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ACTION OF CLASSICAL GROUPS ON VARIETIES ASSOCIATED WITH SKEW FIELD EXTENSIONS[1]

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Abstract. Let $L|K$ be *a finite skew field* extension, K commutative *and V a finite* dimensional vector space over L. We study the action *of* general L-linear groups *on* the set *of* K-subspaces *of V and* the action *of* unitary groups *on* the set *of f'-isotropic* K-subspaces *of V*. *In* the *latter* case let *(V,f)* be a vector space endowed with a *regular ε-hermitian form,* $\varphi : L \longmapsto K$ a K-linear map *and* $f' := \varphi \circ f$. We show that *for* linear *groups* the number *of* orbits depends *only on* the degree *of the field* extension *and on* the dimension *of V*. The orbits are *classified* completely when $[L:K] \leq 3$. *For* unitary groups in general the number *of* orbits depends *also on* the *underlying fields and on* the map φ. We discuss in more detail the quadratic case *for* some *particular fields.*

1 Introduction, Notation and Main Results

In [28] Patrick Rabau studies the action on Grassmannians of general linear groups defined over an extension field of the base field. In [20] this work was extended by him and Dae San Kim to the case of a symplectic group defined over an extension field of the base field acting on singular subspaces of a symplectic space. The present paper generalizes these studies to non commutative extension fields and to arbitrary trace-valued ε-hermitean spaces.

Let $L|K$ be a finite skew field extension, where K is always commutative and V a finite dimensional vector space over L. We assume that the center Z of L is contained in K if L is non commutative. Then $[L:Z]$ is finite and L is a centra1 simple Z-algebra [8, p. 49]. Denote by $\mathrm{comp}_L W$ the greatest L-subspace contained in a K-subspace W. We call $\mathrm{comp}_L W$ the L – component of W. We define the *type of* a vector v *in* W as $\mathrm{tp}_W(v) := \dim_K(W \cap Lv)$. For W holds $\dim_L LW \leq \dim_K W \leq s \dim_L LW$ with $s = [L:K]$. One has $\dim_K W = s \dim_L LW$ iff W is a L-subspace. K-subspaces with $\dim_L LW = \dim_K W$ are called K-substructures. For K-subspaces W, $W_1, \ldots, W_k$ we write $W = W_1 \oplus_L \ldots \oplus_L W_k$, if $W = W_1 \oplus \ldots \oplus W_k$ and $LW_i \cap LW_j = 0$ for all $i, j = 1, \ldots, k$, $i \neq j$. We call this sum direct over L. The set of d-dimensional k-subspaces of a vector space V is a Grassmann *variety* and will be denoted by $\mathcal{G}_k(d,V)$. If we mean the set of all k-subspaces, we write $\mathcal{G}_k(V)$.
To decompose a K-subspace we need besides the just defined L-component and K-substructures (which already suffice in the case of a quadratic extension) two other basic subspaces: subspaces which contain only vectors of type 2 and triangular subspaces. We call a K-subspace W triangular if there is a $s > 0$ such that $\dim_K W = 3s$, $\dim_L LW = 2s$ and $\mathrm{tp}_W(v) = 1$ for all $0 \neq v \in W$. If $s = 1$, we cali W simple triungular.

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First let us consider linear groups. Our main results are:

Theorem 1 *Let $[L:K]=2$ and W be a K-subspace of V. Then there are L-independent vectors $v_1,\dots,v_s;z_1,\dots,z_t$, $s+t\leq n$, such that $W=\langle v_1,\dots,v_s\rangle_L\oplus_L\langle z_1,\dots,z_t\rangle_K$.*

If L is a cubic extension of K, we can choose a K-basis $\{1,\eta,\eta^2\}$ of L such that $\eta^3=a\in K$. Here we always suppose $K|Z$ to be galois.

Theorem 2 *Let $[L:K]=3$ and W be a K-subspace of V. Then there are L-independent vectors $v_1,\dots,v_m;\ e_1,\dots,e_r;u_1,u_1',\dots u_s,u_s';w_1,\dots,w_t$, $m+r+2s+t\leq n$, such that*

$$W = W_1\oplus_L W_2\oplus_L W_3\oplus_L W_4,$$

with

$$\begin{aligned} W_1 &= \langle v_1,\dots,v_m\rangle_L,\\ W_2 &= \bigoplus_{i=1}^{r}\langle e_i,\eta e_i\rangle_K,\\ W_3 &= \bigoplus_{i=1}^{s}\langle u_i,u_i',\eta u_i+\eta^2u_i'\rangle_K,\\ W_4 &= \langle w_1,\dots,w_t\rangle_K.\end{aligned}$$

The subspaces W_2,W_3 and W_4 are not unique. The flag (U_1,U_2,U_3,U_4) with $U_j:=\bigoplus_{i=1}^{j}W_i$ $(j=1,\dots,4)$, however, is completely determined by W.

From Theorem 1 and Theorem 2 follows at once that the number of orbits is finite for quadratic and cubic extensions. By the following theorem there are infinitly many orbits if $[L:K]>3$ and $|K|=\infty$:

Theorem 3 *The number of orbits of K-subspaces of a vector space V over $L|K$ with $|K|=\infty$ is finite if and only if $[L:K]\leq 3$.*

For proofs and discussion of the general linear case see section 2. Now we turn to a unitary group $\mathrm{U}(V,f)$. Let L be a skew field as above which now admits an involution $*$. Let K be such that $K^*=K$ and let $\varphi(\lambda^*)=\varphi(\lambda)^*$ for all $\lambda\in L$. Let (V,f) be a regular E-hermitian trace-valued space. Then (V,f), with $f':=\varphi\circ f$, is also a regular ε-hermitian space over K. We investigate how the group $\mathrm{U}(V,f)$ acts an the set of f'-singular K-subspaces of V. The results of the symplectic case, however, do not remain valid for the general case. In particular the number of orbits depends on the structure of the underlying skew fields and on the map φ. By $\mathrm{wi}(V,f)$ we denote the *Witt-index* of an ε-hermitian space (V,f), that is the dimension of an maximal f-singular subspace of V.

Theorem 4 *Let $[L:K]=2$ and W be a f'-singular K-suhspace of V and $\ker\varphi=K\alpha$. Then there is a hyperbolic sequence $v_1,v_1',\dots,v_m,v_m';u_1,u_1',\dots,u_s,u_s';w_1,w_1',\dots,w_t,w_t'$ of L-independent vectors, a sequence $y_1,\dots,y_p$, $z_1,\dots,z_p$ of L-independent vectors with $f(y_i,y_i)=0$,*

$f(z_i,z_i)\neq 0$, $f(y_i,z_i)=1$ ($i=1,\dots,p$) and an orthogonal sequence $e_1,\dots,e_r$ of L-independent vectors, with $m+s+2t+2p+r\leq n$, such that

$$\begin{aligned} W &= \langle v_1,\dots,v_m\rangle_L\perp_L\langle u_1,\dots,u_s\rangle_K\perp_L \\ &\quad \langle w_1,\alpha^* w_1',\dots,w_t,\alpha^* w_t'\rangle_K\perp_L \\ &\quad \langle y_1,z_1\rangle_K\perp_L\dots\perp_L\langle y_p,z_p\rangle_K\perp_L\langle e_1\rangle_K\perp_L\dots\perp_L\langle e_r\rangle_K. \end{aligned}$$

$p\neq 0$ is possible only in the cases

(A) *$L^+=K$ if L is commutative and*

(B) *$K\subset L^+$ and $K|Z$ not separable, if L is not commutative,*

where $L^+ :=\{h\in L : \lambda^*=h\}$. In these cases $a=1$ always holds.

We discuss the number of orbits for some particular fields in section 4.2.

Theorem 5 *Let $[L:K]=3$ and W be a maximal f'-singular K-subspace of V with $\mathrm{tp}(W)=(m,r,s,t)$ and $\ker\varphi=\langle\alpha,\beta\rangle_K$. Then there exists a decomposition*

$$W=\mathrm{comp}_L W\perp_L W_{2,4}\perp_L W_3'\perp_L W_3''\perp_L(\mathrm{rad}_f W_3\oplus_L W_4'),$$

where:

(a) $\mathrm{rad}_f W=\mathrm{comp}_L W$.

(b) *$W_{2,4}=\langle\alpha w_1,\beta w_1,w_1'\rangle_K\perp_L\dots\perp_L\langle\alpha w_r,\beta w_r,w_r'\rangle_K$, where $w_1,w_1',\dots,w_r,w_r'$ is a hyperbolic sequence in (V,f).*

(c) *$W_3'\perp_L W_3''\perp_L\mathrm{rad}_f W_3=W_3$, where $\mathrm{rad}_f W_3$ is triangular. If $W_4'\perp_L\mathrm{rad}_f W_3$, then $\mathrm{rad}_f W_3=0$. Let $j:=\dim_K\mathrm{rad}_f W_3$. There are L-independent vectors $u_1,v_1\dots u_k,v_k$ and a hyperbolic sequence $x_1,x_1',\dots,x_l,x_l';y_1,y_1',\dots,y_l,y_l'$ in (V,f) such that*

$$\begin{aligned} W_3' &= \perp_{i=1}^k\langle u_i,v_i,\eta u_i+\eta^2 v_i\rangle_K \\ W_3'' &= \perp_{i=1}^l\big(\langle\alpha^* x_i,\beta^* y_i,\beta^* x_i+\alpha^* y_i\rangle_K\oplus_L \\ &\qquad \langle x_i',y_i',\beta\alpha^{-1}x_i'-c\alpha\beta^{-1}y_i'\rangle_K\big), \end{aligned}$$

with $s=j+k+2l$. There are no f-singular triangular subspaces contained in W_3'. In particular $f(u_i,u_i)\neq 0$ or $f(v_i,v_i)\neq 0$ for $i=1,\dots,l$. If L is commutative, then $c=N_{L|K}(\beta\alpha^{-1})$. If L is non commutative, we can suppose that $\beta\alpha^{-1}$ equals η or η^2. Then $c=a$ or $c=a^2$.

(d) *$W_4'\leq W_4$. In particular W_4' is a K-substructure with $\dim_K W_4'=t-r$.*

We discuss this case further in section 5.2. Finally we show in section 6 that for extensions of higher degree under cerain conditions the number of orbits is infinite.

Theorem 6 *If $[L:K]>3$, $\mathrm{wi}(V,f)\geq 1$ and $|K|=\infty$, then the number of orbits of maximal f'-singular K-subspaces under $U(V,f)$ is infinite.*

2 The General Linear Case.

By a vector space V over a skew field k we mean always a left vector space. The group $\mathrm{GL}(k^n)$ acts from the right. The subspace of V generated by a set X is denoted by kX or $\langle x_1,\ldots,x_s\rangle_k$ if $X=\{x_1,\ldots,x_s\}$. By $\dot{k}$ we denote the multiplicative group of k.

2.1 Central simple algebras with involutions.

Before we go on, we want to recall some facts about central simple algebras and involutions. Let $\mathbf{A}$ be a central simple algebra with center Z. Then $[\mathbf{A}:Z]$ is always a square, say n^2 and n is called *degree* of $\mathbf{A}$. By Wedderburn $\mathbf{A}$ is isomorphic to a matrix algebra $\mathrm{M}_t(D)$, where D is a suitable division algebra. D is unique up to isometry and so is t. The number n/t is called *index* of $\mathbf{A}$. If there is a field extension $K|Z$ such that $\mathbf{A}\otimes_Z \mathrm{K}\cong \mathrm{M}_t(\mathrm{K})$ we call K a *splitting field* of $\mathbf{A}$ and say $\mathbf{A}$ *splits over* K. The smallest number m, such that $\mathbf{A}^m=\mathbf{A}\otimes\ldots\otimes A$ splits over Z, is called *exponent* of $\mathbf{A}$. The exponent divides the index and any prime divisor of the index divides the exponent [32, p. 215]. If $\mathbf{A}$ is a skew field, then there is a commutative subfield K such that $K|Z$ is separable and $[\mathrm{K}:Z]^2=[\mathbf{A}:\mathrm{Z}]$. One of the most important results is the theorem of Skolem and Noether, which says that every isomorphism of simple subalgebras of $\mathbf{A}$ can be extended to an inner automorphism of $\mathbf{A}$.

Let $K|Z$ be galois of degree n and σ be generator of $\mathrm{Gal}(K|Z)$. A *cyclic algebra* is an algebra which contains K and has a K-basis of the form $\{1,\eta,\ldots,\eta^{n-1}\}$ with $\eta^n=a\in Z$ and $\eta c=c^{\sigma}\eta$ for all $c\in \mathrm{K}$. We denote this algebra by $(K|Z,\sigma,a)$. In particular it is a central simple Z-algebra [33, p. 316]. By a theorem of Wedderburn any division algebra of degree 2 or 3 is cyclic [32, p. 209]. Algebras of degree 2 are called *qunternion algebras.*

For the whole paper $L|K$ denotes a finite skew field extension with K commutative. If L is not commutative, the center Z of L shall be contained in K. In this case K and $C_L(K)$ are Z-algebras and since $[L:C_L(K)]<\infty$, one has $[L:C_L(K)]=[\mathrm{K}:Z]$ [8, p. 49]. Then $[L:\mathrm{Z}]$ is finite, too, and L is a central simple Z-algebra. In case of a quadratic or cubic extension K is a maximal subfield of L. This yields $\mathrm{K}=C_L(K)$ and $[L:\mathrm{Z}]=[\mathrm{K}:Z]^2$. Moreover, an easy consequence of the Skolem-Noether-Theorem is $N_{\dot{L}}(\dot{K})/\dot{K}\cong \mathrm{Gal}(K|Z)$.

2.2 Basic results.

Lemma 7 *Let W_1 and W_2 be in $\mathcal{G}_K(V)$. Then holds:*

(1) $L(W_1+W_2)=LW_1+LW_2$.

(2) $\mathrm{comp}_L(W_1\cap W_2)=\mathrm{comp}_L W_1\cap \mathrm{comp}_L W_2$.

(3) $L(W_1\cap W_2)\leq LW_1\cap LW_2$.

(4) $\mathrm{comp}_L(W_1+W_2)\geq \mathrm{comp}_L W_1+\mathrm{comp}_L W_2$, *with equality iff W_1 and W_2 are direct over L.*

Proof. trivial ☐

Lemma 8 *Let $W\in \mathcal{G}_K(V)$ and let $\{\eta_1,\cdots\eta_s\}$ be n K-basis of L. Then*

$$\mathrm{comp}_L W=\eta_1^{-1}W\cap\ldots\cap\eta_s^{-1}W.$$

Proof. trivial □

Lemma 9 *Let* $W \in \mathcal{G}_K(V)$ *and* $U \in \mathcal{G}_L(W)$. *Then there exists* $Z \in \mathcal{G}_K(V)$, *such that* $W = U \oplus_L Z$. *For any other subspace* Z' *with* $W = U \oplus_L Z'$ *there is a linear map* $\tau \in \mathrm{GL}_L(V)$ *with* $u\tau = u$ *for all* $u \in U$ *and* $Z\tau = Z'$. *In particular* Z *and* Z' *are in the same orbit under* $\mathrm{GL}_L(V)$.

Proof. see [28, p. 131], Prop. 3.3 □

Lemma 10 *Let* $W, W_1, W_2 \in \mathcal{G}_K(V)$ *with* $W = W_1 \oplus_L W_2$. *Let* $v = v_1 + v_2 \in W$ *with* $0 \neq v_i \in W_i$, $i = 1,2$. *Then*

(1) $\mathrm{tp}_W(v) \leq \min(\mathrm{tp}_{W_1}(v_1), \mathrm{tp}_{W_2}(v_2))$.

(2) *If* $\mathrm{tp}_{W_1}(v_1) = [L:K]$, *then* $\mathrm{tp}_W(v) = \mathrm{tp}_{W_2}(v_2)$.

Proof. see [28, p. 131], Lemma 3.4 □

Lemma 11 *The group* L *acts transitively on the* K*-hyperplanes of* L

Proof. Let H_1 and H_2 be two K-hyperplanes in L and $\{\alpha_1, \ldots, \alpha_{s-1}\}$ be a K-basis of H_1 with $s = [L:K]$. Let $W := \bigcap_{i=1,\ldots,s-1} \alpha_i^{-1} H_2$. Then $\dim_Z W \geq [K:Z] := t$, since $\dim_Z H_2 = t(s-1)$. For all nonzero vectors $\lambda \in W$ holds $\alpha_i\lambda \in H_2$, $i = 1, \ldots, s-1$. This yields $H_1\lambda = H_2$. □

Corollary 12 *Let* $\{\alpha_1, \ldots, \alpha_s\}$ *be a* K-basis *of* L *and let* $v \in W \in \mathcal{G}_K(V)$ *with* $\mathrm{tp}_W(v) = s-1$. *Then* $Lv \cap W = \langle \alpha_1 v', \ldots, \alpha_{s-1} v' \rangle_K$ *for a vector* $v' \in W$.

Lemma 13 *Let* $W \in \mathcal{G}_K(V)$. *The following statements are equivalent*

(i) W *is a K-substructure.*

(ii) *Every K-independent subset of* W *is L-independent.*

(iii) *A K-basis of* W *is L-independent.*

Proof. trivial □

2.3 Quadratic extensions: Proof of Theorem 1.

Let $L|K$ be a quadratic extension and V an n-dimensional vector space over L. We fix $\eta \in L \setminus K$, such that $L = K \oplus K\eta$. Note that in characteristic two the extension $K|Z$ need not be galois.

Lemma 14 *Let* W *be a* K*-subspace of* V. *Then*

(1) $\mathrm{comp}_L W = W \cap \eta W$.

(2) $\dim_L \mathrm{comp}_L W = \dim_K W - \dim_L LW$.

(3) *For all* $x \in W$ *holds:* $x \in \mathrm{comp}_L W \iff \eta x \in W$.

Proof. see [28, p. 134], Theorem 4.1 □

Lemma 15 *Fora K-subspace* W *are* equivalent:

(i) W is K-substructure.

(ii) $\mathrm{comp}_L W = 0$.

(iii) *For all nonzero* vectors $v \in W$ *holds* $\mathrm{tp}_W(v) = 1$.

Proof. trivial □

Lemma 16 Let W, U and Y be K-subspaces with $W = U \oplus Y$ and $\mathrm{comp}_L W \leq U$. Then Y *is* a K-substructure, $W = U \oplus_L Y$ and $\dim_K Y = \dim_L(LW/LU)$.

Proof. see [28, p. 134], Theorem 4.1 □

Proof of Tlzeorem 1. By 16 there is $Y \in \mathcal{G}_K(t,W)$ such that $W = \mathrm{comp}_L W \oplus_L Y$. Choose a L-Basis $\{v_1,\ldots,v_s\}$ of $\mathrm{comp}_L W$ and a K-basis $\{z_1,\ldots,z_t\}$ of Y. By 16 Y is a K-substructure and the z_i are L-independent. □

We define the (GL-) type of a *K-subspace* W of V to be the ordered pair of nonnegative integers

$$\mathrm{tp}(W) := (\dim_L \mathrm{comp}_L W\, , \dim_K(W/\mathrm{comp}_L W))\,.$$

Corollary 17 *Two K-subspaces* of W are *in* the *same* orbit under $\mathrm{GL}_L(V)$ *if and only if they have the same* type.

Proof. Let W and W' be two K-subspaces of V with $\mathrm{tp}(W) = \mathrm{tp}(W')$. Pick suitable L-bases B and B' of V. The element of $\mathrm{GL}_L(V)$ which maps B to B' maps also W to W'. □

Corollary 18 *The number* of orbits of K-subspaces *under* $\mathrm{GL}_L(V)$ equals $\binom{n+2}{2}$.

Proof. The number N of orbits equals $|\{s,t \in \mathbb{N}\cup\{0\} : s+t \leq n\}|$. This yields $N = \sum_{i=0}^{n} |\{s+t = i\}| = \sum_i i+1 = \frac{1}{2}(n+2)(n+1) = \binom{n+2}{2}$. □

2.4 Cubic Extensions: Proof of Theorem 2.

We now consider the case $[L:K] = 3$. Here we assume $K|Z$ to be galois. The fact that L is a centra1 simple Z-algebra yields that L has a K-basis $\{1,\eta,\eta^2\}$ with $\eta^3 = a \in Z$ and $\eta \in N_{\dot{L}}(\dot{K})$. In this whole section we choose a basis as above. Since $N_{\dot{L}}(\dot{K})/\dot{K} \cong \mathbb{Z}_3$, one has $N_{\dot{L}}(\dot{K}) = \dot{K} \oplus \dot{K}\eta \oplus \dot{K}\eta^2$.

By σ we denote the galois-automorphism $k \longmapsto k^\sigma := \eta k \eta^{-1}$, $k \in K$.

Lemma 19 Let $W \in \mathcal{G}_K(V)$ with $\mathrm{comp}_L W = 0$ and *let* $W_2 := \langle\{v \in W : \mathrm{tp}_W(v) = 2\}\rangle_K$. Then

(1) There exist L-independent vectors $e_1,\ldots,e_r$ such that $W_2 = \bigoplus_{i=1}^{r} \langle e_i, \eta e_i\rangle_K$.

(2) $W \cap \eta W$ *is a* K-substructure and $W_2 = (W\cap\eta W) \oplus (W\cap\eta^{-1}W)$.

(3) $\dim_K(W \cap \eta W) = \frac{1}{2}\dim_K W_2$.

Proof. see [28, p. 138], Theorem 5.2 □

Lemma 20 *Let* T *be a simple triangular subspace*

(1) *Let* $\{1,\gamma,\delta\}$ *be a K-rigkt basis of L. Then there exist L-independent vectors x and y such that* $T = \langle x,y,\gamma x+\delta y\rangle_K$.

(2) *Let x, y be two arbitrary L-independent vectors in* T. *Then there is a K-right basis* $\{1,\gamma,\delta\}$ *of L such that* $T = \langle x,y,\gamma x+\delta y\rangle_K$.

Proof. (1) For to show $LT = T \oplus \delta T$ use that $T \cap \delta T = 0$, since T contains no vector of type ≥ 2 and the fact that $\dim_Z T = \dim_Z \delta T = \frac{1}{2}\dim_Z LT$ (in genera1 δT is not a K-subspace). The rest of the proof is like in the commutative case, see [28, p. 137], Lemma 5.1.

(2) Let $T = \langle x,y,z\rangle_K$. Since $z \in LT = \langle x,y\rangle_L$, there are $\gamma,\delta \in L$ such that $z = \gamma x + Sy$. If $\{1,\gamma,\delta\}$ was no K-right basis, it would hold that $6 = p + \gamma q$, $p,q \in K$. Then we have $z = \gamma x + (p+\gamma q)y \in T$. Now $y \in T$ yields $\gamma(x+qy) \in T$ and since $\gamma \notin K$, one gets the contradiction $tp_T(x+qy) \geq 2$. □

Lemma 21 *If x andy are two L-independent vectors, then* $W = \langle x,y,\eta x+\eta^2 y\rangle_K$ *is a simple triangular subspace.*

Proof. Obviously $\dim_L LW = 2$. Let $v = bx + cy + d(\eta x + \eta^2 y)$, $b,c,d \in K$ be a nonzero vector in W and suppose that also $W \ni hv = b'x + c'y + d'(\eta x + \eta^2 y)$, $b',c',d' \in K$. We have to show that $h \in K$.

Let $h := p + q\eta + r\eta^2$, $p,q,r \in K$. Then

$$\begin{aligned} hv &= (p+q\eta+r\eta^2)(bx+cy+d(\eta x+\eta^2 y) \\ &= pbx+pcy+pd(\eta x+\eta^2 y)+ \\ &\quad qb^\sigma \eta x+qc^\sigma \eta y+qd^\sigma(\eta^2 x+ay)+ \\ &\quad rb^{\sigma^2}\eta^2 x+rc^{\sigma^2}\eta^2 y+rd^{\sigma^2}(ax+a\eta y). \end{aligned}$$

Comparing the coefficients yields the following equations

$$\begin{aligned} pb+rd^{\sigma^2}a &= b', && (1)\\ pc+qd^\sigma a &= c', && (2)\\ pd+qb^\sigma &= d', && (3)\\ qc^\sigma+rd^{\sigma^2}a &= 0, && (4)\\ qd^\sigma+rb^{\sigma^2} &= 0, && (5)\\ pd+rc^{\sigma^2} &= d'. && (6) \end{aligned}$$

If $q = 0$ or $r = 0$, one gets easily from the equations above that $\lambda \in K$. So let both q and r be different from zero. We can assume that $r = 1$. One has $d \neq 0$, otherwise would hold $c = 0$ (4) and $b = 0$ (5). Now (4) yields $qc^\sigma = -d^{\sigma^2}a$. Then follows $c^{\sigma^2} = \frac{-da}{q^\sigma}$. From (5) one gets

$b^{\sigma^2} = -qd^{\sigma}$, that is $b^{\sigma} = -q^{\sigma^2}d$. Finally (3) and (6) imply $qb^{\sigma} = c^{\sigma^2}$. Then $-qq^{\sigma^2}d = -\frac{da}{q^{\sigma}}$. This yields $a = qq^{\sigma}q^{\sigma^2} = N_{K|Z}(q)$. But this is a contradiction, since it implies that the skew field L splits [33, p. 318]. □

Lemma 22 *Let W be a triangular subspace. Then*

(1) $LW = W \oplus \lambda W$, *for all* $\lambda \in L \setminus K$.

(2) Let $\{1,\gamma,\delta\}$ *be an arbitrary K-right basis of L. Then for each* $0 \neq x \in W$ *there is exactly one* $y \in W \setminus Lx$ *such that* $\langle x, y, \gamma x + \delta y\rangle_K$ *is the unique simple triangular subspace which contains x.*

Proof. see [28, p. 144], Prop. 5.6 □

Lemma 23 *Let* W *be triangular and let* $T, T' \subset W$ *be simple triangular and* $T \neq T'$. *Then* $LT \cap LT' = 0$.

Proof. see [28, p. 144], Cor. 5.11 □

Lemma 24 *Let* W *be a triangular subspace and T a simple triangular subspace of* W. *Then there is a triangular subspace Y such that* $W = T \oplus_L Y$.

Proof of Theorem 2. Only the proof of (d) differs a bit from the commutative case ([28, p. 134], Theorem 4.1):

(d) Choose a K-substructure $Z \leq W$ such that $LW = LW_2 \oplus LZ$. If $W = W_2 \oplus Z$, we are done with $W_3 = 0$ and $W_4 = Z$. Otherwise one can find K-independent vectors $x_1, \ldots, x_s$ such that $W = W_2 \oplus Z \oplus \langle x_1, \ldots, x_s\rangle_K$. In general the sum of W_2 and $Z \oplus \langle x_1, \ldots, x_s\rangle_K$ will not be direct over L. We now show that $x_1, \ldots, x_s$ can always be chosen in such a manner that the above sum is direct over L. For all $\tilde{x}_i \in W$ holds

$$\tilde{x}_i = \lambda \tilde{w}_i + \mu \tilde{z}_i,$$

with $\lambda, \mu \in L$, $\tilde{w}_i \in \langle e_1, \ldots, e_r\rangle_K$ and $\tilde{z}_i \in Z$.

Let $\lambda := p + q\eta + r\eta^2$ and $\mu := p' + q'\eta + r'\eta^2$, $p, q, r, p', q', r' \in K$. Adding to each $\tilde{x}_i$ the vectors $-(p + q\eta)\tilde{w}_i \in W_2$ and $-p'\tilde{z}_i \in Z$ we get

$$x_i := r\eta^2 \tilde{w}_i + (q'\eta + r'\eta^2)\tilde{z}_i \in W.$$

Since $\eta \in N_L(\dot{K})$, follows $y_i := \eta^{-2} r \eta^2 \tilde{w}_i \in \langle e_1, \ldots, e_r\rangle_K$, $z_i := \eta^{-1} q' \eta \tilde{z}_i \in Z$ and $z_i' := \eta^{-2} r' \eta^2 \tilde{z}_i \in Z$. Then we can write

$$x_i = \eta z_i + \eta^2(z_i' + y_i),$$

with $z_i, z_i' \in Z$, $y_i \in \langle e_1, \ldots, e_r\rangle_K$, $i = 1, \ldots, s$.

The rest of the proof is exactly as in the commutative case.

□

In the cubic case we define the *(GL-) type of a K-subspace* $\mathrm{tp}(W)$ to be the ordered quadruple of nonnegative integers

$$\left(\dim_L \mathrm{comp}_L W, \frac{1}{2}\dim_K\left(\frac{U_2}{\mathrm{comp}_L W}\right), \frac{1}{3}\dim_K\left(\frac{U_3}{U_2}\right), \dim_K\left(\frac{W}{U_3}\right)\right).$$

U_2 is the subspace generated by all vectors $v \in W$ with $\mathrm{tp}_W(v) \geq 2$. U_3 is the sum of U_2 and all simple triangular subspaces contained in W. By Theorem 2 this type is well defined.

Corollary 25 *Two K-subspaces of W are in the same orbit under $\mathrm{GL}_L(V)$ iff they have the same type.*

Proof. The proof is analogous to the proof of 17. □

Corollary 26 *For the number N_d of orbits of d-dimensional K-subspaces under $\mathrm{GL}_L(V)$ holds*

$$N_d = \frac{1}{24}\left(2d^3 + 15d^2 + 34d + 24\right),$$

if d is even and

$$N_d = \frac{1}{24}\left(2d^3 + 15d^2 + 34d + 21\right),$$

if d is odd. For the number N of all orbits we have

$$N = \frac{1}{48}\left(n^4 + 12n^3 + 50n^2 + 84n + 48\right),$$

if n is even and

$$N = \frac{1}{48}\left(n^4 + 12n^3 + 50n^2 + 84n + 45\right),$$

if n is odd.

Proof. We have $N = |\{m+r+2s+t \leq n : m,r,s,t \in \mathbb{N} \cup \{0\}\}|$, $N_d = |\{m+r+2s+t = d : m,r,s,t \in \mathbb{N} \cup \{0\}\}|$ and $N = \sum_{d=0}^{n} N_d$. We calculate N_d. For convenience we define $N_d := 0$ if $d < 0$. Then $N_d = \{m+r+2s+t = d\} = \sum_{i=0}^{d} |\{r+2s+t = i\} = \sum_{i=0}^{d}\sum_{j=0}^{i} |\{2s+t = j\}$ $= \sum_{i=0}^{d}\left(\lfloor \frac{i}{2} \rfloor + 1\right)\left(\lceil \frac{i}{2} \rceil + 1\right)$.

Let $G_d := N_d$ if d is even and $U_d := N_d$ if d is odd. By induction one proves that

$$\begin{aligned} G_d &= \frac{1}{24}(d+2)(d+4)(2d+3) \qquad \text{and} \\ U_d &= \frac{1}{24}(d+1)(d+3)(2d+7). \end{aligned}$$

For the number N of all orbits holds

$$\begin{aligned} N &= \sum_{k=0}^{\frac{n}{2}} G_{2k} + U_{2k-1} \quad \text{if } n \text{ is even and} \\ N &= \sum_{k=0}^{\frac{n-1}{2}} G_{2k} + U_{2k+1} \quad \text{if } n \text{ is odd.} \end{aligned}$$

Using the formulas $\sum_{d=1}^{n} d = \frac{n(n+1)}{2}$, $\sum_{d=1}^{n} d^2 = \frac{n(n+1)(2n+1)}{6}$ and $\sum_{d=1}^{n} d^3 = \frac{n^2(n+1)^2}{4}$ we get

$$\begin{aligned} N &= \left(\frac{n}{2}+1\right)\left(\frac{n^3+10n^2+30n+24}{24}\right) \quad \text{if } n \text{ is even and} \\ &= \left(\frac{n+1}{2}\right)\left(\frac{n^3+11n^2+39n+45}{24}\right) \quad \text{if } n \text{ is odd.} \end{aligned}$$

□

2.5 Extensions of higher degree: Proof of Theorem 3

To prove Theorem 3 it suffices to show that there are infinitely many orbits of 2-dimensional K-subspaces of L under the multiplicative group $\dot{L} = \mathrm{GL}_L(L)$ of L. Let $s := [L:K] > 3$ and $m := [K:Z]$ and consider L as an s-dimensional K-vector space. We show that $\mathcal{G}_K(d,L)$, $1 \le d \le s$, is a projective variety over Z with dimension $dm(s-d)$. Recall that the Grassmann variety $\mathcal{G}_Z(md,L)$ already is a projective variety over Z with dimension $dm(sm-dm)$, see e.g. [2, p. 135] or [17, p.14].

Let $\{e_1,\ldots,e_n\}$ be a K-basis of L and $\{v_1,\ldots,v_m\}$ be a Z-basis of K, such that $v_p v_k = \sum_{q=1}^{m} c_{pkq} v_q$, $c_{pkq} \in Z$, $p,k = 1,\ldots,m$. Let

$$W_0 = \langle v_1 e_1,\ldots,v_m e_1,\ldots,v_1 e_d,\ldots,v_m e_d\rangle_Z$$

and

$$W_0' = \langle v_1 e_{d+1},\ldots,v_m e_{d+1},\ldots,v_1 e_s,\ldots,v_m e_s\rangle_Z.$$

Any subspace $W \in \mathcal{G}_Z(dm,V)$ which the projection maps isomorphically onto W_0 has a unique basis of the form

$$\{v_1 e_1 + x_{11}(W),\ldots,v_m e_d + x_{md}(W)\},$$

with

$$x_{pi}(W) = \sum_{j>d,q=1,\ldots,m} a_{piqj} v_q e_j \in W_0,$$

$a_{piqj} \in Z$, $i = 1,\ldots,d$, $p = 1,\ldots,m$.

In the proofs cited above is shown that the image of $\mathcal{G}_Z(dm,L)$ under the injection $\psi : W \longmapsto Z v_1 e_1 \wedge \ldots \wedge v_m e_d$ in the projective space $\mathbb{P}(\bigwedge^{md} L)$ is essentially the graph of a morphism from the space of the a_{piqj} to another linear space. In particular it is closed, hence is a projective variety. The dimension one gets by counting the free a_{piqj}.

Now we must show that the fact that W is a K-subspace can be expressed in polynomial conditions on the a_{piqj}. Since

$$W = \langle v_1 e_1 + x_{11}(W),\ldots,v_1 e_d + x_{1d}(W)\rangle_K,$$

there is $c_{pi} \in K$ such that

$$v_p e_i + x_{pi}(W) = c_{pi}\,(v_1 e_i + x_{1i}(W)).$$

Without loss we can suppose $v_1 = 1$. Thus $c_{pi} = v_p$, $i = 1,\ldots,d$. We have

$$\begin{aligned}
\sum_{j>d,q} a_{piqj} v_q e_j &= v_p \sum_{j>d,k} a_{1ikj} v_k e_j \\
&= \sum_{j>d,k} a_{1ikj} \left(\sum_q c_{pkq} v_q\right) e_j \\
&= \sum_{j>d,q} \left(\sum_k a_{1ikj} c_{pkq}\right) v_q e_j.
\end{aligned}$$

Comparing coefficients yields $a_{piqj} = \sum_k c_{pkq} a_{1ikj}$, $i = 1, \ldots, d$. This shows that $\psi(\mathcal{G}_K(d,V))$ can be considered as the graph of a morphism from the space of the a_{1iqj} to another linear space. The dimension follows by counting the free a_{1iqj}. Thus $\mathcal{G}_K(d,V)$ has the structure of a projective variety with dimension $dm(s-d)$.

Now the group $\dot{L}$ is an sm-dimensional irreducible algebraic group over Z, since L is a Z algebra [2, p. 51]. $\dot{L}$ acts on $\mathcal{G}_K(2,L)$. Suppose there are only finitely many orbits. Then at least one orbit must be Zariski-dense in $\mathcal{G}_K(2,L)$. Pick a subspace W contained in such a dense orbit. Denote by R the closure of this orbit and consider the orbit map

$$\gamma_W : \begin{array}{ccc} \dot{L} & \longrightarrow & \mathcal{G}_K \\ \lambda & \longmapsto & W\lambda \end{array} .$$

Let S be the stabilizer of W. We have $\dim B = 2m(s-2)$ and $\dim S \geq 1$, since Z is contained in S. By applying the formula (see [3, p.12])

$$\dim \dot{L} = \dim B + \dim S$$

we get

$$2ms - 4m \leq ms - 1.$$

This yields the contradiction $s \leq 3$.

REMARK: This method works also in the commutative case. $\dot{L}$ is an algebraic group over K with $\dim \dot{L} = s$ and, as mentioned above, $\mathcal{G}_K(2,\mathrm{L})$ is a projective variety over K with $\dim \mathcal{G}_K(2,L) = 2(s-2)$. Since K is contained in the stabilizer of any subspace, we get in the same way the contradiction $s \leq 3$.

3 The Hermitean Case: Further Notation And Basic Results.

Let L be a skew field with an involution $*$ and let Z be the center of L. Recall that an *involution* on L is an anti automorphism of L such that $\alpha^{**} = a$ for all $a \in L$. It is easy to see that $Z^* = Z$. If $*|_Z$ is the identity, $*$ is called involution of the *first* kind. Otherwise $*$ is called involution of the *second* kind. In this case $Z|Z_0$ is a separable quadratic extension, where Z_0 denotes the fixed field of $*|_Z$. Let $L^+ := \{a \in \mathrm{L} : \alpha^* = a\}$ and $L^- := \{a \in L : \alpha^* = -\alpha\}$. The latter we define only for characteristics unequal 2. If the degree of L is n, the following holds [33, p. 303].

(1) If $\operatorname{char} Z \neq 2$, then $L = L^+ \oplus L^-$.

(2) If $*$ is of the second kind, then $\dim_{Z_0} L^+ = \dim_{Z_0} L^- = n^2$.

(3) If $*$ is of the first kind, then $\dim_Z L^+ = \frac{1}{2}n(n+1)$ or $= \frac{1}{2}n(n-1)$. If $\operatorname{char} Z = 2$, always $\dim_Z L^+ = \frac{1}{2}n(n+1)$.

An involution of the first kind is called *orthogonal* if $\dim_Z L^+ = \frac{1}{2}n(n+1)$ and *symplectic* if $\dim_Z L^+ = \frac{1}{2}n(n-1)$. Involutions of the second kind are also called *unitary*. By a theorem of Albert [32, p. 232] a central simple algebra has exponent 2 if and only if it admits an involution of the first kind.

Let V be a finite-dimensional left-vector space over L. A *sesquilinear form* f is a map $f : V \times V \longrightarrow L$ such that

$$\begin{aligned} f(x+y,z) &= f(x,z)+f(y,z), \\ f(x,y+z) &= f(x,y)+f(x,z), \\ f(\lambda x,y) &= \lambda f(x,y), \\ f(x,\lambda y) &= f(x,y)\lambda^*, \end{aligned}$$

for all $x,y,z \in V$ and all $\lambda \in L$. A sesquilinear form f is called *ε-hermitian* ($\mathrm{E} = \pm 1$) if $f(x,y) = \varepsilon f(y,x)^*$ for all $x,y \in V$. An E-hermitian form f is called *symmetric* if $* = \mathrm{id}$ and $\mathrm{E} = +1$, *skew symmetric* if $* = \mathrm{id}$ and $\mathrm{E} = -1$ and *symplectic* (or *alternating)* if $f(x,x) = 0$ for all x E V.

Now let U be a subspace of an E-hermitian space (V,f). We write $x \perp y$ if $f(x,y) = 0$ and $x \perp U$ if $f(x,u) = 0$ for all u E U. The *orthogonal subspace* of U in V is the space $U^\perp = \{ v \in V : v \perp u \, \forall u \text{ E } U \}$. The *radical* of U is the space $\mathrm{rad}_f U := \{ u \in U = U \cap U^\perp \}$. A space (V,f) is called *regular* iff $\mathrm{rad}_f V = 0$. Otherwise it is called *degenerate.* A vector $v \in \mathrm{V}$ is called *isotropic* if $f(v,v) = 0$. A subspace U is called *isotropic space* if it contains an isotropic vector. Otherwise it is called *anisotropic*. A subspace U is *singular* if $f(u,u') = 0$ for all u,u' E U.

A sequence $e_1,\ldots,e_r$ in (V,f) is called *orthogonal* iff $e_i \perp e_j$ for $\mathrm{i} \neq j$. A sequence $v_1, v_1', \ldots, v_r, v_r'$ is called *hyperbolic* iff $f(v_i,v_i') = 1, f(v_i,v_j) = f(v_i',v_j') = 0$ and $f(v_i,v_j') = 0$ if $i \neq j$. Two vectors x,y E V are a *hyperbolic* pair if $f(x,x) = f(y,y) = 0$ and $f(x,y) = 1$. The plane $\mathrm{H} := \langle x,y \rangle$ is called *hyperbolic* plane. A space is called *hyperbolic* if it is the orthogonal sum of hyperbolic planes. If a orthogonal or a hyperbolic sequence form a basis of V, it is called an *orthogonal basis* or a *symplectic* ***basis*** respectively. An ε-hermitian form f is called *trace-valued* if for all $\mathrm{X} \in V$ holds $f(x,x)$ E $\{ h + \varepsilon\lambda^* : \lambda \text{ E } L \}$.

An *isometry* between two spaces (V_1,f_1) and (V_2,f_2) is an injective linear map $\tau : V_1 \longrightarrow V2$, such that $f_2(x\tau,y\tau) = f_1(x,y)$ for all x,y E V_1. The spaces (V_1,f_1) and (V_2,f_2) are called *isometric*, we write $(V_1,f_1) \sim (V_2,f_2)$, if there is a bijective isometry $V_1 \longrightarrow V_2$. The bijective isometries of a space (V,f) onto itself form the *Unitary Group* $\mathrm{U}(V,f)$.

From now on we consider only regular and trace-valued forms. If we want to emphasize the particular form f, we write f-orthogonal, f-isotropic, f-singular, etc..

Recall that if an ε-hermitian space is not symplectic, one can always find an orthogonal basis. If the space is symplectic, it is an orthogonal sum of hyperbolic planes.

Closely related to the theory of symmetric bilinear spaces is the theory of quadratic forms. They are equivalent concepts if the characteristic is unequal 2. Let V be a vector space over a field K. A *quadraticform* is a map $q : V \longrightarrow K$ with

$$q(\lambda x) = \lambda^2 q(x)$$

for all $x \in V$, λ E K such that the map $b_q : V \times V \longrightarrow K$,

$$b_q(x,y) := q(x+y) - q(x) - q(y),$$

is a bilinear form over K. The map b_q is called the *associated bilinearform* of q. The pair (V,q) is *called quadratic space.* If b is a bilinear form, the map $q_b : V \longrightarrow K$, $q_b(x) := b(x,x)$, is called the *associated quadratic form* of b. It holds that $b_{q_b}(x,y) = b(x,y) + b(y,x)$ and $q_{b_q}(x) = 2q(x)$.

Let (V,q) be a quadratic space. **A** vector $x \in V$ is called *isotropic* if $q(x) = 0$ and *anisotropic* if $q(x) \neq 0$. Two vectors x and y are called *orthogonal* if $b_q(x,y) = 0$.

3.1 Statement of the problem and basic results.

Let (V,f) be a regular trace-valued ε-hermitian space over $(L,*)$ and K a commutative subfield of L such that $K^* = K$. If L is non commutative, let always $Z := Z(L) \subset K$. $*$ induces an involution on K, which we also denote by $*$. Let $\varphi : L \longrightarrow K$ be a K-linear map with $\varphi(\lambda^*) = \varphi(\lambda)^*$. Since $\varphi(\lambda c) = \varphi((c^*\lambda^*)^*) = \varphi(c^*\lambda^*)^* = (c^*\varphi(\lambda^*))^* = \varphi(\lambda)c$ for all $c \in K$, $\lambda \in L$, φ is two-sided K-linear. Define $f' := \varphi \circ f$. It is easy to see that (V,f') is a regular ε-hermitian space over $(K,*)$. Since every isometry of (V,f) is an isometry of (V,f'), too, one gets an embedding $U(V,f) \hookrightarrow U(V,f')$.

If orthogonality refers to the form f', we write $x\perp' y$, $U\perp' W$, $U^{\perp'}$, etc.. We write $U \perp_L W$ if $U \perp W$ and $U \oplus_L W$.

Lemma 27 *Let U be a L-subspace of V.*

(1) *For $x \in V$ holds that $x \perp' U$ iff $x \perp U$.*

(2) *It holds that $U^{\perp'} = U^{\perp}$. In particular U is f-singular iff U is f'-singular and U is f-regular iff U is f'-regular.*

Proof. see [20, p.285] Lemma 4.1 □

Lemma 28 *Let W be a f'-singular K-subspace of V. Then*

(1) $\mathrm{comp}_L W \leq \mathrm{rad}_f W$.

(2) *If W is maximal f'-singular in LW, then* $\mathrm{comp}_L W = \mathrm{rad}_f W$

In particular $\mathrm{comp}_L W$ *is f-singular.*

Proof. see [20, p.285] Lemma 4.1 □

Lemma 29 *Let W be f'-singular and let X be a K-subspace of W. If there is a f'-regular L-subspace U of V such that $X \subset U$ and $\dim_K U = 2\dim_K X$, then there exists another K-subspace Y of V such that $W = X \perp_L Y$. In particular $Y = U^{\perp} \cap W$ is such a subspace.*

Proof. Since $V = U \perp_L U^{\perp}$, it is enough to show that $W \leq X + (W \cap U^{\perp})$. Let $W \ni x = x_1 + x_2$, $x_1 \in U$ and $x_2 \in U^{\perp}$. Then $x \perp' X$, since W is f'-singular and $x_2 \perp X$, since $X \subset U$. This yields $x_1 \perp' X$, that is $x_1 \in U \cap X^{\perp'}$. We now show that $U \cap X^{\perp'} = X$. Obviously it holds that $X \leq U \cap X^{\perp'}$. Since U is f'-regular, we have

$$\dim_K X^{\perp'} \cap U = \dim_K U - \dim_K X = \dim_K X.$$

Therefore, equality follows by dimensional reasons. This implies $x_1 \in X$. □

For a K-subspace W we denote by $R(W) := \{f(w,w) : w \in W\}$ the set of all elements of L which are *represented* by $f|_W$.

Lemma 30 *Let W he a f'-singular K-suhspace and let $R(W) \cap \ker\varphi = 0$. Then W is f-singular f one of the following conditions holds:*

(1) $* = \mathrm{id}$ *and f is not symplectic.*

(2) L is commutative and there is a $\zeta \in K$ with $\zeta^* = -\zeta$ **or** *$\zeta^* = \zeta + 1$ in characteristic 2 respectively.*

(3) L is not commutative and there is a $\zeta \in Z$ with $\zeta^* = -\zeta$ or $\zeta^* = \zeta + 1$ in characteristic 2 respectively.

Proof. In all cases we can by "Hilbert 90" suppose that $\varepsilon = 1$. For all $x \in W$ it holds that $f(x,x) = 0$. Then for any $y \in W$ holds $0 = f(x+y,x+y) = f(x,y) + f(y,x)$, that is $f(x,y) = -f(y,x)$. Case (1) is clear. In *(2)* there exists $\zeta \in K$ with $\zeta^* = -\zeta$ if char $K \neq 2$. Thus $0 = f(x+\zeta y, x+\zeta y) = -2\zeta f(x,y)$, hence $f(x,y) = 0$. If char K $= 2$, there exists $\zeta^* = \zeta + 1$ and we get similarly $f(x,y) = 0$. The proof of **(3)** is the same with $\zeta \in Z$. This yields $f(x,y) = 0$. □

4 The Hermitean Case: Quadratic Extensioiis

In this section we consider extensions with $[L:K] = 2$. If L is not commutative, it is a quaternion algebra over the center Z of L. The basic difference to the symplectic case is that in the general situation $f(w,w)$, $w \in V$, may be different froni zero. Since W is f'-singular, we must investigate the set $R(W) \cap \ker\varphi$. In general not much is known about $R(W)$. We have $R(W) \subset L^+$ if f is 1-hermitian and $R(W) \subset L^-$ if f is (-1)-hermitian. For this reason we consider the sets $S^+ := L^+ \cap \ker\varphi$ and $S^- := L^- \cap \ker\varphi$. Note that S^- is defined only for characteristic unequal *2*.

Recall that if $*$ is a nontrivial involution on the field K, then $K|K^+$ is a separable quadratic extension and $* \in \mathrm{Gal}(K|K^+)$. Without loss we can suppose that $K = K^+(\eta)$ and $\eta^* = -\eta$ if char K $\neq 2$ and $\eta^* = \eta + 1$ if char $K = 2$.

4.1 Lemmas and proof of Theorem 4

If L is commutative, three cases can occur: $* = \mathrm{id}$, $* \neq \mathrm{id}$ with $L^+ = K$ and $* \neq \mathrm{id}$ with $L^+ \neq K$. In the first case clearly $L^+ \cap \ker\varphi = \ker\varphi$. So $\ker\varphi = K\alpha$ is possible for any $\alpha \in L$. If $* \neq \mathrm{id}$, L^+ is a field and $L|L^+$ is a separable quadratic extension.

If $L^+ = K$, we can suppose that $L = K \oplus K\xi$ with $\xi^* = -\xi$ if char $L \neq 2$ and $\xi^* = \xi + 1$ if char $L = 2$.

Lemma 31 *If L is commutative und $L^+ = K$, the following holds:*

(1) $S^+ = 0$ *and* $S^- = K\xi$ *if* char $K \neq 2$.

(2) $S^+ = K$ if char $K = 2$.

Proof. (1) $-\varphi(\xi) = \varphi(-\xi) = \varphi(\xi^*) = \varphi(\xi)$, that is $\ker\varphi = K\xi$.
(2) $\varphi(\xi) = \varphi(\xi)^* = \varphi(\xi^*) = \varphi(\xi) + \varphi(1)$, that is $\ker\varphi = K$. □

If $L^+ \neq K$, it holds that $K^+ = L^+ \cap K$. K^+ is a field with $[K:K^+] = 2$. Choose $\xi \in L^+$ such that $L = K \oplus K\xi$. By "Hilbert 90' we can suppose that f is 1-hermitian.

Lemma 32 *If L is commutative and $L^+ \neq K$, it holds that $S^+ = K^+\alpha$, $a \in K^+ \oplus K^+\xi$*

Proof. Let char $K \neq 2$. Pick $\eta \in K \setminus K^+$ with $\eta^* = -\eta$. Write $K = K^+ \oplus K^+\eta$. Let $\varphi(1) = x + y\eta \in K$. Now $\varphi(1) = \varphi(1^*) = \varphi(1)^*$ yields $y = 0$. In the same way follows $\varphi(\xi) = x \in K^+$. Then it holds that $z - x\xi \in \ker\varphi$. The proof for char $K = 2$ is the same with $\eta^* = \eta + 1$. $\square$

If L is non commutative, L is a yuaternion algebra over Z.

Lemma 33 *For every maximal subfield K of a quaternion algebra L there exists a Z-basis $\{1, \eta, \xi, \eta\xi\}$ of L with $K = \langle 1, \eta \rangle_Z$ and*

(1) $\eta^2 = a \in Z$, $\xi^2 = b \in Z$ *and* $\eta\xi = -\xi\eta$ *if* char $Z \neq 2$,

(2) $\eta^2 + \eta = a \in Z$, $\xi^2 = b \in Z$ *and* $\eta\xi = \xi\eta + \xi$ *if* char $Z = 2$ *and* $K|Z$ *is separable*

(3) $\eta^2 = a \in Z$, $\xi^2 + \xi = b \in Z$ *and* $\eta\xi = \xi\eta + \eta$ *if* char $Z = 2$ *and* $K|Z$ *is not separable.*

Proof.
Se: [33, p. 300, p. 312]. $\square$

We call a basis like in the previous lemma a *standard basis* of L. In the following we always choose such a basis if L is not commutative.

Lemma 34 *Let L be a quaternion algebra over Z and let $K|Z$ be separable. Then* $\langle \xi, \eta\xi \rangle_Z^* = \langle \xi, \eta\xi \rangle_Z$.

Proof. First we consider tlie case char $K \neq 2$. Suppose $\eta^* = x + y\eta$, $x, y \in Z$. Since $(\eta^*)^2 = (\eta^2)^* = a^* \in Z$, follows $x^2 + 2xy\eta + y^2 a \in Z$. Then $x = 0$, for $y = 0$ leads to the contradiction $\eta^* \in Z$. This yields $\eta^* = y\eta$. From $\eta = \eta^{**} = yy^*\eta$ one gets $yy^* = 1$. Then the following holds:

$$\eta\xi^* = (\xi\eta^*)^* = (y\xi\eta)^* = (-y\eta\xi)^* = -\xi^*\eta^* y^* = -\xi^*\eta y y^* = -\xi^*\eta.$$

Let $\xi^* := p + q\eta + r\xi + s\eta\xi$, $p, q, r.s \in Z$. Then

$$\eta\xi^* = p\eta + qa + r\eta\xi + sa\xi = -\xi^*\eta = -(p\eta + qa + r\xi\eta + s\eta\xi\eta)$$

This implies $p = q = 0$ and we get $(\eta\xi)^* = \xi^*\eta^* = (r\xi + s\eta\xi)y\eta = -yr\eta\xi - sya\xi$. This proves $\langle \xi, \eta\xi \rangle_Z^* = \langle \xi, \eta\xi \rangle_Z$.

The case char $K = 2$ is similar: From $(\eta^*)^2 + \eta^* \in Z$ follows $x^2 + y^2(a + \eta) + x + y\eta \in Z$. Then $y = i$. Since $\eta = \eta^{**} = x + x^* + \eta$, $x = x^*$. We have

$$\eta\xi^* = (\xi\eta^*)^* = (\xi x + \xi\eta)^* = (\xi x + \eta\xi + \xi)^* = x^*\xi^* + \xi^*\eta^* + \xi^* =$$
$$= x\xi^* + \xi^* x + \xi^*\eta + \xi^* = \xi^*\eta + \xi^*.$$

This yields

$$\eta\xi^* = p\eta + q\eta^2 + r\eta\xi + s\eta^2\xi \text{ and } \xi^*\eta = p\eta + q\eta^2 + r\eta\xi + r\xi + s\eta^2\xi + s\eta\xi.$$

One gets $\xi^* = r\xi + s\eta\xi$. Thisimplies $(\eta\xi)^* = (r\xi + s\eta\xi)(\eta + x) = r\xi\eta + s\eta\xi\eta + rx\xi + sx\eta\xi$. Since $\xi\eta = \eta\xi + \xi$ and $\eta\xi\eta = (\eta^2 + \eta)\xi = a\xi$, it holds that $(\eta\xi)^* \in \langle\xi, \eta\xi\rangle_Z$. □

Lemma 35 Let L be *a* quaternion algebra over Z. Then there is *a* standard basis $\{1, \eta, \xi, \eta\xi\}$ *of* L *such* that:

(1) $L^+ = \langle 1, \eta, \xi\rangle_Z$ and $L^- = \langle\eta\xi\rangle_Z$ *if* $*$ *is* orthogonal, $K \subset L^+$ *and* $K|Z$ separable.

(2) $L^+ = (1, \xi, \eta\xi\rangle_Z$ and $L^- = \langle\eta\rangle_Z$ *if* $*$ is orthogonal and $K \not\subset L^+$. $K|Z$ is *always* separable.

(3) $L^+ = Z$ and $L^- = \langle\eta, \xi, \eta\xi\rangle_Z$ *if* $*$ is symplectic. char $K = 2$ is notpossible in this case.

(4) *If* $*$ *is unitary* and $K|Z$ separable, *it* holds that $L^+ = Z^+ \oplus Z^+\eta \oplus Z^+\xi \oplus Z^-\eta\xi$ *f* char $K \neq 2$. *If* char $K = 2$, *one has* $L^+ = Z^+ \oplus Z^+\eta \oplus Z^+\xi \oplus Z^+(\eta + \zeta)\xi$, *where* $\zeta \in Z$ with $\zeta^* = \zeta + 1$.

Proof. Let $\{1, \eta, \rho, \eta\rho\}$ be an arbitrary standard basis of L. Choose ξ E $L^+ \setminus$ K and let $\xi = x + y\eta + z\rho + t\eta\rho$, x, y, z, t E Z. If $K|Z$ is separable, we have $z\rho + t\eta\rho$ E L^+ by 34.

(1) Without loss let $L^+ \ni \xi = z\rho + t\eta\rho$. If char $K \neq 2$, $\eta\xi = \eta(z\rho + t\eta\rho) = -z\rho\eta - t\rho\eta = -\xi\eta$ and $\xi^2 = z^2\rho^2 + zt\rho\eta\rho + zt\eta\rho^2 + t^2\eta\rho\eta\rho$ E Z. If char $K = 2$, we have $\eta\xi = \eta(z\rho + t\eta\rho) = z(\rho\eta + \rho) + t\eta(\rho\eta + \rho) = \xi\eta + \xi$ and $\xi^2 = z^2\rho^2 + zt\rho\eta\rho + zt\eta\rho^2 + t^2\eta\rho\eta\rho = z^2\rho^2 + zt\,(\rho\eta + \eta\rho)\,\rho + t^2\eta\,(\eta\rho + \rho)\,\rho = z^2\rho^2 + zt\rho^2 + t^2\,(\eta^2 + \eta)\,\rho^2 \in Z$.

(2) Since $*|_K$ equals the nontrivial automorphism of Gal$(K|Z)$, we have $\eta^* = -\eta$ if char $K \neq 2$ and $\eta^* = \eta + 1$ if char $K = 2$. In particular $K|Z$ is separable. Like in (1) one proves that $\{i, \eta, \xi, \eta\xi\}$ is a standard basis.

(3) Since in characteristic 2 always $L^+ = 3$, we must have char $K \neq 2$ in this case. $L^+ = Z$ is clear. Like in (2) one gets $\eta^* = -\eta$. Every vector in L^- has the form $x\eta + z\rho + t\eta\rho$ with $x, z, t \in Z$. Since $x\eta \in L^-$ and $\dim_Z L^- = 3$, $\xi := z\rho + t\eta\rho$ is also in L^-. The rest of the proof is like (1).

(4) We show that we can suppose $\eta^* = \eta$. Let $\{1, o, \ldots\}$ be an arbitrary standard basis. If char $K \neq 2$, we have $\sigma^* = y\sigma$ with $y \in Z$ and $yy^* = i$ (see proof of 34). By "Hilbert 90" there is $u \in Z$ with $y = u^*u^{-1}$. With $\eta := u^*\sigma$ we are done. If char $K = 2$, we have $o = o + x$ with $x \in Z$ and $x + x^* = 0$ (see proof of 34). Since $Z|Z^+$ is separable, there exists $\zeta \in Z$ with $\zeta^* = \zeta + 1$. Then $\eta := \sigma + x\zeta$ is fixed under $*$. Like in (1) follows $\xi \in L^+$. □

Lemma 36 *For* $0 \neq a \in R(W) \cap \ker\varphi$ *holds* a E $N_L(\dot{K})$.

Proof. Let $f(w, w) = a$. Then for all $k \in K$ holds $f(kw, kw) \in \ker\varphi = Ka$. Hence $k\alpha k^* = ca$, $c \in K$. Since k and k^* are arbitrary, we get $\alpha K = Ka$. □

Lemma 37 *If* $R(W) \cap \ker\varphi \neq 0$, $\ker\varphi = K$ or $\ker\varphi = K\xi$. *If* $K|Z$ is not separable, only $\ker\varphi = K$ *is* possible.

Proof. Since $N_{\dot L}(\dot K)/\dot K \cong \mathrm{Gal}(K|Z)$, we have $\mathrm{Gal}(K|Z) \cong \mathbb{Z}/2\mathbb{Z}$ if $K|Z$ is separable. Otherwise $\mathrm{Gal}(K|Z)$ is trivial. In the separable case it is easy to prove that $\xi \in N_{\dot L}(\dot K)$, hence $N_{\dot L}(\dot K) = \dot K \cup \dot K\xi$. □

Now we suppose that we have $\ker\varphi = K$ or $\ker\varphi = K\xi$. But not always these two cases can occur as the next lemma shows.

Lemma 38 (1) $*$ *is orthogonal and* $K \subset L^+$*:* $\ker\varphi = K\xi$ *if* $K|Z$ *is separable and* $\ker\varphi = K$ *if* $K|Z$ *is not separable.*

(2) $*$ *is orthogonal and* $K \not\subset L^+$*:* $\ker\varphi = K\xi$.

(3) $*$ *is symplectic:* $\ker\varphi = K\xi$.

(4) $*$ *is unitary:* $\ker\varphi = K$ *or* $\ker\varphi = K\xi$.

Proof. (1) If $\mathrm{char}\, K \neq 2$, it holds that $-\varphi(\eta\xi) = \varphi((\eta\xi)^*) = \varphi(\eta\xi)^* = \varphi(\eta\xi)$. Thus $\eta\xi \in \ker\varphi$ and hence $\ker\varphi = K\xi$. If $\mathrm{char}\, K = 2$ and $K|Z$ is separable, we have $\varphi(\eta\xi) = \varphi((\eta\xi)^*) = \varphi(\xi\eta) = \varphi(\eta\xi) + \varphi(\xi)$. Hence $\varphi(\xi) = 0$. In the non separable case $\varphi(\eta\xi) = \varphi((\eta\xi)^*) = \varphi(\xi\eta) = \varphi(\eta\xi) + \varphi(\eta)$ holds. Hence $\varphi(\eta) = 0$.

*(2)*By 35 $L^+ = \langle 1,\xi,\eta\xi\rangle_Z$. Suppose $\ker\varphi = K$. Since $\varphi(L^+) = K^+ = Z$ and $\dim_Z L^+ = 3$, there are two Z-independent vectors in $L^+ \cap \ker\varphi$. But this is impossible, since $K \cap L^+ = Z$.

(3) By 35 $L^- = \langle \eta,\xi,\eta\xi\rangle_Z$. Since $\varphi(L^-) = K^- = Z\eta$ and $\dim_Z L^- = 3$, the assumption $\ker\varphi = K$ provides a contradiction as in (2).

(4) Here is nothing to prove. □

Proof of Theorem 4. (a) Let $v_1,\ldots,v_m$ be a L-basis of $\mathrm{comp}_L W$. By 28 $\mathrm{comp}_L W$ is f-singular. Thus there exists a hyperbolic sequence $v_1, v_1', \ldots, v_m, v_m'$ in (V,f). By 29 we have $W = \mathrm{comp}_L W \perp_L Y$ for a suitable K-subspace Y of W. So without loss we can suppose $\mathrm{comp}_L W = 0$.

(b) Let $W = \mathrm{rad}_f W \oplus Y$ with $Y \leq V$. By 28 and 16 it holds that $W = \mathrm{rad}_f W \oplus_L Y$, hence $W = \mathrm{rad}_f W \perp_L Y$. Let $\{u_1,\ldots,u_s\}$ be a K-basis of $\mathrm{rad}_f W$. Since $\mathrm{rad}_f W$ is a K-substructure, $u_1,\ldots,u_s$ are linear independent over L (13). Then there exists a hyperbolic sequence $u_1,u_1',\ldots,u_s,u_s'$ in (V,f). So without loss let $\mathrm{rad}_f W = 0$.

(c) Suppose there is $w_1 \in W$ such that $f(w_1,w_1) = 0$. Since $\mathrm{rad}_f W = 0$, there exists $\tilde x_1 \in W$ such that $f(w_1,X) \neq 0$. We can suppose that $f(w_1,\tilde x_1) = \alpha$. If $f(\tilde x_1,\tilde x_1) = 0$, define $x_1 := \tilde x_1$. Otherwise $f(\tilde x_1,\tilde x_1) = c\alpha$, $c \in K$ and $f(\tilde x_1,\tilde x_1) = \lambda + \varepsilon\lambda^*$, $\lambda \in \dot L$, since f is trace-valued. Define

$$x_1 := -\lambda\alpha^{-1} w_1 + \tilde x_1.$$

Then $f(x_1,x_1) = -\lambda - \varepsilon\lambda^* + \lambda + \varepsilon\lambda^* = 0$. Let $w_1' := (\alpha^*)^{-1}x_1$. Then (w_1,w_1') is a hyperbolic pair. If $x_1 \in W$, by 29 we have

$$W = \langle w_1, \alpha^* w_1'\rangle_K \perp_L Y,$$

for a suitable K-subspace Y of W, since $\langle w_1,x_1\rangle_K$ is maximal f'-singular in the hyperbolic (thus regular) L-subspace $\langle w_1,w_1'\rangle_L$. By induction we get

$$W = \langle w_1, \alpha^* w_1'\rangle_K \perp_L \ldots \perp_L \langle w_t, \alpha^* w_t'\rangle_K \perp_L W_A,$$

where W_A is a K-subspace of W which contains no f-isotropic vector.

It remains to investigate the conditions for x_1 to be in W. Since W is a K-substructure and $\tilde{x}_1, w_1 \in W, x_1 \in W$ holds iff $\lambda\alpha^{-1} \in K$. This is always the case when the characteristic is odd, since $\lambda = -\frac{1}{2}f(\tilde{x}_1,\tilde{x}_1) = \frac{1}{2}c\alpha$. Now consider the cases in characteristic 2. Let $\gamma := f(\tilde{x}_1,\tilde{x}_1)$.

If L is commutative and $*|_K \neq$ id, then $K|K^+$ is separable. Hence there is a $\zeta \in$ K such that $\zeta^* = \zeta + 1$. With $\lambda := \zeta\gamma$ follows $\lambda + \lambda^* = \gamma$ and $\lambda\alpha^{-1} = c\zeta \in K$. Let now L be non commutative. Note that $\alpha \in N_L(\dot{K})$ (36) if there exists $v \in W$ with $f(v,v) \neq 0$. If $*$ is orthogonal, K $\subset L$ and $K|Z$ separable, then $\ker\varphi = K\xi$ (38) that is $\alpha = \xi$ and $L^+ = \langle 1,\eta,\xi\rangle_K$ (35). Hence $\gamma = c\xi$ with $c \in Z$, since f is trace-valued. Moreover, $(\eta\xi)^* = \xi\eta = \eta\xi + \xi$. With $\lambda := c\eta\xi$ follows $\lambda + \lambda^* = c\xi$ and $\lambda\xi^{-1} = c\eta \in K$. If $*$ is orthogonal and K $\not\subset L^+$, we have $L^+ = \langle 1,\xi,\eta\xi\rangle_K$ (35) aiid $\ker\varphi = K\xi$ (38). **An** easy calculation shows $\lambda + \lambda^* \in$ K for all $\lambda \in L$. Since f is trace-valued. we have $f(\tilde{x}_1,\tilde{x}_1) \in K \cap K\xi = 0$. Hence $x_1 \in$ W. In characteristic 2 there are no symplectic involutions. If $*$ is unitary, $Z|Z^+$ is separable and there is $\zeta \in Z$ such that $\zeta^* = \zeta + 1$. With $\lambda := \zeta\gamma$ follows $x_1 \in$ W.

It remains to consider the cases **(A)** and (B). In both cases it holds that $\ker\varphi = K$ and $*|_K =$ id (31 and 38). Suppose there is $y_1 \in W$ such that $f(y_1,y_1) = 0$. **As** above there is $z_1 \in W$ such that $f(y_1,z_1) = 1$. If $f(z_1,z_1) \neq 0$, we have $f(z_1,z_1) = \lambda + \lambda^*$. In order to find a vector $t_1 \in \langle y_1,\zeta_1\rangle_K$ such that (y_1,t_1) is a hyperbolic pair take $t_1 := \lambda y_1 + z_1$ as above. But then it must hold that $\lambda \in$ K and we get the contradiction $f(z_1,z_1) = 0$.

(d) If $W_A \neq 0$, pick $0 \neq e_1 \in W_A$. Since $\langle e_1\rangle_K$ is maximal f'-singular in the f-regular L-subspace $\langle e_1\rangle_L$, the assertion follows by induction as in (c). $\square$

4.2 Qrbits of f'-singular suhspaces.

We define the f-type $\mathrm{tp}_f(W)$ of W to be the 5-tuple (m,s,t,p,r). **A** necessary condition for two subspaces W and W' to be in the same orbit under $\mathrm{U}(V,f)$ is $\mathrm{tp}_f(W) = \mathrm{tp}_f(W')$. In general, however, this condition is not sufficient. Let W_I be the subspace generated by all f-isotropic vectors of W,

$$\begin{aligned} W_I &:= \langle v_1,\ldots,v_m\rangle_L \perp_L \langle u_1,\ldots,u_s\rangle_K \\ &\perp_L \langle w_1,\alpha^* w_1',\ldots,w_t,\alpha^* w_t'\rangle_K \perp_L \langle y_1,\ldots,y_p\rangle_K \end{aligned}$$

and let

$$W_A := \langle z_1,\ldots,z_p,e_1,\ldots,e_r\rangle_K$$

We have $\mathrm{comp}_L W \leq \mathrm{rad}_f W \leq W_I$ and, if $p = 0$, $W = W_I \perp_L W_A$.

Lct now $p + r = 0$. Then two f'-singular subspaces W and W' are in the same orbit under $\mathrm{U}(V,f)$ iff $\mathrm{tp}_f(W) = \mathrm{tp}_f(W')$. For if $\mathrm{tp}_f(W) = \mathrm{tp}_f(W')$, every isometry $\sigma : \mathrm{W} \longrightarrow W'$ can be extended to an isometry $\Sigma \in \mathrm{U}(V,f)$. This, for example, occurs in the symplectic case [20]. Let $\nu := \mathrm{wi}(V,f)$ and $\nu' := \mathrm{wi}(V,f')$. **A** necessary and sufficient condition for $(m,s,t,0,0)$ to be the f-type of a subspace is $m + s + t \leq \nu$. For then $\dim_L LW = m + s + 2t \leq n$ and $\dim_K W = 2m + s + 2t \leq \nu'$. For all $d \geq 0$ let N_d be the number of orbits of d-dimensional K-subspaces if $p + r = 0$. To simplify our notation we define $N_d := 0$ if $d < 0$. If $p + r = 0$, we have $0 \leq d \leq 2\nu$ and the following holds [20]

$$N_d = \begin{cases} \binom{\lfloor \frac{d}{2} \rfloor + 2}{2} & \text{if} \quad 0 \leq d \leq \nu, \quad \text{and} \\ \binom{\lfloor \frac{d}{2} \rfloor + 2}{2} - \binom{d - \nu + 1}{2} & \text{if} \quad \nu < d < 2\nu. \end{cases}$$

For the number of orbits we get

$$N = |\{m + s + t \leq \nu : m, s, t \in \mathbb{N} \cup \{0\}\}| =$$

Now consider the general case where $p + r > 0$. If two subspaces W and W' are in the same orbit under $\mathrm{U}(V, f)$, they must have the same type and the spaces

$$\langle y_1, z_1 \rangle_K \perp_L \ldots \perp_L \langle y_p, z_p \rangle_K \perp_L \langle e_1 \rangle_K \perp_L \ldots \perp_L \langle e_r \rangle_K$$

and

$$\langle y'_1, z'_1 \rangle_K \perp_L \ldots \perp_L \langle y'_p, z'_p \rangle_K \perp_L \langle e'_1 \rangle_K \perp_L \ldots \perp_L \langle e'_r \rangle_K$$

must be isometric. Recall that $\ker \varphi = K\alpha$. Without loss we can assume that $\alpha^* = a$ or $\alpha^* = -\alpha$ respectively. By defining $\tilde{f} := f|_{W_A} \alpha^{-1}$ we get a map

$$\tilde{f} : \begin{array}{ccc} W_A \times W_A & \longrightarrow & K \\ (x, y) & \longmapsto & f(x, y)\alpha^{-1}. \end{array}$$

An easy calculation shows that $\tilde{f}$ is a I-hermitian formover $(K, * \circ \sigma)$, whereo $\in \mathrm{Gal}(K|Z)$ (put $Z = \mathrm{K}$ if L is commutative). Moreover, two $p+$ r-dimensional anisotropic subspaces W_A and W'_A are in the same orbit under $U(V, f)$ iff the induced forms $\tilde{f}$ and $\tilde{f}'$ are isometric over $(K, * \circ \sigma)$. The problem to decide whether two subspaces are in the same orbit leads to the classification of hermitian forms over fields. Since very little is known about the case of general fields, we are able to treat this problem only for some special fields.

Before we do so, let us consider how the form $\tilde{f}$ depends on L and $*$. The following table gives an overview:

Proof of the Table. See 31 and 32 in the commutative case and 35 and 38 in the non commutative case.

If L is commutative, $L = \langle 1, \xi \rangle_K$. Obviously we have here $\sigma = \mathrm{id}$.

$* = \mathrm{id}$. Since $S^+ = K\alpha$ and $S^- = 0$, r > 0 is only possible if j is symmetric. In char $\mathrm{K} \neq 2$ it holds that $a_i \in K$ and $\tilde{f}$ is a symmetric bilinear form over K. Since in char $\mathrm{K} = 2$ always $a_i\alpha = \lambda + \lambda = 0$, the situation r > 0 cannot occur in this case.

$* \neq \mathrm{id}$ and $K = L^+$. If char $K \neq 2$, we have $S^+ = 0$ and $S^- = K\xi$. Thus $r > 0$ is only possible if f is a (-1)-hermitian form. Then $a_i \in K$ and $\tilde{f}$ is symmetric. In cliar $K = 2$ f is symmetric but non trace-valued (case(A)).

$* \neq \mathrm{id}$ and K $\neq L^+$. It suffices to consider 1-hermitian forms. In all cases it holds that $S^+ = K^+\alpha$, where $\alpha = z + x\xi$, $z, x \in K^+$. The a_i are in $\dot{K}^+$ and $\tilde{f}$ is a non symmetric form over

Table 1:

$*$	L^+	char K	$K\|Z$	$\tilde{f}$
$=\mathrm{id}$		$\neq 2$		symmetric, trace-valued
$\neq \mathrm{id}$	$K=L^+$	$\neq 2$		symmetric, trace-valued
$\neq \mathrm{id}$	$K=L^+$	$=2$		symmetric, non trace-valued
$\neq \mathrm{id}$	$K\neq L^+$	$\neq 2$		non symmetric, trace-valued
$\neq \mathrm{id}$	$K\neq L^+$	$=2$		non symmetric, trace-valued
orthogonal	$K\subset L^+$	$\neq 2$		non symmetric, trace-valued
orthogonal	$K\subset L^+$	$=2$	separabel	non symmetric, trace-valued
orthogonal	$K\subset L^+$	$=2$	non separabel	symmetric, non trace-valued
orthogonal	$K\not\subset L^+$	$\neq 2$		symmetric, trace-valued
symplectic		$\neq 2$		symmetric, trace-valued
unitary		$\neq 2$		non symmetric, trace-valued
unitary		$=2$	separabel	non symmetric, trace-valued
unitary		$=2$	non separabel	non symmetric, trace-valued

$(K,*)$. In characteristic 2 we have $K=K^+\oplus K^+\eta$, where $\eta^*=\eta+1$. Then $a_i=a_i\eta+(a_i\eta)^*$, hence $\tilde{f}$ is trace-valued.

Now let L be non commutative, that is $L=\langle 1,\eta,\xi,\eta\xi\rangle_Z$.

$*$ orthogonal and $K\subset L^+$. If char $K\neq 2$, we have $S^+=Z\xi$ and $S^-=Z\eta\xi$. The a_i are in Z and $a\in\{\xi,\eta\xi\}$. Hence $\sigma\neq\mathrm{id}$. Since $*|_K=\mathrm{id}$, $\tilde{f}$ is a non symmetric form over (K,σ). If char $K=2$, we have $S^+=Z\xi$ if $K|Z$ is separable. Like in the odd characteristic case one gets a non symmetric form $\tilde{f}$ over (K,σ). Since $K=Z\oplus Z\eta$ and $\eta^\sigma=\eta+1$, the form $\tilde{f}$ is trace-valued. If $K|Z$ is not separable, $\tilde{f}$ symmetric but non trace-valued (case (B)).

$*$ orthogonal and $K\not\subset L^+$. We have $S^+=K\xi$ and $S^-=0$. The a_i are in K and $*|_K=\sigma$. Hence $\tilde{f}$ is symmetric. In characteristic 2 we have always $r=0$ (see the proof of Theorem 4(c)).

$*$ symplectic. This case occurs only if char $K\neq 2$. We have $S^+=0$ and $S^-=K\xi$. The a_i are in K and $*|_K=\sigma$. Thus $\tilde{f}$ is symmetric.

$*$ unitary. It suffices to consider 1-hermitian forms. If char $K\neq 2$, we have $S^+=Z^+\oplus Z^+\eta=K^+$ or $S^+=Z^+\xi\oplus Z^-\eta\xi$. In the first case $\tilde{f}$ is a non symmetric form over $(K,*)$. In the second case let $a_i=b_{i1}+b_{i2}\eta$ and $a_i'=b_{i1}'+b_{i2}'\eta$, $b_{i1},b_{i1}'\in Z^+$ and $b_{i2},b_{i2}'\in Z^-$. Then

$$b_{i1}\xi+b_{i2}\eta\xi=\sum_j a_{ij}(b_{i1}'\xi+b_{i2}'\eta\xi)a_{ij}^*.$$

Multiplication from the right by ξ^{-1} yields

$$b_{i1}+b_{i2}\eta=\sum_J a_{ij}(b_{i1}'+b_{i2}'\eta)(a_{ij}^*)^\sigma.$$

Let F be the fixed field of the involution $*\circ\sigma$. Since $*\circ\sigma\neq\mathrm{id}$, we have $[K:F]=2$. Since $Z^+\subset F$ and $Z^-\eta\subset F$, $F=Z^+\oplus Z^-\eta$. Hence $\tilde{f}$ is a non symmetric form over $(K,*\circ\sigma)$.

The case char $K = 2$ is not essentially different: if $K|Z$ is separable, the cases $\ker\varphi = K$ and $\ker\varphi = K\xi$ can occur. In the first case we have $S^+ = Z^+ \oplus Z^+\eta = K^+$. In the second case we have $S^+ = Z^+\xi \oplus Z^+ (\zeta+\eta)\xi$, $\zeta \in Z$ and $\zeta^* = \zeta + 1$. Like in odd characteristics we get that $\tilde{f}$ is a non symmetric Form over $(K,*)$ or over $(K,*\circ\sigma)$ respectively. Since $a_i \in K$ implies $(a_i \in K, a_i = (a_i + (\zeta a_i)^*$ or $a_i = \zeta a_i + (\zeta a_i)^{*\circ\sigma}$ respectively. Thus $\tilde{f}$ in both cases in trace-valued. If $K|Z$ is not separable, we have $\ker\varphi = K$, hence $S^+ = K$. Since there is a $\zeta \in Z$ with $\zeta^* = \zeta + 1$, the form $\tilde{f}$ is trace-valued and non symmetric over $(K,*)$. $\square$

We have seen that $\tilde{f}$ is either a symmetric or non symmetric form over K. Now let R_d be the number of orbits of d-dimensional f'-singular subspaces if $p + r \geq 0$. Recall that N_d denotes the number of orbits if $p + r = 0$. In order to calculate R_d one has to solve two problems: On the one hand we must investigate how many forms $\tilde{f}$ can be realized, given an ε-hermitian space (V,f) and a subspace W. On the other hand one must know which of these forms are isometric. For convenience suppose $p = 0$. Let $W_0 := LW_A = \langle e_1 \ldots, e_r\rangle_L$ and let $V = W_0 \oplus W_0'$, where $W_0' = \langle e_{r+1}, \ldots, e_n\rangle_L$. Then every r-dimensional subspace W' has the form $\langle \delta_1 e_1 + x_1(W'), \ldots, \delta_r e_r + x_r(W')\rangle_K$ with $\delta_i \in \{0,1\}$ and $x_i(W') = \sum_{j=r+1}^n \lambda_{ij} e_j$, $i = 1, \ldots, r$. Let $L = K\alpha \oplus K\beta$ ($\beta = 1$ if $\alpha \notin K$). If at least one subspace W exist with $\tilde{f} \sim [a_1, \ldots, a_r]$, then $(V,f) \sim [a_1\alpha, \ldots, a_r\alpha, a_{r+1}\alpha + b_{r+1}\beta, \ldots, a_n\alpha + b_n\beta]$. Then we have for all basis vectors of W'

$$f(\delta_i e_i + x_i, \delta_i e_i + x_i) = \delta_i a_i \alpha + \sum_{j=r+1}^{n} \lambda_{ij}(a_j\alpha + b_j\beta)\lambda_{ij}^*.$$

In order for a subspace W' with a form $\tilde{f}' \sim [a_1', \ldots, a_r']$ to be realized, given a subspace W with $\tilde{f} \sim [a_1, \ldots, a_r]$, the equation

$$a_i'\alpha = \delta_i a_i \alpha + \sum_{j=r+1}^{n} \lambda_{ij}(a_j\alpha + b_j\beta)\lambda_{ij}^*$$

must be solvable over L for i = $1, \ldots, r$. Very little is known yet about the problem under which conditions this holds true for the general case. However, every form can be realized if W_0' contains a hyperbolic sequence $y_1, y_1', \ldots, y_r, y_r'$. With $x_i := \gamma y_i + y_i'$, $\gamma \in L$ one gets $f(e_i + x_i, e_i + x_i) = a_i\alpha + \gamma + \gamma^*$. Since we consider only trace-valued forms, we have $a_i\alpha$ and $a_i'\alpha \in \{\lambda + \lambda^* : \lambda \in L\}$, and the equation $a_i\alpha = a_i'\alpha + \gamma + \gamma^*$ can be solved.

Since in general we cannot decide which forms can be realized, we are only able to give an upper limit for the number of orbits. In the following we consider some special fields. The invariants of a form f we use for classification are the dimension $\dim(f)$, the *determinant* $\det(f)$, the *signature* $\operatorname{sig}(f)$ and the *Hasse-invariant* $s(f)$. Recall that $s(f)$ denotes the equivalence class of the Hasse-algebra $S(f) := \otimes_{i,j} (\alpha_i, \alpha_j)$, where $f \sim [a_1, \ldots, a_n]$. Here (α_i, α_j) denotes the quaternion algebra with standard basis $\{1, \eta, \xi, \eta\xi\}$ such that $\eta^2 = \alpha_i$ and $\xi^2 = \alpha_j$.

Let K be a field with characteristic unequal **2**. If every 3-dimensional symmetric bilinear space over K is isotropic, the forms over K are completely classified by their dimension and determinant [33, **p.** 38]. If every 5-dimensional symrnetric bilinear space over K is isotropic, the forms over K are completely classified by their dimension, determinant and Hasse-invariant [33, p. 91].

For a quadratic form q in characteristic 2 we use two other invariants. These are the *Clifford-invariant* $c(q)$ which is the equivalence class of the Clifford-algebra $C(q)$ in the Brauergroup [33, p. 333] and the *Arf-invariant* $\Delta(q)$ [33, p. 340].

Let $h : V \times V \longrightarrow K$ be a hermitian form over a field K with nontrivial involution $*$. Let $k := K^+$. Then $K = k(\eta)$, where $k(\eta)|k$ is a separable quadratic extension. Let $\eta^2 =: a \in k$ if char $k \neq 2$ and $\eta^2 + \eta =: a \in k$ if char $k = 2$. By defining $q_h(x) := h(x,x)$ we get canonically a quadratic form $q_h : V \longrightarrow k$. A theorem of Jacobson [26, p. 115] says:

(1) A hermitian form h over K is isotropic iff q_h is isotropic over k.

(2) Two hermitian forms h_1 and h_2 are isometric over K iff q_{h_1} and q_{h_2} are isometric over k.

The following relations hold for the invariants of h and q_h [23, p. 261 ff], [33, p. 350]:

$$\begin{aligned}
\dim(q_h) &= 2\dim(h), \\
\det(q_h) &= (-a)^{\dim(h)}, \\
s(q_h) &= (-a, \det(h)), \\
\mathrm{sig}(q_h) &= 2\mathrm{sig}(h). \\
c(q_h) &= (a, \det(h)) \qquad \textit{and} \\
\Delta(q_h) &= \dim(h)a.
\end{aligned}$$

Above we have shown that $\tilde{f}$ is either a symmetric bilinear form over K or a hermitian form over $K|k$. We keep the notation $K^+ = k$. For simplicity let $K := k$ if f is symmetric.

k is quadratically closed, char $k \neq 2$. Here f must be symmetric, since a quadratically closed field cannot have a quadratic extension field. Every 2-dimensional space is isotropic, thus $f \sim [1]$ and $\mathrm{tp}_f(W) = (m,s,t1)$. Hence there are N_d orbits if $r = 0$ and N_{d-1} orbits if $r = 1$ and we have

$$R_d \leq N_d + N_{d-1}.$$

For example, quadratically closed fields k occur if L is the real quaternion skew field $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}ij$, $k = \mathbb{C}$ and $*$ the involution which fixes i and j.

k is real closed, char $k \neq 2$. Recall that a field is called ***real*** if -1 is not a sum of squares. A real field is called *real closed* if no proper algebraic extension field is real. If $\tilde{f}$ is symmetric, every indefinite form with $\dim(\tilde{f}) \geq 2$ is isotropic. Hence $\tilde{f} \sim [1,\ldots,1]$ or $\tilde{f} \sim [-1,\ldots,-1]$. We get

$$R_d \leq N_d + 2\sum_{r=1}^{d} N_{d-r}.$$

If $\tilde{f}$ is non symmetric, K is algebraically closed and every indefinite form is isotropic. Thus $\tilde{f} \sim [1,\ldots,1]$ or $\tilde{f} \sim [-1,\ldots,-1]$. Since without loss we can take $a = -1$, we have $q_{\tilde{f}} \sim [1,\ldots,1]$ or $q_{\tilde{f}} \sim [-1,\ldots,-1]$

k is a finite field, char $k \neq 2$. Every 3-dimensional quadratic form over a finite field is isotropic and there are exactly two non isometric anisotropic quadratic forms [33, p. 39]. Hence, if $\tilde{f}$ is symmetric, we have exactly two anisotropic forms for $r = 1$ and $r = 2$. Th

$$R_d \leq N_d + 2N_{d-1} + 2N_{d-2}.$$

Let now $\tilde{f}$ be non symmetric. Since every 3-dimensional form $q_{\tilde{f}}$ is isotropic, W_A contains an isotropic vector if $r > 1$. Hence $r \leq 1$ and since the dimension is the only invariant in this case, we get

$$R_d \leq N_d + N_{d-1}.$$

k is a local field, char $\neq 2$. By a local field we mean a finite extension of the p-adic numbers $\mathbb{Q}_p$ or the field of Laurent series $\mathbb{F}_q((X))$. Let $\bar{k}$ denote the residue class field of k. Recall that for the number g of square classes of k holds $g = 4$ if char $\bar{k} \neq 2$ and $g = 2^{[k:\mathbb{Q}_2]}$ if char $\bar{k} = 2$. Here k is a finite extension of $\mathbb{Q}_2$ [33, p. 217]. Moreover, there is up io isomorphy only one non split quaternion algebra over k. This allows us to replace the Hasse-invariant by the Hasse-symbol, which we also denote by s. If f is a symmetric bilinear form with diagonal representation $[\alpha_1, \ldots, \alpha_n]$, the Hasse-symbol is the product $s(f) := \prod_{i<j} s(\alpha_i, \alpha_j)$, where $s(\alpha, \beta)$ is the Hilbert-symbol defined by

$$s(\alpha,\beta) = \begin{cases} 1 & \text{if} \quad (\alpha,\beta) \text{ splits} \\ -1 & \text{if} \quad (\alpha,\beta) \text{ not splits.} \end{cases}$$

Every 5-dimensional form over a local field is isotropic and there is up to isometry exactly one anisotropic 4-dimensional form [33, p. 217]. Therefore, forms over local fields can be classified by their diinension, determinant and Hasse-symbol. Any combination of these three invariants is possible except when dim $= 1$ or dim $= 2$ and det $\in -\dot{k}^2$. Then $s = (\text{det}, -1)$ [25, p. 171]. Let $\tilde{f}$ be symmetric. Since any 5-dimensional form is isotropic, we have $r \leq 4$. If $r = 4$, there is only one anisotropic form. If $r = 3$, the invariants det and s are independent. Since $s = \pm 1$, the number of orbits is $\leq 2gN_{d-3}$ for $r = 3$. If $r = 2$, we have $s = (\text{det}, -1)$ if det $\in -k^2$. Hence the number of orbits is $\leq (2g-1)\, N_{d-2}$ for $r = 2$. If $r = 1$, we have $s = (\text{det}, -1)$. Thus here the number of orbits is $\leq gN_{d-1}$. For the number of all orhits we get

$$R_d \leq N_d + gN_{d-1} + (2g-1)N_{d-2} + 2gN_{d-3} + N_{d-4}.$$

If $\tilde{f}$ is non symmetric, every 5-dimensional form $q_{\tilde{f}}$ is isotropic. Hence $r \leq 2$. The only invariants are $\dim(\tilde{f})$ and $\det(\tilde{f})$, since $s(q_{\tilde{f}}) = (-a, \det(\tilde{f}))$. For non symmetric forms the determinant is an element of $k/N_{K|k}(K)$. Since for local fields $\dot{k}/N_{K|k}(\dot{K}) \cong \text{Gal}(K|k)$ [24, p 315], we have $|\dot{k}/N_{K|k}(\dot{K})| = 2$ and hence we get

$$R_d \leq N_d + 2\,(N_{d-1} + N_{d-2}).$$

k is a global field, char $k \neq 2$. **A** global field is a finite extension of $\mathbb{Q}$ or a finite extension of $\mathbb{F}_q(X)$. The completions of a global field at all discrete valuations are the local fields. Symmetric bilinear forms over global fields are isometric iff the corresponding forms at each valuation are isometric. This is the famous local-global principle of Hasse and Minkowski [33, p. 223]. If $\tilde{f}$ is symmetric, the invariants are the dimension, the determinant, the Hasse-symbols at each non archimedean valuation and the signatures at each archimedean valuation. If $\tilde{f}$ is non symmetric, the Hasse-symbols are completely determined by the determinant. Since the number of square classes is not finite in general, the number of orbits does not have to be finite either. We give an example in which infinitely many orbits occur:

Consider the quaternion algebra $L = (-1,-1)_{\mathbb{Q}}$. L is a skew field with center $Z = \mathbb{Q}$. Let $\{1, \eta, \xi, \eta\xi\}$ be the corresponding standard basis and let $K := \mathbb{Q}(\eta)$. By **(4)** we have

$S^+ = \mathbb{Q}\xi$. Let (V,f) be a 2-dimensional hermitian vector space over L and let $f \sim [1,1]$ for a L-basis $\{e_1, e_2\}$. Let $\alpha, \beta \in L$ with $\alpha = a_1 + a_2\eta + a_3\xi + a_4\eta\xi$ and $\beta = b_1 + b_2\eta + b_3\xi + b_4\eta\xi$, $a_i, b_i \in \mathbb{Q}$ $(i = 1, \ldots, 4)$. For $V \ni w := \alpha e_1 + \beta e_2$ holds $f(w,w) = \alpha\alpha^* + \beta\beta^*$. Since for all $\lambda \in L$, $\lambda = x_1 + x_2\eta + x_3\xi + x_4\eta\xi$, holds $\lambda\lambda^* = x_1^2 - x_2^2 - x_3^2 + x_4^2 + 2\,(x_1x_2 - x_3x_4)\,\eta + 2\,(x_1x_3 + x_2x_4)\xi$, we get

$$f(w,w) = 2(a_1^2 + a_2^2 + b_1^2 + b_2^2)\xi,$$

if we choose $a_1 = a_3$, $a_2 = a_4$, $b_1 = b_3$ and $b_2 = b_4$. Thus $f(w,w) \in S^+$ for any a_1, a_2, b_1, b_2. We show that there are infinitely many orbits of such subspaces.

By a theorem of Hilbert and Siegel [31] every positive element of $\mathbb{Q}$ can be written as a sum of four squares. Pick two primes $p, q \equiv 3 (\mathrm{mod}\, 4)$. There exist infinitely many primes of this kind. Choose w and $w' := (a_1' + a_2'\eta + a_2'\eta\xi)e_1 + (b_1' + b_2'\eta + b_1'\eta + b_2'\eta\xi)\,e_2$ such that

$$f(w,w) = 2p\xi \qquad \text{and} \qquad f(w',w') = 2q\xi.$$

If the K-subspaces Kw and Kw' are isometric, we must have $f(w,w) = f(cw',cw')$ for some $c \in K$. Now we show that this is impossible.

Suppose there is such a c. Then $2p\xi = c(2q\xi)c^*$, hence

$$p = N_{K|\mathbb{Q}}(c)q \qquad \text{or} \qquad p = (c_1^2 + c_2^2)q$$

if $c = c_1 + c_2\eta$, $c_1, c_2 \in \mathbb{Q}$. Both c_1 and c_2 must be different from zero. For instance if $c_2 = 0$, follows

$$ps^2 = qr^2$$

if $c_1 = \frac{r}{s}$, $r, s \in \mathbb{Q}$, $(r,s) = 1$. Thus $q|s$ and hence $s = m_1q^{n_1}$, $(m_1, q) = 1$, $n_1 \neq 0$ and $r = m_2p^{n_2}$, $(m_2, p) = 1$, $n_2 \neq 0$. Thus we have

$$m_1^2q^{2n_1-1} = m_2^2p^{2n_2-1}.$$

This yields $q|m_2$, hence $q|r$ which contradicts $(r,s) = 1$.

Without loss let c_1 and c_2 be positive. Let $c_i = \frac{r_i}{s_i}$ with $r_i, s_i \in \mathbb{N}$ and $(r_i, s_i) = 1$, $i = 1, 2$. We get

$$\frac{p}{q} = \frac{r_1^2}{s_1^2} + \frac{r_2^2}{s_2^2} = \frac{(r_1s_2)^2 + (r_2s_1)^2}{(s_1s_2)^2}$$

This yields $q|s_1s_2$, hence $s_1 = m_1q^{n_1}$ and $s_2 = m_2q^{n_2}$ with $(m_i, q) = 1$ and $n_i > 0$ for at least one i. Now we have

$$\frac{p}{q} = \frac{r_1^2m_2^2q^{2n_2} + r_2^2m_1^2q^{2n_1}}{m_1^2m_2^2q^{2(n_1+n_2)}},$$

or equivalently

$$pm_1^2m_2^2q^{2(n_1+n_2)-1} = r_1^2m_2^2q^{2n_2} + r_2^2m_1^2q^{2n_1}. \tag{1}$$

If $n_1 \neq n_2$, $n_1 < n_2$, say, multiplication of (1) by q^{-2n_1} yields $q|r_2^2m_1^2$. This contradicts $(r_2, m_1) = 1$. Let now be $n_1 = n_2 =: n$. Thus (1) becomes

$$pm_1^2m_2^2q^{2n-1} = r_1^2m_2^2 + r_2^2m_1^2 \tag{2}$$

This is equivalent to
$$m_1^2\left(pm_2^2q^{2n-1}-r_2^2\right)=r_1^2m_2^2.$$

Since $(m_1,r_1)=1$, we have $m_1|m_2$. In the same way we get $m_2|m_1$, hence $m_1=m_2=:m$. Finally we have the equation
$$pm^2q^{2r-1}=r_1^2+r_2^2,$$

which contradicts the following theorem from elementary number theory: a natural number n is a sum of two squares in $\mathbb{N}$ iff in the decomposition of n the exponent of each prime p with $p\equiv 3(\mathrm{mod}4)$ is even [31].

Fields of characteristic 2. First let us exclude the cases **(A)** and (B). Hence $p=0$ and f is always non symmetric. Recall that a field F in characteristic 2 is called perfect if $\dot{F}=\dot{F}^2$. Regular quadratic forms over perfect fields are completely classified by the dimension and the Arf-invariant [33, p. 342]. In particular finite fields are perfect. If $[F:F^2]=2$, $\dim(q)$, $\Delta(q)$ and $c(q)$ are a complete set of invariants for a regular quadratic form 4. $[F:F^2]=2$ holds for algebraic function fields and for local fields [1, p. 167].

k is a finite field, char $=2$. If k is finite, $r\leq 1$ must hold, since we have, as in odd characteristics, $\dim(q_{\tilde f})=2\dim(\tilde f)$ and $q_{\tilde f}$ is isotropic if $\dim(q_{\tilde f})\geq 3$. Since the Arf-invariant $\Delta(q_{\tilde f})$ is completely determined by $\dim(\tilde f)$, we get
$$R_d\leq N_d+N_{d-1}.$$

Local and *global fields,* char $=2$. Local and global fields in characteristic 2 are C_2-fields. Recall that a field is called C_i-field if for every homogeneous polynomial P of degree d in $n>d^i$ variables the equation $P(X_1,\ldots,X_n)=0$ has a nontrivial solution. Therefore, 5-dimensional forms are isotropic [1, p. 164], hence $r\leq 2$. The invariants of $q_{\tilde f}$ are $\dim(q_{\tilde f})$, $\Delta(q_{\tilde f})$ and $c(q_{\tilde f})$. These are completely determined by $\dim(\tilde f)$ and $\det(\tilde f)$. Since $\det(\tilde f)\in \dot{k}/N_{K|k}(\dot K)$ and $|\dot{k}/N_{K|k}(\dot K)|=2$, it holds that
$$R_d\leq N_d+2\left(N_{d-1}+N_{d-2}\right)$$

Now we tum to the cases **(A)** and (B). Only here $p\neq 0$ is possible and f is always a non trace-valued symmetric form.

Finite fields, ***(A)*** and *(B).* If K is finite, it is a perfect field. Then every equation of the form $cX^2+dY^2=0$ has a nontrivial solution over K. Thus $p+r\leq 1$ and $f\sim[1]$ if $p+r=1$. We have N_d orbits if $p=r=0$, N_{d-1} orbits if $r=1$ and N_{d-2} orbits if $p=1$. For the number of ali orbits we get
$$R_d\leq N_d+N_{d-1}+N_{d-2}.$$

Local and *global fields,* ***(A)*** and *(B).* Since every 5-dimensional form is isotropic, we have $p+r\leq 4$. For local and global fields in characteristic 2 it holds that $[K:K^2]=2$. **A** simple calculation shows that $|\dot K/\dot K^2|=\infty$. Thus infinitely many orbits are possible. This we want to illustrate by the following example: Let $V:=\langle e_1,e_2\rangle_L$ and $f\sim[1,k]$ with $k\in K\backslash K^2$. Let $W:=W(p,q):=\langle pe_1+qe_2\rangle_K$, $p,q\in K$. Two subspaces W and $W':=W(p',q')$ are in the same orbit iff the corresponding forms $\tilde f$ and $\tilde f'$ lie in the same square-class of K. But the forms f can assume values in every square-class of K, since $f(pe_1+qe_2,\ pe_1+qe_2)=p^2+kq^2$ and $K=K^2\oplus kK^2$.

5 The Hermitean Case: Cubic Extensions

In this section we consider the case $[L:K]=3$. If L is commutative, the involution $*$ is either the identity or it holds that $K \not\subset L^+$, hence $[K:K^+]=2$. Recall that a central simple algebra has exponent 2 iff it admits an involution of the first kind [32, p. 232]. Thus if L is non commutative, $*$ must be a unitary involution. In both the commutative and the non commutative case it suffices to consider 1-hermitian forms.

If L is commutative, there is $\eta \in L^+ \setminus K$. Thus $\{1,\eta,\eta^2\}$ is a K-basis of L with $\eta^* = \eta$. In the non commutative case one can find a K-basis $\{1,\eta,\eta^2\}$ of L such that $\eta \in N_L(\dot{K})$ and $\eta^3 = a \in Z$. Since $N_L(\dot{K}) = K \oplus K\eta \oplus K\eta^2$ and η^* is contained in the normalizer, too, we have $\eta^* = c\eta$ or $\eta^* = d\eta^2$, $c,d \in K$. For this whole section $\{1,\eta,\eta^2\}$ shall be a K-basis of L as above. Furthermore we fix $\alpha,\beta \in L$ such that $\ker\varphi = \langle\alpha,\beta\rangle_K$. Observe that $\ker\varphi$ is both a left- and right vector space, since $\varphi: L \longrightarrow K$ is two-sided K-linear. Moreover, we have $\ker\varphi = (\ker\varphi)^*$.

5.1 Lemmas and proof of Theorem 5

The next lemma shows that in the non commutative case not every selection of α and β is possible.

Lemma 39 *If L is non commutative, then* $\ker\varphi = \langle 1,\eta\rangle_K$, $\ker\varphi = \langle 1,\eta^2\rangle_K$ *or* $\ker\varphi = \langle \eta,\eta^2\rangle_K$.

Proof. Since $\ker\varphi = \langle\alpha,\beta\rangle_K$ is both a K-right vector space and a K-left vector space for all $k \in K$, $\alpha k = p\alpha + q\beta$ and $\beta k = r\alpha + s\beta$, $p,q,r,s \in K$. Without loss we can assume that $\alpha = 1$, η, η^2, $1+b\eta$ or $1+c\eta^2$ for some $b,c \in K$.

Let $\beta := x + y\eta + z\eta^2$. Then $\beta k = kx + k^\sigma y\eta + zk^{\sigma^2}\eta^2 = r\alpha + sx + sy\eta + sz\eta^2$. Let $\alpha = 1$. Without loss we can suppose that $x=0$. If $y=0$ or $z=0$, then $\beta \in \{\eta,\eta^2\}$. So we take $y \neq 0$ and $z \neq 0$. Comparing coefficients yields $yk'' = sy$ and $zk^{\sigma^2} = sz$. But this is only possible for $k \in Z$. In the same way follow the assertions for $\alpha = \eta$ and $\alpha = \eta^2$.

If $\alpha = 1 + b\eta$ without loss we can suppose that $z \neq 0$. If $\alpha k = p\alpha + q\beta$, then $q = 0$. Thus $k + bk^\sigma\eta = p + pb\eta$. This yields $p = k$ and $k \in Z$. If $\alpha = 1 + c\eta^2$, we get in the same way that $k \in Z$. $\square$

We shall apply frequently the following simple lemma:

Lemma 40 *Let W be a f'-singular subspace of V and let $x \in W$. Then $x^\perp$ meets every 3-dimensional K-subspace of W non trivially.*

We now consider the special subspaces mentioned in the introduction.

Lemma 41 *Let W be a f'-singular subspace. Then* $\mathrm{comp}_L W \oplus_L W_2$ *is f-singular.*

Proof. It suffices to proof this lemma for vectors contained in W_2, since $\mathrm{comp}_L W \subset \mathrm{rad}_f W$. By 19 we have $W_2 = \bigoplus_{i=1}^{r} \langle e_i, \eta e_i\rangle_K$. We show that the inner products of the basis vectors vanish. Let $\lambda := f(e_i,e_j)$. Then $\lambda, \eta\lambda, \lambda\eta^*$ and $\eta\lambda\eta^*$ are contained in $\ker\varphi$. If $\lambda \neq 0$, we have $\ker\varphi = \langle\lambda, \eta\lambda\rangle_K =: U$. The space U is both a left- and a right-vector space over L. Since $\lambda\eta^*$ and $\eta\lambda\eta^* \in U$, the space U is invariant under multiplication by $(\eta^*)^2$ from the right, too. Let $0 \neq u \in U$ and $\mu = p + q\eta^* + r(\eta^*)^2$. Then $u\mu = up + uq\eta^* + ur(\eta^*)^2$. Since every summand lies in U, also $u\mu \in U$. Note that $\{1, \eta^*, (\eta^*)^2\}$ is a K-basis of L, too. Thus $u\mu \in U$ for all $\mu \in L$. This yields the contradiction $uL \subset U$. □

Lemma 42 *Let* W *be a f'-singular simple triangular subspace and* $x \in W$. *Let* $T \subset$ W *be* a *unique simple triangular subspace containing* x *and* $Y \subset$ W *an arbitrary simple triangular subspace. Then* $x \perp Y$ *iff* $T \perp Y$.

Proof. Let $T = \langle x, y, \eta x + \eta^2 y\rangle_K$. Since by 40 $y^\perp \cap Y \neq 0$, there are vectors u, v such that $Y = \langle u, v, \eta u + \eta^2 v\rangle_K$ and $f(y,u) = 0$. Let $\rho := f(y,v)$. In $\ker\varphi$ are contained: ρ, $f(y, \eta u + \eta^2 v) = \rho(\eta^2)^*$, $f(\eta x + \eta^2 y, v) = \eta^2\rho$ and $f(\eta x + \eta^2 y, \eta u + \eta^2 v) = \eta^2\rho(\eta^2)^*$. If $\rho \neq 0$, ρ and $\eta^2\rho$ form a basis of $\ker\varphi$. Then $\ker\varphi$ is invariant under multiplication by $(\eta^2)^* = (\eta^*)^2$ and $\eta^* = \frac{1}{a^*}(\eta^*)^2(\eta^*)^2$ from the right. Like in 41 we get $\rho = 0$. □

Lemma 43 *Let* $T := \langle x, y, \eta x + \eta^2 y\rangle_K$ *be a f'-singular simple triangular subspace.*

(1) *If* $f(x,x) = f(y,y) = 0$, *then* $f(x,y) = 0$.

(2) *If* $f(x,y) = f(y,y) = 0$, *then* $f(x,x) = 0$.

(3) *If* $f(x,y) = f(x,x) = 0$, *then* $f(y,y) = 0$.

Proof. (1) Let $\lambda := f(x,y)$. In $\ker\varphi$ are contained: λ, $f(x, \eta x + \eta^2 y) = \lambda(\eta^*)^2$, $f(y, \eta x + \eta^2 y) = \lambda^*\eta^*$ and $f(\eta x + \eta^2 y, \eta x + \eta^2 y) = \eta\lambda(\eta^*)^2 + \eta^2\lambda^*\eta^*$. Note that if $\gamma \in \ker\varphi$, then $\gamma^* \in \ker\varphi$. If L is commutative, we have $\eta \in L^+$. Hence $\lambda, \eta\lambda$ and $\eta^2\lambda \in \ker\varphi$. This yields $\lambda = 0$.

Suppose now L is non commutative. Here $\ker\varphi = \langle 1, \eta\rangle_K$, $\ker\varphi = \langle 1, \eta^2\rangle_K$ or $\ker\varphi = \langle \eta, \eta^2\rangle_K$. Suppose $\ker\varphi = \langle 1, \eta\rangle_K$. Then $\lambda = p + q\eta$, $p, q \in K$, hence $\eta\lambda = p^\sigma\eta + q^\sigma\eta^2 \in \ker\varphi$. This yields $q = 0$ and $\lambda \in K$. Recall that $\eta^* = c\eta$ or $\eta^* = d\eta^2$. We get $\ker\varphi \ni \lambda(\eta^*)^2 = \lambda cc^\sigma\eta^2$ or $\ker\varphi \ni (\eta\lambda)^* = \lambda^* d\eta^2$. This implies $\lambda = 0$. The other cases are proved similar.

(2) Let $\lambda := f(x,x)$. Then $\lambda, \eta\lambda$ and $\eta\lambda\eta^* \in \ker\varphi$. If L is commutative, $\eta\lambda\eta^* = \eta^2\lambda$, hence $\lambda = 0$. The non commutative case works as in (1).

(3) like (2). □

Corollary 44 A *f-degenerate f'-singular simple triangular subspace T is also f-singular.*

Proof of Theorem 5. In the proof we need that both $\{1, \alpha\beta^{-1}, \beta\alpha^{-1}\}$ and $\{1, \beta^{-1}\alpha, \alpha^{-1}\beta\}$ is a K-left basis of L. This is clear for the commutative case. For the non commutative case by 39 we can choose α and β such that this is true. Note that in characteristic 2 a trace-valued symmetric bilinear form is symplectic. Thus we can assume that char K $\neq 2$ when $* = \mathrm{id}$.

(a) By 28 it holds that $\mathrm{comp}_L W = \mathrm{rad}_f W$ if W is maximal f'-singular.

(b) Without loss we can suppose that $\mathrm{tp}(W) = (0, r, s, t)$. Choose $\tilde{w}_1 \in$ W with $\mathrm{tp}_W(\tilde{w}_1) = 2$. By 41 we have $f(\tilde{w}_1, \tilde{w}_1) = 0$. Since $\mathrm{rad}_f W = 0$, there is $\tilde{w}'_1 \in$ W such that $f(\tilde{w}_1, \tilde{w}'_1) \neq 0$. Choose $w_1 \in L\tilde{w}_1$ such that $f(w_1, \tilde{w}'_1) = 1$. Since $f(Lw_1 \cap W, \tilde{w}'_1) \subset \ker\varphi = K\alpha \oplus K\beta$, follows

$$\mathrm{W} \cap L\tilde{w}_1 = \langle \alpha w_1, \beta w_1\rangle_K.$$

If already $f(\tilde{w}_1', \tilde{w}_1') = 0$, define $w_1' := \tilde{w}_1'$. Otherwise $f(\tilde{w}_1', \tilde{w}_1') = \lambda + h * \lambda \in \dot{L}$. Choose

$$w_1' := -\lambda w_1 + \tilde{w}_1'.$$

Then (w_1, w_1') is a hyperbolic pair. It remains to show that $w_1' \in$ W. If char K $\#$ 2, we have $h = \frac{1}{2} f(\tilde{w}_1', \tilde{w}_1')$. Since $f(\tilde{w}_1', \tilde{w}_1') \in \ker \varphi$, there are $p, q \in K$ such that $\lambda = p\alpha + q\beta$. Hence $w_1' = -(p\alpha + q\beta) w_1 + \tilde{w}_1' \in W$, because $\langle \alpha w_1, \beta w_1 \rangle_K \subset$ W. We now show that in characteristic 2, too, h is contained in $\ker \varphi$. Then follows analogously that $w_1' \in$ W. Without loss let $* \neq$ id. Then there is $\zeta \in K$ (if L is commutative) or $\zeta \in Z$ (if L is non commutative) such that $\zeta^* = \zeta + 1$. Let $f(\tilde{w}_1', \tilde{w}_1') =: \gamma$. Now γ E S^+ and $K\gamma$ C $\ker\varphi$ imply $\zeta\gamma$ E $\ker \varphi$. Defining $\lambda := \zeta\gamma$ we have $\lambda + \lambda^* = \zeta\gamma + \zeta\gamma + \gamma = \gamma$.

Since $\langle \alpha w_1, \beta w_1, w_1' \rangle_K$ is maximal f'-singular in the hyperbolic L-subspace $\langle w_1, w_1' \rangle_L$, 29 yields W $= \langle \alpha w_1, \beta w_1, w_1' \rangle_K \perp_L Y$ for a suitable K-subspace Y of W. Then $\mathrm{tp}(Y) = (0, r - 1, s, t - 1)$ and by induction we get $W = W_{2,4} \perp_L W'$. In particular $r \leq t$.

(c) Without loss let $\mathrm{tp}(W) = (0, 0, s, t')$, where $t' := t - r$. By 42 and 43 follows that $\mathrm{rad}_f W_3$ is triangular. Let $W_3 = \hat{W}_3 \perp_L \mathrm{rad}_f W_3$. Then $\hat{W}_3$ is f-regular and triangular. $\hat{W}_3$ is maximal f'-singular in the f-regular L-subspace $L\hat{W}_3$, since $\dim_K \hat{W}_3 = 3(s - j)$ and $\dim_K L\hat{W}_3 = 6(s - j)$. Thus 29 yields $W = \hat{W}_3 \perp_L Y$ for a suitable K-subspace Y of W. Since $\mathrm{rad}_f W_3 \leq$ Y, there is a subspace W_4' such that Y $= \mathrm{rad}_f W_3 \oplus_L W_4'$. Then W_4' is a K-substructure with $\dim_K W_4' = t'$. If $\mathrm{rad}_f W_3 \perp_L W_4'$, then $\mathrm{rad}_f W_3 \leq \mathrm{rad}_f W = 0$.

Without loss suppose that W_3 is f-regular. We have a decomposition

$$W_3 = \bigoplus_{i=1}^{s} T_i \quad \text{with } T_i := \langle \tilde{x}_i, \tilde{y}_i, \eta \tilde{x}_i + \eta^2 \tilde{y}_i \rangle_K.$$

Let $T_1 \ldots T_k$, say, be f-regular and $T_{k+1} \ldots T_s$ not f-regular. Moreover, choose the decomposition such that k is minimal. By 44 $T_{k+1} \ldots T_s$ are f-singular. For i $= 1, \ldots, k$ holds $T_i = \langle u_i, v_i, \eta u_i + \eta^2 v_i \rangle_K$. By 43 (2) either $f(u_i, u_i)$ $\#$ 0 or $f(v_i, v_i) \neq 0$. Since T_i is maximal f'-singular in the f-regular subspace LT_i, by 29 we get W3 $= T_1 \perp_L \ldots \perp_L T_k \perp_L (\bigoplus_{i=k+1}^{s} Ti)$.

Consider now the f-singular T_i. Let $T := \langle \tilde{x}, \tilde{y}, \eta \tilde{x} + \eta^2 \tilde{y} \rangle_K$ be such a subspace. Choose $\alpha^* x \in T$. Since $\{1, \beta^*(\alpha^*)^{-1}, \alpha^*(\beta^*)^{-1}\}$ is a K-right basis of L ($\{1, \alpha^{-1}\beta, \beta\alpha^{-1}\}$ isleftbasis), by 22 there is exactly one $\beta^* y \in T$ such that

$$T := \langle \alpha^* x, \beta^* y, \beta^* x + \alpha^* y \rangle_K.$$

There is a simple triangular subspace T' such that $f(T, T') \neq 0$, otherwise it would hold that $T \subset \mathrm{rad}_f W_3 = 0$. By 23 we have $LT \cap LT' = 0$. By 40 there exist nonzero vectors x', y' E T' such that $f(y, x') = 0 = f(x, y')$. Hence 42 implies $f(x, x') \neq 0$. Thus

$$\begin{aligned} f(x', \alpha^* x) &= f(x', x)\alpha \quad \mathrm{E} \quad K\alpha \oplus K\beta \qquad \text{and} \\ f(x', \beta^* x + \alpha^* y) &= f(x', x)\beta \quad \mathrm{E} \quad K\alpha \oplus K\beta. \end{aligned}$$

This yields

$$f(x', x) \ \mathrm{E} \ (K \oplus K\beta\alpha^{-1}) \cap (K\alpha\beta^{-1} \oplus K) = K,$$

since $\{1, \beta\alpha^{-1}, \alpha\beta^{-1}\}$ is a K-left basis of L, too. As we consider only 1-hermitian forms we can suppose $f(x, x') = 1 = f(x', x)$ and analogously $f(y, y') = 1 = f(y', y)$. Since x' and y'

are linear independent over L and T' is f-singular, we get a hyperbolic sequence x, x', y, y'. Now $T \oplus_L T'$ is maximal f'-singular in the hyperbolic L-subspace $LT \oplus LT' = \langle x, x', y, y' \rangle_L$. Hence there is a triangular subspace $Y \leq W_3$ such that $W = (T \oplus_L T') \perp_L Y$ with $\mathrm{tp}(Y) = (0, 0, s-k-2, 0)$. By induction we get $W = (T_1 \oplus_L T_1') \perp_L \ldots \perp_L (T_l \oplus_L T_l')$, where $s = k + 2l$.

It remains to normalize T': By 20 (2) T' has a basis of the form $\{x', y', \gamma x' + \delta y'\}$, where $\{1, \gamma, \delta\}$ is a K-right basis of L. Then

$$\begin{array}{rclcll} f(\gamma x' + \delta y', \alpha^* x) & = & \gamma\alpha \in K\alpha \oplus K\beta & \Longrightarrow & \gamma \in K \oplus K\beta\alpha^{-1} & \text{and} \\ f(\gamma x' + \delta y', \beta^* y) & = & \delta\beta \in K\alpha \oplus K\beta & \Longrightarrow & \delta \in K \oplus K\alpha\beta^{-1}. & \end{array}$$

Thus for suitable $p, q, r, s \in K$

$$(p + q\beta\alpha^{-1})x' + (r + s\alpha\beta^{-1})y' \in T'.$$

Then there is $c \in K$ such that

$$z' := \beta\alpha^{-1}x' - c\alpha\beta^{-1}y' \in T'.$$

Since x', y' and z' are linear independent over K, $T' = \langle x', y', z' \rangle_K$. We get

$$T \oplus_L T' = \langle \alpha^* x, \beta^* y, \beta^* x + \alpha^* y \rangle_K \oplus_L \langle x', y', \beta\alpha^{-1}x' - c\alpha\beta^{-1}y' \rangle_K.$$

Now

$$f(z', \beta^* x + \alpha^* y) = \beta\alpha^{-1}\beta - c\alpha\beta^{-1}\alpha \in K\alpha \oplus K\beta.$$

With $\lambda := \beta\alpha^{-1}$ and multiplication by $\alpha^{-1}\lambda$ from the right we get

$$\lambda^3 - c \in K\lambda \oplus K\lambda^2.$$

Hence there are $b, d \in K$ such that

$$\lambda^3 - c = b\lambda + d\lambda^2$$

If L is commutative, then $c = N_{L|K}(\beta\alpha^{-1})$, since the minimal polynomial of λ has degree 3. If L is non commutative, we can assume without loss that $\{\alpha, \beta\} \in \{\{1, \eta\}, \{1, \eta^2\}, \{\eta, \eta^2\}\}$. Then $\beta\alpha^{-1} = \eta$ or $\beta\alpha^{-1} = \eta^2$ respectively. Thus we have $c = a$ or $c = a^2$.

(d) follows from (b). □

5.2 Orbits of f'-singular subspaces

We define the f-type $\mathrm{tp}_f(W)$ of W in the cubic case to be the 6-tuple

$$\mathrm{tp}_f(W) := (m, r, k, l, j, t-r).$$

Let $W = W_1 \oplus_L W_2 \oplus_L W_3 \oplus_L W_4$ and $\hat{W} = \hat{W}_1 \oplus_L \hat{W}_2 \oplus_L \hat{W}_3 \oplus_L \hat{W}_4$ be in the same orbit under $U(V, f)$. Then there is an isometry $\tau : \hat{W} \mapsto W$. Since τ preserves the GL-type, we have by Theorem 2 that $(\hat{W}_1 \oplus_L \hat{W}_2 \oplus_L \hat{W}_3)_\tau = W_1 \oplus_L W_2 \oplus_L W_3$. By 41 the subspaces $W_1 \oplus_L W_2$ and $\hat{W}_1 \oplus_L \hat{W}_2$ are f-singular. Thus by Witt's cancelation theorem $L\hat{W}_3$ and LW_3 are isometric. This yields $\mathrm{rad}_f W_3 = \mathrm{rad}_f \hat{W}_3$. From this follows that if two subspaces are in the same orbit under

$\mathrm{U}(V,f)$, they must have the same f -type. Like in the quadratic case this condition is not sufficient. Moreover there must be an isometry $W_3'\perp_L(\mathrm{rad}_f W_3 \oplus_L W_4') \longrightarrow \hat{W}_3'\perp_L(\mathrm{rad}_f \hat{W}_3 \oplus_L \hat{W}_4')$.

We now discuss the question when two subspaces of the same type are in the same orbit. Let $U := W_3'\perp_L(\mathrm{rad}_f W_3 \oplus_L W_4')$ and $\hat{U} := \hat{W}_3'\perp_L(\mathrm{rad}_f \hat{W}_3 \oplus_L \hat{W}_4')$. Since U is f'-singular, $f|_U \in K\alpha \oplus K\beta$. Define f_α to be the α-component and f_β to be the β-component of $f|_U$, that is $f_\alpha(u,v) = p$ and $f_\beta(u,v) = q$ if $f(u,v) = p\alpha + q\beta$. First must be shown whether $f_\alpha, f_\beta : U \times U \longrightarrow$ K are hermitian forms at all. We have to show that there exist involutions $\dagger$ and $\ddagger$ on K such that $f_\alpha(u,kv) = f_\alpha(u,v)k^\dagger$ and $f_\beta(u,kv) = f_\beta(u,v)k^\ddagger$ for all $u,v \in U$ and $k \in$ K. We consider only f_α, since it will be the same for f_β. Let $f(u,v) = p\alpha + q\beta$. Then $f_\alpha(u,kv) = p\alpha k^* = p(\alpha k^*\alpha^{-1})\alpha$. Therefore we must show whether ihe map $k \longmapsto \alpha k^*\alpha^{-1}$ is an involution on K. Obviously this is the case when L is commutative. Thus here we have $\dagger = \ddagger = *|_K$. The same is true in the non commutative case if $a = 1$. If $a \neq 1$ we can assume $a = \eta$ or $a = \eta^2$, hence qk* $= k^{*\circ\sigma}\eta$ and $\eta^2 k^* = k^{*\circ\sigma^2}\eta^2$. There are the cases $\eta^* = c\eta$ and $\eta^* = d\eta^2$, $c,d \in$ K. In the first case we have $(k^\sigma)^* = (k^*)^{\sigma^2}$ and $(k^{\sigma^2})^* = (k^*)^\sigma$. In the second case it holds that $(k^\sigma)^* = (k^*)^\sigma$ and $(k^{\sigma^2})^* = (k^*)^{\sigma^2}$ for all $k \in K$. Thus for $\eta^* = c\eta$ we have $k^{(*\circ\sigma)^2} = (((k^*)^\sigma)^*)^\sigma = ((k^{\sigma^2})^{*^2})^\sigma = k^{\sigma^3} = k$ and $(kl)^{(*\circ\sigma)} = l^{(*\circ\sigma)}k^{(*\circ\sigma)}$ for all $k,l \in K$. Hence $*\circ\sigma$ is an involution. In the same way one shows that $*\circ\sigma^2$ is an involution. If $\eta^* = d\eta^2$, then $k^{(*\circ\sigma)^2} = ((k^\sigma)^{*^2})^\sigma = k^{\sigma^2}$. Since not for all $k \in$ K holds that $k^{\sigma^2} = k$, $*\circ\sigma$ is no involution. In the same way follows that $*\circ\sigma^2$ is no involution.

Since in the non commutative case not both a and β equal 1, the maps f_α and f_β are only hermitian forms if $\eta^* = c\eta$. In order to proceed as in the quadratic case, we suppose this to be the case. Since U (and $\hat{U}$) in general contains triangular subspaces, we cannot conclude like in the quadratic case that U and $\hat{U}$ are in the same orbit iff f_α and f_β are simultaneously isometric (over K) to $\hat{f}_\alpha$ and $\hat{f}_\beta$. But this is the case if both U and $\hat{U}$ are K-substructures. However, in general the forms f_α and f_β (and $\hat{f}_\alpha$ and $\hat{f}_\beta$) may be isotropic and are not given in diagonal form, since K-substructures in the cubic case in general cannot be diagonalized over K.

If $R(W)\cap\ker\varphi = 0$, then W is f-singular by 30. If this is the case for all subspaces, the number of orbits is finite and independent of the underlying fields. We now consider two further special cases: Let n be odd. Denote by v the Witt-index of (V,f) and by v' the Witt-index of (V,f'). We can assume without loss that $\mathrm{comp}_L W = 0$. If v' is maximal, that is if v' $= 3n/2$, it holds that $W = W^{\perp'}$, since W $\subset W^{\perp'}$ and $\dim_K V = \dim_K W + \dim_K W^{\perp'}$. From $W^\perp \subset W^{\perp'}$ follows $W^\perp \subset \mathrm{comp}_L W$, hence $W^\perp = 0$. Since $V = LW\perp W^\perp$, we have $V =$ LW. The equations

$$\begin{array}{lclcll} \dim_L LW & = & r+2s+t & = & n & \text{and} \\ \dim_K W & = & 2r+3s+t & = & 3n/2, & \end{array}$$

yield $r = t$. Thus $W_4' = 0$ and hence $\mathrm{rad}_f W_3 = 0$. Moreover, if v is maximal, that is v $= n/2$, we have v $= r + 2l$. Since $n/2 = r + s$, follows s $= 2l$, hencc $k = 0$. Then all simple triangular subspaces of W are f-singular. Thus the condition $\mathrm{tp}_f(W) = \mathrm{tp}_f$ (6') is sufficient for W and W to be in the same orbit under $U(V,f)$.

6 The Herinitean Case: Extensions Of Higher Degree

In this section we consider skew field extensions $L|K$ with $[L:K]=s\geq 4$. We show that the number of orbits of f'-singular K-subspaces is infinite provided the Witt-index of (V,f) is greater than zero and K is infinite. To us no counterexample is known when the Witt-index of (V,f) equals zero. Our conjecture is that in this case, too, the number of orbits is always infinite. However, this cannot be proved by the methods used to prove Theorem 6. We give an example that illustrates that there may occur infinitely many orbits when $\mathrm{wi}(V,f)=0$.

Proof of Theorern 6. Let W be a 2-dimensional K-subspace of L. The map $\varphi: L \longrightarrow K$ induces via $\phi: L\times L \longrightarrow K$, $\phi(\alpha,\beta):=\varphi(\alpha\beta^*)$ a regular I-hermitian form over K. Then there is an unique $(s-2)$-dimensional K-subspace W' of L with $\phi(W,W')=0$.

First suppose that the Witt-index of (V,f) equals 1. Let (e,e') be a hyperbolic pair in (V,f). The space $X:=We\oplus W'e'$ is f'-singular and contained in a maximal f'-singular subspace U. Since $\dim_K\langle e,e'\rangle_L = 2\dim_K X$, by 29 follows $U=X\perp_L Y$ for a K-subspace Y of U. The space Y is f-anisotropic, for if Y contained a f-isotropic vector y the space $\langle e,y\rangle_L$ would be f-singular which contradicts the assumption $\mathrm{wi}(V,f)=1$.

Let $\hat{W}$ be another 2-dimensional K-subspace of L and $\hat{W}'$ the unique $(s-2)$-dimensional K-subspace such that $\phi(\hat{W},\hat{W}')=0$. Let $\hat{X}:=\hat{W}e\oplus\hat{W}'e'$. Then $\hat{X}\perp_L Y$ and $\hat{U}:=\hat{X}\perp_L Y$ is maximal f'-singular. We have $LU=$ Li? and if τ is an isometry in (V,f) with $U\tau=\hat{U}$, τ is an isometry in (LU,f), too. Thus we can suppose $V=LU$. Let $\{y_1,\dots,y_r\}$ be a maximal L-independent set in Y. Then $\{e,e',y_1,\dots,y_r\}$ is a L-basis of V. Let

$$\begin{aligned} e\tau &= \alpha e+\beta e' + \sum_{j=1}^{r}\rho_j y_j \in \hat{U} \quad \text{and} \\ e'\tau &= \gamma e+\delta e' + \sum_{j=1}^{r}\sigma_j y_j \in \hat{U} \end{aligned}$$

Now $\alpha=0$ $(\gamma=0)$ implies $\rho_j=0$ $(\sigma_j=0)$ for $j=1,\dots,r$, since Y is f-anisotropic. Not both α and γ are zero, otherwise we would have the contradiction $1=f(e,e')=f(e\tau,e'\tau)=f(\beta e',Fe')=0$.

Since

$$\begin{aligned} (We)\tau &= (W\alpha)e+\dots &= \hat{W}e+\dots \quad \text{and} \\ (W'e')\tau &= (W'\gamma)e+\dots &= \hat{W}e+\dots, \end{aligned}$$

either $W\alpha=\hat{W}$ or $W'\gamma=\hat{W}$, where $\gamma \# 0$ is possible only if $s=4$ for dimensional reasons. Since by Theorem 3 the number of orbits of 2-dimensional K-subspaces is infinite, there are infinitely many orbits of maximal f'-singular K-subspaces.

Suppose now $\mathrm{wi}(V,f)=m>1$ and let $e_1,e_1',\dots,e_m,e_m'$ be a hyperbolic sequence in (V,f). Let $X:=(We_1\oplus W'e_1')\perp_L\dots\perp_L(We_m\oplus W'e_m')$ and $\hat{X}:=(\hat{W}e_1\oplus\hat{W}'e_1')\perp_L\dots\perp_L(\hat{W}e_m\oplus \hat{W}'e_m')$. As above there is a K-subspace Y such that $U:=X\perp_L Y$ and $\hat{U}:=\hat{X}\perp_L Y$ are maximal f'-singular. Let $\tau\in \mathrm{U}(LU,f)$ and let $\{y_1,\dots,y_r\}$ be as above. Let

$$e_1\tau = \sum_{i=1}^{m}(\alpha_i e_i+\beta_i e_i') + \sum_{j=1}^{r}\rho_j y_j \in \hat{U} \qquad \text{and}$$

$$e'_1\tau \quad = \quad \sum_{i=1}^{m}(\gamma_i e_i + \delta_i e'_i) + \sum_{j=1}^{s} \rho_j y_j \in \hat{U}.$$

As above we get that not all α_i and γ_i can vanish and in the same way follows that there are infinitely many orbits. ☐

Note that in the proof we cannot use Witt's cancelation theorem to conclude that the spaces X and $\hat{X}$ are f-isometric, because f is a form over L, but X and $\hat{X}$ are K-subspaces. The following example illustrates that there may occur infinitely many orbits when $\mathrm{wi}(V,f) = 0$:

Let L be an infinite field and char $L \neq 2$. Let $\{1,\eta,\eta^2,\eta^3\}$ be a K-basis of L such that $\ker\varphi = \{1,\eta,\eta^2\}$, $\varphi(\eta^3) = 1$ and $\eta^4 = a \in K$. Let ϕ be as above. Then $\mathrm{wi}(L,\phi) = 2$. This is clear, since the space $\langle 1,\eta\rangle_K$ is ϕ-singular.

Moreover, for all $c \in$ K the space $W(c) := \langle 1+c\eta^2, \eta - c\eta^3\rangle_K$ is ϕ-singular, since

$$\begin{array}{rclcl}
\phi(1+c\eta^2, \mathrm{I}+c\eta^2) & = & \varphi(1+2c\eta^2+c^2\eta^4) & = & 0, \\
@(\mathrm{i}+c\eta^2,\eta - c\eta^3) & = & \varphi(\eta + c\eta^3 - c\eta^3 + c^2\eta^5) & = & 0 \quad \text{and} \\
\phi(\eta - c\eta^3, \eta - c\eta^3) & = & \varphi(\eta^2 - 2c\eta^4 + c^2\eta^6) & = & 0.
\end{array}$$

We now show that there is an infinite sequence $(c_i)_{i\in I}$ in K such that $W(c_i)$ and $W(c_j)$ are not in the same orbit under $\dot{L}$ if $i \# j$. We need the following easy lemma, see [28, p. 129]:

Lemma 45 ***Let*** K *be an infinite field and $P(X,Y)$ a nonzero polynomial over* K. ***Then there*** *is an infinite sequence $(a_i)_{i\in I}$ in* K *such that for all* $\mathrm{i}, j \in I$ *holds $P(a_i,a_j) \neq 0$ if $i \neq j$.*

We suppose that the spaces $W(c)$ and $W(d)$ are in the same orbit under $\dot{L}$. Then there are $\lambda \in L$ and $p,q,r,s \in K$ such that

$$\begin{array}{rcl}
(1+c\eta^2)\lambda & = & p(1+d\eta^2)+q(\eta - d\eta^3) \quad \text{and} \\
(\eta - c\eta^3)\lambda & = & r(1+d\eta^2)+s(\eta - d\eta^3).
\end{array}$$

Elimination of λ yields

$$\begin{array}{rcl}
0 & = & p(\eta + d\eta^3 - c\eta^3 - acd\eta) + q(\eta^2 - ad - ac + acd\eta^2) \\
 & - & r(\mathrm{i} + d\eta^2 + c\eta^2 + acd) - s(\eta + c\eta^3 - d\eta^3 - acd\eta).
\end{array}$$

We get the four equations

$$qa(c+d)+r(1+acd) \quad = \quad 0, \tag{1}$$
$$p(1-acd)-s(1-ncd) \quad = \quad 0, \tag{2}$$
$$q(1+acd)-r(c+d) \quad = \quad 0: \tag{3}$$
$$p(d-c)+s(d--c) \quad = \quad 0. \tag{4}$$

First suppose $q = r = 0$. Then either p or s must be different from zero. Since d # c, froin (4) follows that $s = -p$. Then (2) yields

$$2p(1-acd) = 0.$$

By 45 there exists an infinite sequence (c_i) such that $i - ac_ic_j \neq 0$. Hence there are infinitely many orbits.

Now let $q = 0$ and $r \neq 0$. Then (1) and (3) yield $1 - ac^2 = 0$. Hence $a = \frac{1}{c^2}$. Since $\eta^4 = a$, we have $\eta^2 = \pm\frac{1}{c}$. This is a contradiction, since $\eta^2 \notin K$. For $q \neq 0$ and $r = 0$ follows in the same way that η^2 E K.

It remains to consider the case $q \neq 0$ and $r \neq 0$. Now $1 + acd = 0$ iff $c + d = 0$. In this case we get as above the contradiction $\eta^2 \in K$. Thus both $1 + acd$ and $c + d$ are different from zero. From (1) and (3) follows

$$a(c+d)^2 + (1+acd)^2 = 0.$$

By 45 we get that there are infinitely many orbits.

Now let $\{e_1, \ldots, e_n\}$ be a L-basis of V and let $f \sim [i, \ldots, 1]$. Then all subspaces of the form

$$U(c) := \perp_{i=1}^{n} W(c)e_i$$

are maximal f'-singular. If $U(c)$ is in the same orbit as $U(d)$, there is λ E $\dot{L}$ such that $W(c)\lambda = W(d)$. Since there is an infinite sequence (c_i) such that $W(c_i)\lambda \neq W(c_j)$ for all $\lambda \in \dot{L}$ if $i \neq j$, there are infinitely many orbits.

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