A);*

(2) $\tilde{\nabla} g \equiv 0$;

(3) $\tilde{\nabla} R \equiv 0$;

(4) $\tilde{\nabla} T \equiv 0$;

(5) $\tilde{\nabla}\phi \equiv 0$;

(6) $\tilde{\nabla} A \equiv 0$;

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2 Preliminaries

Let $\bar{M} := CP^n(4)$ be an n-dimensional complex projective space of constant holomorphic sectional curvature 4 and let g and J be its Fubini-Study metric and complex structure, respectively. Further, let M be a connected real hypersurface of $\bar{M}$. We denote the induced Riemannian metric on M by the same letter g and a local unit normal vector field along M in $\bar{M}$ by ν .

The Gauss and Weingarten formulas are:

$$\bar{\nabla}_X Y = \nabla_X Y + g(AX,Y)\nu, \tag{2.1}$$

$$\bar{\nabla}_X \nu = -AX. \tag{2.2}$$

Here $\bar{\nabla}$ and ∇ denote the Levi-Civita connection on $\bar{M}$ and M, respectively, and A is the shape operator of M in $\bar{M}$.

Let $\bar{R}$ and R be the curvature tensors of $\bar{M}$ and M. The first structure equation becomes

$$\bar{R}(X,Y)Z = R(X,Y)Z - g(AY,Z)AX + g(AX,Z)AY + g((\nabla_X A)Y - (\nabla_Y A)X, Z)\nu.$$

Next, let (ϕ,ξ,η,g) be the almost contact metric structure naturally defined on M, that is,

$$\xi = -J\nu, \quad \eta(X) = g(X,\xi), \quad JX = \phi X + \eta(X)\nu.$$

These structure tensors satisfy the following equations:

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta\circ\phi = 0. \tag{2.3}$$

From (2.1) , (2.2) and $\bar{\nabla}J \equiv 0$ we get

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX,Y)\xi, \tag{2.4}$$

$$\nabla_X \xi = \phi AX. \tag{2.5}$$

Using (2.5), we obtain

$$(\nabla_X \eta)Y = g(\phi AX, Y). \tag{2.6}$$

In our case the Gauss and Codazzi equations become:

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y \\ &\quad -2g(\phi X,Y)\phi Z + g(AY,Z)AX - g(AX,Z)AY, \end{aligned} \tag{2.7}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi. \tag{2.8}$$

Let C^{n+1} be the complex $(n+1)$-space and $z_0, z_1, \ldots, z_n$ the natural coordinate system. $S^{2n+1}(r)$ is the sphere of radius r in C^{n+1} defined by

$$S^{2n+1}(r) = \{(z_0, z_1, \ldots, z_n) \mid z_0\bar{z}_0 + z_1\bar{z}_1 + \cdots + z_n\bar{z}_n = r^2\}.$$

The Riemannian metric tensor on $S^{2n+1}(r)$ is induced from the following metric $\langle\, , \rangle$ on C^{n+1} :

$$\langle z, w\rangle = \mathrm{Re} \sum_{j=0}^{n} z_j \bar{w}_j,$$

where $z = (z_0, z_1, \ldots, z_n), w = (w_0, w_1, \ldots, w_n) \in C^{n+1}$.

Then we can consider the Hopf fibration

$$S^1 \to S^{2n+1}(1) \xrightarrow{\pi} CP_n.$$

The product of two spheres $S^{2p+1}(\cos t) \times S^{2q+1}(\sin t)$ can be embedded in $C^{n+1} := C^{p+1} \times C^{q+1}$, where $n = p+q+1$. Then we may regard $S^{2p+1}(\cos t) \times S^{2q+1}(\sin t)$ as a submanifold in S^{2n+1}. Pushing down this submanifold to $\bar{M}$ according to the following diagram, we get a homogeneous real hypersurface in $\bar{M}$:

$$\begin{array}{ccc} S^{2p+1}(\cos t) \times S^{2q+1}(\sin t) & \longrightarrow & S^{2n+1}(1) \\ \downarrow & & \downarrow \\ M & \longrightarrow & \bar{M} \end{array}$$

This is a tube of radius t over the totally geodesic CP_p, that is, a real hypersurface of type(A) [3].

Now we recall the following:

Lemma 4 ([6]) *If ξ is a principal curvature vector, then the corresponding principal curvature α is constant.*

Lemma 5 ([6]) *Assume that ξ is a principal curvature vector with corresponding principal curvature α. If $AX = \lambda X$ for X orthogonal to ξ, then we have*

$$A\phi X = \frac{\alpha\lambda + 2}{2\lambda - \alpha}\phi X.$$

Theorem 6 ([8]) *Let M be a real hypersurface of $\bar{M}$. Then the following statements are equivalent:*

(1) *M is locally congruent to the homogeneous real hypersurface of type(A);*

(2) $A\phi = \phi A$.

Theorem 7 ([5]) *Let M be a real hypersurface of $\bar{M}$. Then the shape operator satisfies $g((\nabla_X A)Y, Z) = 0$ for X, Y, Z orthogonal to ξ and ξ is a principal curvature vector if and only if M is locally congruent to one of the homogeneous hypersurfaces of type(A) or of type(B).*

3 Proof of Theorem

Our purposes are to prove (2) $\Rightarrow$ (1), (3) $\Rightarrow$ (1), (4) $\Rightarrow$ (1), (5) $\Rightarrow$ (1), and (6) $\Rightarrow$ (1).

First, we prove (2) $\Rightarrow$ (1).

From (1.1) and $\tilde{\nabla} g \equiv 0$ we have

$$\eta(X)g((\phi A - A\phi)Y, Z) = 0$$

for $X, Y, Z \in TM$. Since this implies $A\phi = \phi A$, the result follows from Theorem 4.

Next, we prove (5) $\Rightarrow$ (1).

From (1.1), (2.3), (2.4) and $\tilde{\nabla}\phi \equiv 0$ we have

$$0 = (\tilde{\nabla}_X \phi)Y = \eta(X)\phi(A\phi - \phi A)Y$$

for $X, Y \in TM$. Putting $X = Y = \xi$, we get

$$A\xi = \eta(A\xi)\xi.$$

So, ξ is principal. Putting $X = \xi$ and applying ϕ yields

$$0 = (A\phi - \phi A)Y.$$

Since this implies $A\phi = \phi A$, (5) $\Rightarrow$ (1) follows again from Theorem 4.

Now we turn to the proof of (3) $\Rightarrow$ (1).

From the definition of $\tilde{\nabla} R$ and $\tilde{\nabla} R \equiv 0$ we have

$$0 = (\nabla_W R)(X,Y)Z - T_W(R(X,Y)Z) + R(T_W X, Y)Z + R(X, T_W Y)Z + R(X,Y)T_W Z$$

for $W, X, Y, Z \in TM$. Calculating the inner product of the right-hand members of the above equation and Z, we have

$$g(T_W(R(X,Y)Z), Z) + g(R(X,Y)Z, T_W Z) = 0.$$

From (2.7) we have

$$\begin{aligned}
&g(Y,Z)g(T_W X, Z) - g(X,Z)g(T_W Y, Z) + g(\phi Y, Z)g(T_W(\phi X), Z) \\
&- g(\phi X, Z)g(T_W(\phi Y), Z) - 2g(\phi X, Y)g(T_W(\phi Z), Z) + g(AY, Z)g(T_W(AX), Z) \\
&- g(AX, Z)g(T_X(AY), Z) + g(Y,Z)g(X, T_W Z) - g(X,Z)g(Y, T_W Z) \\
&+ g(\phi Y, Z)g(\phi X, T_W Z) - g(\phi X, Z)g(\phi Y, T_W Z) - 2g(\phi X, Y)g(\phi Z, T_W Z) \\
&+ g(AY, Z)g(AX, T_W Z) - g(AX, Z)g(AY, T_W Z) = 0. \qquad (3.1)
\end{aligned}$$

Next, we prove that ξ is principal. Putting $W = Z = \xi$ in (3.1) we get

$$-\eta(Y)g(\phi A\xi, X) + \eta(X)g(\phi A\xi, Y) - \eta(AY)g(\phi A\xi, AX) + \eta(AX)g(\phi A\xi, AY) = 0. \qquad (3.2)$$

Putting $Y = \xi$ in (3.2) we have

$$\phi A\xi + \eta(A\xi)A\phi A\xi = 0.$$

If $\eta(A\xi)=0$, then $\phi A\xi=0$ and this shows that ξ is principal. Further, if $\eta(A\xi)\neq 0$ we have

$$A\phi A\xi=-\frac{1}{\eta(A\xi)}\phi A\xi. \tag{3.3}$$

Putting $Y=\phi A\xi$ in (3.2) and taking account of (3.3), we have

$$(\eta(X)-\frac{\eta(AX)}{\eta(A\xi)})g(\phi A\xi,\phi A\xi)=0.$$

If $g(\phi A\xi,\phi A\xi)=0$, then $\phi A\xi=0$ and hence, ξ is principal. If $\eta(X)-\eta(AX)/\eta(A\xi)=0$, we have

$$\eta(A\xi)\eta(X)-\eta(AX)=g(X,\eta(A\xi)\xi-A\xi)=0$$

and ξ is again principal.

From Lemma 1, if $A\xi=\alpha\xi$, α is constant.

By a straightforward calculation, taking account of (2.7) and $A\xi=\alpha\xi$, we get for $(\tilde{\nabla}_\xi R)(X,\xi)\xi$ and $R(\tilde{\nabla}_\xi X,\xi)\xi$ the following calculations:

$$\begin{aligned}(\tilde{\nabla}_\xi)(R(X,\xi)\xi) &= \tilde{\nabla}_\xi X-g(\nabla_\xi X,\xi)\xi+\alpha\tilde{\nabla}_\xi(AX)-\alpha g(\nabla_\xi(AX),\xi)\xi,\\ R(\tilde{\nabla}_\xi X,\xi)\xi &= \tilde{\nabla}_\xi X-g(\nabla_\xi X,\xi)\xi+\alpha A\tilde{\nabla}_\xi X-\alpha g(A\nabla_\xi X,\xi)\xi.\end{aligned} \tag{3.4}$$

From (1.1),(2.8) and $A\xi=\alpha\xi$ we get

$$\begin{aligned}\tilde{\nabla}_\xi(AX)-A\tilde{\nabla}_\xi X &= (\nabla_\xi A)X-T_\xi AX+AT_\xi X\\ &= \alpha\phi AX-2A\phi AX+\phi X+\phi A^2X.\end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we have

$$\begin{aligned}0=(\tilde{\nabla}_\xi R)(X,\xi)\xi &= (\tilde{\nabla}_\xi)(R(X,\xi)\xi)-R(\tilde{\nabla}_\xi X,\xi)\xi-R(X,\tilde{\nabla}_\xi\xi)\xi-R(X,\xi)\tilde{\nabla}_\xi\xi\\ &= \alpha(\phi X+\alpha\phi AX-2A\phi AX+\phi A^2X).\end{aligned} \tag{3.6}$$

Since α is constant, we have only to consider the two cases, namely $\alpha=0$ or $\alpha\neq 0$.

Case 1: $\alpha=0$

Putting $Y=W=\xi$ in (3.1), we get

$$\eta(Z)g((A\phi-\phi A)Z,X)=0.$$

Putting $Z=Z'+\xi$ where Z' is an arbitrary vector orthogonal to ξ, we have

$$g((A\phi-\phi A)Z',X)=0.$$

Hence $A\phi=\phi A$ for any vector orthogonal to ξ and thus, $A\phi=\phi A$. Theorem 4 then implies that M is of type (A).

Case 2: $\alpha\neq 0$

(3.6) is equivalent to

$$\phi X+\alpha\phi AX-2A\phi AX+\phi A^2X=0. \tag{3.7}$$

We define V_λ by

$$V_\lambda := \{X \in TM | X \perp \xi, AX = \lambda X\}.$$

Choosing $X \in V_\lambda$ in (3.7) and taking account of Lemma 2, we have

$$(2\lambda + \alpha)(\lambda^2 - \alpha\lambda - 1) = 0.$$

Following the same procedure for $X \in V_{\frac{\alpha\lambda+2}{2\lambda-\alpha}}$ in (3.7) and taking again account of Lemma 2, we have

$$(4\alpha\lambda - \alpha^2 + 4)(\lambda^2 - \alpha\lambda - 1) = 0.$$

From this, it follows that λ is constant. If $\lambda^2 - \alpha\lambda - 1 \neq 0$, then $2\lambda + \alpha = 0$ and $4\alpha\lambda - \alpha^2 + 4 = 0$. This yields $\lambda^2 - \alpha\lambda - 1 = 0$ which gives a contradition.

So, $\lambda^2 - \alpha\lambda - 1 = 0$ and hence $\lambda = \frac{\alpha\lambda+2}{2\lambda-\alpha}$. This implies $A\phi = \phi A$ and thus, M is of type (A).

Next, we consider (4) $\Rightarrow$ (1).

From the definition of $\tilde{\nabla}T$ and $(\tilde{\nabla}_X T)_Y Z \equiv 0$ we have

$$(\nabla_X T)_Y Z = T_X T_Y Z - T_Y T_X Z - T_{T_X Y} Z. \tag{3.8}$$

From $T_\xi \xi = 0$, putting $X = Y = Z = \xi$, we have

$$\begin{aligned} 0 = (\nabla_\xi T)_\xi \xi &= -T_{\nabla_\xi \xi}\xi - T_\xi \nabla_\xi \xi \\ &= +g(\phi A\xi, \phi A\xi)\xi. \end{aligned}$$

Hence $\phi A\xi = 0$, that is, $A\xi = \alpha\xi$ where α is constant (Lemma 1).

Choosing X, Y orthogonal to ξ then we have

$$T_X Y = -g(\phi AX, Y)\xi,$$

From this and for X, Y, Z orthogonal to ξ, the right-hand of (3.8) becomes

$$T_X T_Y Z - T_Y T_X Z - T_{T_X Y} Z = -g(\phi AY, Z)\phi AX + g(\phi AX, Z)\phi AY - g(\phi AX, Y)\phi AZ. \tag{3.9}$$

From (2.6) and for the same choice of vectors X, Y, Z, the left-hand of (3.8) becomes

$$\begin{aligned} (\nabla_X T)_Y Z &= \nabla_X (T_Y Z) - T_{\nabla_X Y} Z - T_Y \nabla_X Z \\ &= -g(\phi(\nabla_X A)Y, Z)\xi - g(\phi AY, Z)\phi AX - g(\phi AX, Y)\phi AZ + g(\phi AX, Z)\phi AY. \end{aligned} \tag{3.10}$$

From (3.9) and (3.10) and for X, Y, Z orthogonal to ξ, we have

$$g((\nabla_X A)Y, \phi Z)\xi = 0.$$

It then follows from Theorem 5 that M is of type (A) or type (B).

Putting $X = Y = \xi$ in (3.8) and since the right-hand of (3.8) vanishes, we have

$$\begin{aligned} 0 = (\nabla_\xi T)_\xi Z &= \nabla_\xi(T_\xi Z) - T_{\nabla_\xi \xi} Z - T_\xi \nabla_\xi Z \\ &= -\phi(\nabla_\xi A)Z \\ &= \alpha AZ + \phi A \phi AZ + Z - (\alpha^2+1)\eta(Z)\xi. \end{aligned}$$

Choosing $Z \in V_\lambda$ and taking account of Lemma 2, we then have

$$\alpha(\lambda^2 - \alpha\lambda - 1) = 0.$$

If $\alpha \neq 0$, then $\lambda = \frac{\alpha\lambda+2}{2\lambda-\alpha}$, and hence $A\phi = \phi A$, or equivalentiy, M is type (A). Next, let $\alpha = 0$. From Theorem 1 it follows that there exists no real hypersurface of type(B) with $\alpha = 0$. That is, M is of type (A).

Finally, we prove (6) $\Rightarrow$ (1). In the following we define U and α by $U = \phi A\xi$ and $\alpha = \eta(A\xi)$.

From the definition of $\tilde{\nabla}A$ and $\tilde{\nabla}A \equiv 0$ we have

$$\begin{aligned} (\nabla_X A)Y &= \eta(AY)\phi AX - \eta(X)\phi A^2 Y - g(\phi AX, AY)\xi \\ &\quad -\eta(Y)A\phi AX + \eta(X)A\phi AY + g(\phi AX, Y)A\xi. \end{aligned} \tag{3.11}$$

Putting $X = \xi$ and $Y = X$ in (3.11), we have

$$(\nabla_\xi A)X = \eta(AX)U - \phi A^2 X - g(AU, X)\xi - \eta(X)AU + A\phi AX + g(U, X)A\xi. \tag{3.12}$$

Taking the ξ-component, we obtain

$$g((\nabla_\xi)X, \xi) = -2g(AU, X) + \alpha g(U, X). \tag{3.13}$$

Putting $Y = \xi$ in (3.11) we have

$$(\nabla_X A)\xi = \alpha\phi AX - \eta(X)\phi A^2\xi + g(AX, U)\xi - A\phi AX + \eta(X)AU.$$

From this and (2.8) we have

$$\begin{aligned} (\nabla_\xi A)X &= (\nabla_X A)\xi + \phi X \\ &= \alpha\phi AX - \eta(X)\phi A^2\xi + g(AX, U)\xi - A\phi AX + \eta(X)AU + \phi X. \end{aligned} \tag{3.14}$$

Taking the ξ-component, we get

$$g((\nabla_\xi A)X, \xi) = 2g(AU, X). \tag{3.15}$$

From (3.13) and (3.15) we have

$$AU = \frac{1}{4}\alpha U. \tag{3.16}$$

On the other hand, from the symmetry of ∇A and (3.14) we have

$$\begin{aligned} 0 &= g((\nabla_\xi A)X,U) - g((\nabla_\xi A)U,X) \\ &= \alpha g(\phi AX,U) - \alpha g(\phi AU,X) - \eta(X)g(\phi A^2\xi,U) - 2g(A\phi AX,U) + 2g(\phi X,U). \end{aligned} \tag{3.17}$$

Putting $X = \xi$ in (3.17) and using (3.16), we get

$$g(\phi A^2\xi,U) = \frac{1}{2}\alpha g(U,U). \tag{3.18}$$

Putting $X = A\xi$ in (3.17) and using again (3.16), we obtain

$$\frac{1}{4}\alpha^2 g(U,U) - \frac{1}{2}\alpha g(\phi A^2\xi,U) + 2g(U,U) = 0.$$

From this and (3.18), we get

$$2g(U,U) = 0.$$

Hence $U = \phi A\xi = 0$ and so, ξ is principal. From Lemma 1, if $A\xi = \alpha\xi$, α is constant. Then (3.12) and (3.14) become

$$\begin{aligned} (\nabla_\xi A)X &= -\phi A^2X + A\phi AX, \\ (\nabla_\xi A)X &= \alpha\phi AX - A\phi AX + \phi X. \end{aligned}$$

From this we get

$$\phi X + \alpha\phi AX - 2A\phi AX + \phi A^2X = 0.$$

We can prove $A\phi = \phi A$ in a similar way as for the case 2 in the proof of (3) $\Rightarrow$ (1).

Thus our main theorem is proved. □

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