Note di Matematica Vol. 18 - n. 2, 227-234 (1998)

# THE SUBMANIFOLDS $X_m$ OF THE MANIFOLD $*g-MEX_n$ II. FUNDAMENTAL EQUATIONS ON $X_m$ OF $*g-MEX_n$

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**Abstract.** *In our previous paper [4], we studied the induced connection of the $^*g$-ME-connection on a submanifold $X_m$ embedded in a manifold $^*g$-$MEX_n$ together with the generalized coefficients $\Omega_{ij}$ of the second fundamental form of $X_m$, with emphasis on the proof of a necessary and sufficient condition for the induced connection of $X_m$ in $^*g-MEX_n$ to be a $^*g$-ME-connection. This paper is a direct continuation of [4]. In this paper, we derive the generalized fundamental equations on $X_m$ of $^*g$-$MEX_n$, such as the generalized Gauss formulae, the generalized Weingarten equations, and the Gauss-Codazzi equations. Furthermore, we also present surveyable tensorial representations of curvature tensors $R^{\mu}_{\omega\mu\lambda}$ of $^*g$-$MEX_n$ and $R^h_{ijk}$ of $X_m$.*

## 1 The generalized fundamental equations on a submanifold $X_m$ of $^*g$-MEX$_n$

This section is a direct continuation of our previous paper [4], which will be denoted by I in the present section. All considerations in this section are based on the results and symbolism of I. Whenever necessary, they will be quoted in the present section.

In this section, we derive the generalized fundamental equations on a submanifold $X_m$ of $^*g$-MEX$_n$, such as the generalized Gauss formulae, the generalized Weingarten equations, and the Gauss-Codazzi equations. Furthermore, in Theorem 8 we also present surveyable tensorial representations of curvature tensors $R^{\nu}_{\omega\mu\lambda}$ of $^*g-MEX_n$ and $R^h_{ijk}$ of $X_m$. The convenient and powerful $C$-nonholonomic frame of reference in $^*g-MEX_n$ at points of $X_m$, introduced in I-Section 1(b), will be employed throughout the present section. Particularly, we note in virtue of Definition I-11 that under the present conditions the $^*g-ME$-connection of a given $^*g-MEX_n$ is unique.

**Theorem 1 (The generalized Gauss formulae on $X_m$ of $^*g$-MEX$_n$)** *At points of $X_m$ of $^*g$-$MEX_n$, the following relation holds:*

$$D^{\circ}_j B^{\nu}_i = \sum_x(-\Lambda^x_{ij} + 2\varepsilon_x X_x {}^*g_{ij})N^{\nu}_x \tag{1.1}$$

*Proof.* This relation is a consequence of I-(3.4) and I-(3.8).

In the derivation of the generalized Weingarten equations, a representation of the vector $D^{\circ}_j N^{\alpha}_x$, it is convenient to introduce the following abbreviations:

$$M^{\nu}_{jx} = D^{\circ}_j N^{\nu}_x \tag{1.2}$$

$$H_\gamma^{xy} = \varepsilon_y(\nabla_\gamma N_x^\alpha)N_\alpha^y \qquad 1.3$$

**Theorem 2** *The vector $H_\gamma^{xy}$ is skew-symmetric with respect to x and y. That is*

$$H_\gamma^{xy} = -H_\gamma^{yx}, \quad H_\gamma^{xx} = 0 \qquad 1.4$$

*Proof.* The second relation of (1.4) is a consequence of the first. In virtue of I-(2.16a) and I-(2.24), the first relation of (1.4) follows as in the following way:

$$\begin{aligned} 0 &= \nabla_\gamma(h_{\alpha\beta} N_x^\alpha N_y^\beta) \\ &= \nabla_\gamma(N_x^\alpha)N_\alpha^y + \nabla_\gamma(N_y^\beta)N_\beta^x \\ &= H_\gamma^{xy} + H_\gamma^{yx} \end{aligned}$$

In a sequence of the following four theorems, we derive the generalized Weingarten equations on $X_m$ of $^*g$-MEX$_n$.

**Theorem 3** *The vector $M_{jx}^\nu$ may be decomposed as*

$$M_{jx}^\nu = M_{jx}^i B_i^\nu + \sum_y M_{jx}^y N_y^\nu \qquad 1.5$$

*the first vector on the right being tangential to $X_m$ and the second vector normal to $X_m$. Here*

$$M_{jx}^i = M_{jx}^\alpha B_\alpha^i, \quad M_{jx}^y = M_{jx}^\alpha N_\alpha^y \qquad 1.6$$

*Proof.* This Theorem is an immediate consequence of I-Theorem 6.

**Theorem 4** *At points of $X_m$ of $^*g$-MEX$_n$, the induced vector $M_{jx}^i$ of $M_{jx}^\nu$ may be given by*

$$M_{jx}^i = \varepsilon_x {}^*h^{ih} \Lambda_{hj}^x - 2X^i {}^*k_{xj} \qquad 1.7a$$

*or equivalently*

$$M_{jx}^i = \varepsilon_x {}^*h^{ih} \Omega_{hj}^x + 2[X_x(\delta_j^i - {}^*k_j{}^i) - X^i {}^*k_{xj}] \qquad 1.7b$$

*Proof.* In order to prove the relation (1.7a), we first note that $M_{jx}^i$ is the induced tensor of $D_\gamma N_x^\alpha$ in virtue of (1.2) and (1.6). That is,

$$M_{jx}^i = (D_\gamma N_x^\alpha)B_\alpha^i B_j^\gamma \qquad 1.8$$

Making use of I-(2.25), I-(2.24a), and I-(3.9), we also note that

$$(\nabla_\gamma N_x^\alpha)B_\alpha^i B_j^\gamma = (\nabla_\gamma N_x^{\beta *} h_{\beta\varepsilon})({}^*h^{\varepsilon\alpha} B_\alpha^i)B_j^\gamma = \varepsilon_x {}^*h^{ih} \Lambda_{hj}^x \qquad (1.9)$$

Consequently, making use of I-(3.1), I-(2.20a), I-(2.23), and (1.9), the relation (1.7a) follows from (1.8) as in the following way:

$$\begin{aligned} M_{jx}^i &= \left[\partial_\gamma N_x^\alpha + ({}^*\{{}^{\,\alpha}_{\beta\gamma}\} + 2\delta_\beta^\alpha X_\gamma - 2\,{}^*g_{\beta\gamma} X^\alpha)N_x^\beta\right] B_\alpha^i B_j^\gamma \\ &= (\nabla_\gamma N_x^\alpha)B_\alpha^i B_j^\gamma - 2(X^\alpha B_\alpha^i)({}^*k_{\beta\gamma} N_x^\beta B_j^\gamma) \\ &= \varepsilon_x {}^*h^{ih} \Lambda_{hj}^x - 2X^i {}^*k_{xj} \end{aligned}$$

Substitution of I-(3.8) into (1.7a) gives (1.7b).

**Theorem 5** *At points of $X_m$ of $^*g$-MEX$_n$, the C-nonholonomic components $M^y_{jx}$ of $M^\nu_{jx}$ may be given by*

$$M^y_{jx} = \varepsilon_y H^{xy}_\gamma B^\gamma_j + 2(\delta^y_x X_j + \varepsilon_x {}^*k_j{}^x X^y) \tag{1.10}$$

*Proof.* In virtue of (1.2) and (1.6), we first note that $M^y_{jx}$ is *the induced vector* of $(D_\gamma N^\alpha_x)N^y_\alpha$. Hence, making use of I-(2.33), I-(2.23), I-(2.20a), and (1.3), the representation (1.10) follows as in the following way:

$$\begin{aligned} M^y_{jx} &= ((D_\gamma N^\alpha_x)N^y_\alpha B^\gamma_j \\ &= \left[\partial_\gamma N^\alpha_x + ({}^*\{{}^{\alpha}_{\beta\gamma}\} + 2\delta^\alpha_\beta X_\gamma - 2\,{}^*g_{\beta\gamma} X^\alpha)N^\beta_x\right] N^x_\alpha B^\gamma_j \\ &= (\nabla_\gamma N^\alpha_x)N^y_\alpha B^\gamma_j + 2(N^\alpha_x N^y_\alpha)(X_\gamma B^\gamma_j) + 2\varepsilon_x({}^*k_\gamma{}^\beta N^x_\beta B^\gamma_j)(X^\alpha N^y_\alpha) \\ &= \varepsilon_y H^{xy}_\gamma B^\gamma_j + 2(\delta^y_x X_j + \varepsilon_x {}^*k^j{}_x X^y) \end{aligned}$$

Now, we are ready to present the following representation of the generalized Weingarten equations by simply substituting (1.7a, b) and (1.10) into (1.5). We formally state

**Theorem 6 (The generalized Weingarten equations on $X_m$ of $^*g$-MEX$_n$)** *At points of $X_m$ of $^*g$-MEX$_n$, the following equations hold:*

$$\begin{aligned} D^\circ_j N^\nu_x &= (\varepsilon_x {}^*h^{ih} \Lambda^x_{hj} - 2X^i\, {}^*k_{xj})B^\nu_i + \\ &\quad + \Sigma_x \left[\varepsilon_y H^{xy}_\gamma B^\gamma_j + 2(\delta^y_x X_j + \varepsilon_x {}^*k_j{}^x X^y)\right] N^\nu_y \end{aligned} \tag{1.11a}$$

*or equivalently*

$$\begin{aligned} D^\circ_j N^\nu_x &= \left[\varepsilon_x {}^*h^{ih}\, \Omega^x_{hj} + 2(X_x(\delta^i_j - {}^*k_j{}^i) - X^i\, {}^*k_{xj})\right] B^\nu_i \\ &\quad + \Sigma_y \left[\varepsilon_y H^{xy}_\gamma B^\gamma_j + 2(\delta^y_x X_j + \varepsilon_x {}^*k_j{}^x X^y)\right] N^\nu_y \end{aligned} \tag{1.11b}$$

Our next considerations concern the derivation of the generalized Gauss-Codazzi equations for $X_m$ of $^*g$-MEX$_n$. For this purpose, we need the following curvature tensors:

$$R_{\omega\mu\lambda}{}^\nu = 2(\partial_{[\mu}\Gamma_{|\lambda|}{}^\nu{}_{\omega]} + \Gamma_\alpha{}^\nu{}_{[\mu}\Gamma_{|\lambda|}{}^\alpha{}_{\omega]}) \tag{1.12}$$

$$\bar{R}_{ijk}{}^h = 2(\partial_{[j}\Gamma_{|k|}{}^h{}_{i]} + \Gamma_p{}^h{}_{[j}\Gamma_{|k|}{}^p{}_{i]}) \tag{1.13}$$

$$H_{\omega\mu\lambda}{}^\nu = 2(\partial_{[\mu}\,{}^*\{{}^{\ \ \nu}_{\omega]\lambda}\} + {}^*\{{}^{\ \nu}_{\alpha[\mu}\}\,{}^*\{{}^{\ \ \alpha}_{\omega]\lambda}\}) \tag{1.14}$$

$$\bar{H}_{ijk}{}^h = 2(\partial_{[j}\,{}^*\{{}^{\ h}_{i]k}\} + {}^*\{{}^{\ h}_{p[j}\}\,{}^*\{{}^{\ p}_{i]k}\}) \tag{1.15}$$

where $\Gamma^\nu_{\lambda\mu}$ is the $^*g$-ME-connection of $^*g$-MEX$_n$ and $\Gamma^k_{ij}$ is the induced connection on $X_m$ of $^*g$-MEX$_n$. The tensors $R_{\omega\mu\lambda}{}^\nu$ and $\bar{R}_{ijk}{}^h$ are called *$^*g$-curvature tensors* of $^*g$-MEX$_n$ and $X_m$, respectively. It should be noted that $\bar{R}_{ijk}{}^h$ and $\bar{H}_{ijk}{}^h$ are not the induced tensors of $R^\nu_{\omega\mu\lambda}$ and $H^n_{\omega\mu\lambda}u$, respectively.

The following Theorem gives precise tensorial representations of $^*g$-curvature tensors.

**Theorem 7** *The* $^*g$*-curvature tensors* $R_{\omega\mu\lambda}{}^{\nu}$ *and* $\bar{R}_{ijk}{}^{h}$ *may be given by*

$$R_{\omega\mu\lambda}{}^{\nu} = H_{\omega\mu\lambda}{}^{\nu} + 4(\delta^{\nu}_{[\omega}\,\partial_{\mu]}X_{\lambda} - X^{\nu}\,\nabla_{[\omega}\,{}^*k_{\mu]\lambda} - {}^*g_{\lambda[\omega}\,\nabla_{\mu]}X^{\nu} + 2\,{}^*g_{\lambda[\omega}\,{}^*g_{|\alpha|\mu]}\,X^{\alpha}X^{\nu}) \tag{1.16}$$

$$\bar{R}_{ijk}{}^{h} = \bar{H}_{ijk}{}^{h} + 4(\delta^{h}_{[i}\,\partial_{j]}X_{k} - X^{h}\,\nabla_{[i}{}^*k_{j]k} - {}^*g_{k[i}\,\nabla_{j]}X^{h} + 2\,{}^*g_{k[i}\,{}^*g_{|p|j]}\,X^{p}X^{h}) \tag{1.17}$$

*Proof.* In virtue of I-(2.33), (1.12), and (1.14), the representation (1.16) may be proved as in the following way:

$$\begin{aligned} R_{\omega\mu\lambda}{}^{\nu} &= 2\partial_{[\mu}({}^*\{{}^{\ \nu}_{\omega]\lambda}\} + 2\delta^{\nu}_{\omega]}\,X_{\lambda} - 2\,{}^*g_{|\lambda|\omega]}\,X^{\nu}) + \\ &\quad +2({}^*\{{}^{\ \nu}_{\alpha[\mu}\} + 2\delta^{\nu}_{\alpha}\,X_{[\mu} - 2X^{\nu}\,{}^*g_{\alpha[\mu}) \times \\ &\quad \times({}^*\{{}^{\ \alpha}_{\omega]\lambda}\} + 2X_{\omega]}\,\delta^{\alpha}_{\lambda} - 2\,{}^*g_{\omega]\lambda}\,X^{\alpha}) \\ &= H_{\omega\mu\lambda}{}^{\nu} + 4\delta^{\nu}_{[\omega}\,\partial_{\mu]}X_{\lambda} - 4X^{\nu}(\partial_{[\mu}{}^*g_{|\lambda|\omega]} + {}^*g_{\alpha[\mu}\,{}^*\{{}^{\ \alpha}_{\omega]\lambda}\}) - \\ &\quad -4\,{}^*g_{\lambda[\omega}\,(\partial_{\mu]}X^{\nu} + {}^*\{{}^{\ \nu}_{\mu]\alpha}\}X^{\alpha}) + 8\,{}^*g_{\lambda[\omega}\,{}^*g_{|\alpha|\mu]}\,X^{\alpha}X^{\nu} \\ &= H_{\omega\mu\lambda}{}^{\nu} + 4(\delta^{\nu}_{[\omega}\,\partial_{\mu]}X_{\lambda} - X^{\nu}\,\nabla_{[\mu}{}^*g_{|\lambda|\omega]} - \\ &\quad -{}^*g_{\lambda[\omega}\,\nabla_{\mu]}X^{\nu} + 2\,{}^*g_{\lambda[\omega}\,{}^*g_{|\alpha|\mu]}\,X^{\alpha}X^{\nu}) \end{aligned}$$

Similarly, the representation (1.17) may be obtained from (1.13) in virtue of I-(3.11) and (1.15).

Now, we are ready to display the Gauss-Codazzi equations for $X_m$ of $^*g$-MEX$_n$.

**Theorem 8** *At points of* $^*g$*-MEX*$_n$*, the* $^*g$*-curvature tensors* $R_{\omega\mu\lambda}{}^{\nu}$ *of* $^*g$*-MEX*$_n$ *and* $\bar{R}_{ijk}{}^{h}$ *of* $X_m$ *are involved in the following equations:*

**(The generalized Gauss equations on $X_m$ of $^*g$-MEX$_n$ )**

$$\bar{R}_{ijk}{}^{h} = R_{\beta\gamma\varepsilon}{}^{\alpha}\,B^{\alpha}_{i}\,B^{\gamma}_{j}\,B^{\varepsilon}_{k}\,B^{h}_{\alpha} + 2\textstyle\sum_x \Omega^{x}_{k[i}[\Omega_{|p|j]}\,{}^*h^{hp}\,\varepsilon_x + +2(\delta^{h}_{j]} - {}^*k_{j]}{}^{h})X_x + 2\,{}^*k_{j]x}\,X^{h}] \tag{1.18}$$

**(The generalized Codazzi equations on $X_m$ of $^*g$-MEX$_n$ )**

$$2\nabla_{[k}\Omega^{x}_{|i|j]} = R_{\beta\gamma\varepsilon}{}^{\alpha}\,B^{\beta}_{k}\,B^{\gamma}_{j}\,B^{\varepsilon}_{i}\,N^{x}_{\alpha} + 2\textstyle\sum_y \Omega^{y}_{i[k}(B^{\gamma}_{j]}\,H^{yx}_{\gamma\varepsilon_x} + +2\,{}^*k_{j]y}\,X^{x}) + 4X^{h}\,{}^*g_{i[j}\,\Omega^{x}_{|h|k]}$$

*Proof.* We first note that I-(3.11) gives

$$S_{jk}{}^{h} = 2\delta^{h}_{[j}\,X_{k]} - 2{}^*k_{jk}\,X^{h} \tag{1.20}$$

In virtue of I-(3.6), I-(3.4), (1.12), (1.13) and (1.20), it follows that

$$\begin{aligned} 2D^\circ_{[k}(D^\circ_{j]}B^\alpha_i) &= 2\partial_{[k}(D^\circ_{j]}B^\alpha_i) + 2\Gamma^\alpha_{\beta\gamma}B^\gamma_{[k}(D^\circ_{j]}B^\beta_i) - \\ &\quad -2\Gamma^p_{i[k}(D^\circ_{j]}B^\alpha_p) - 2\Gamma^h_{[jk]}(D^\circ_h B^\alpha_i) \\ &= 2\partial_{[k}(B^\alpha_{j]i} + B^\gamma_{j]}B^\beta_i\Gamma^\alpha_{\beta\gamma} - \Gamma^h_{|i|j]}B^\alpha_h) + \\ &\quad +2\Gamma^\alpha_{\beta\gamma}B^\gamma_{[k}(B^\beta_{j]i} + B^\phi_{j]}B^\theta_i\Gamma^\beta_{\theta\phi} - \Gamma^h_{|i|j]}B^\beta_h) - \\ &\quad -2\Gamma^p_{i[k}(B^\alpha_{j]p} + B^\gamma_{j]}B^\beta_p\Gamma^\alpha_{\beta\gamma} - \Gamma^h_{|p|j]}B^\alpha_h) + \\ &\quad +4(\delta^h_{[j}X_{k]} - {}^*k_{jk}X^h)\textstyle\sum_x \Omega^x_{ih}N^\alpha_x \\ &= -R_{\varepsilon\gamma\beta}{}^\alpha B^\beta_i B^\gamma_j B^\varepsilon_k + \bar{R}_{kji}{}^h B^\alpha_h + \\ &\quad +4\textstyle\sum_x(\Omega^x_{i[j}X_{k]} - {}^*k_{jk}\Omega^x_{ih}X^h)N^\alpha_x \end{aligned} \tag{1.21}$$

On the other hand, the relation I-(3.4) gives

$$\begin{aligned} 2D^\circ_{[k}(D^\circ_{j]}B^\alpha_i) &= -2\textstyle\sum_x D^\circ_{[k}(\Omega^x_{|i|j]}N^\alpha_x) \\ &= 2\textstyle\sum_x(D^\circ_{[j}\Omega^x_{|i|k]})N^\alpha_x + 2\textstyle\sum_x\Omega^x_{i[k}(D^\circ_{j]}N^\alpha_x) \end{aligned} \tag{1.22}$$

In virtue of I-(2.33), the first term in the right-hand side of (1.22) may be written as

$$\begin{aligned} \text{First term} &= 2\textstyle\sum_x(\partial_{[j}\Omega^x_{|i|k]} - \Gamma^h_{i[j}\Omega^x_{|h|k]} - \Gamma^h_{[kj]}\Omega^x_{ih})N^\alpha_x \\ &= 2\textstyle\sum_x(\nabla_{[j}\Omega^x_{|i|k]} + 4\Omega^x_{i[j}X_{k]} + 2X^h\,{}^*g_{i[j}\Omega^x_{|h|k]} - \\ &\quad -2X^h\,{}^*k_{jk}\Omega^x_{ih})N^\alpha_x \end{aligned} \tag{1.23a}$$

Making use of the relations (1.11b) and

$$\varepsilon_x{}^*k_j{}^x = {}^*k_{jx}\text{[1]}$$

the second term in the right-hand side of (1.22) is equal to

$$\begin{aligned} \text{Second term} &= 2\textstyle\sum_x\Omega^x_{i[k}[\Omega^x_{|h|j]}\,{}^*h^{ph}\varepsilon_x + 2(\delta^p_{j]} - {}^*k_{j]}{}^p)X_x + \\ &\quad +2{}^*k_{j]x}X^p]B^\alpha_p + +4\textstyle\sum_x\Omega_{i[k}X_{j]}N^\alpha_x + \\ &\quad +2\textstyle\sum_{x,y}\Omega_{i[k}(B^\gamma_{j]}H^{xy}_{\gamma\varepsilon_y} + 2{}^*k_{j]x}X^y)N^\alpha_y \end{aligned} \tag{1.23b}$$

Substitution of (1.23) into (1.22) gives

$$\begin{aligned} 2D^\circ_{[k}D^\circ_{j]}B^\alpha_i &= 2\textstyle\sum_x\Omega^x_{i[k}(\Omega^x_{|h|j]}\,{}^*h^{ph}\varepsilon_x + 2(\delta^p_{j]} - {}^*k_{j]}{}^p)X_x + \\ &\quad +2\,{}^*k_{j]x}X^p)B^\alpha_p + 2\textstyle\sum_x(\nabla_{[j}\Omega^x_{|i|k]} + \\ &\quad +2X^h\,{}^*g_{i[j}\Omega^x_{|h|k]} + 2\Omega^x_{i[j}X_{k]} - 2X^h\,{}^*k_{jk}\Omega^x_{ih})N^\alpha_x + \\ &\quad +2\textstyle\sum_{x,y}\Omega_{i[k}(B^\gamma_{j]}H^{xy}_{\gamma\varepsilon_y} + 2\,{}^*k_{j]x}X^y)N^\alpha_y \end{aligned} \tag{1.24}$$

[1]The relation I-(2.24b) show that

$$\varepsilon_x{}^*k_j{}^x = \varepsilon_x{}^*k_\alpha{}^\beta B^\alpha_j N^x_\beta = {}^*k_{\alpha\beta}B^\alpha_j(\varepsilon_x N^x_\beta) = {}^*k_{\alpha\beta}B^\alpha_j N^\beta_x = {}^*k_{jx}$$

Consequently, comparing (1.21) and (1.24), we finally have

$$\begin{aligned}\bar{R}_{kji}{}^{h}\,B_h^{\alpha} \quad &= R_{\beta\gamma\varepsilon}{}^{\alpha}\,B_i^{\beta}\,B_j^{\gamma}\,B_k^{\varepsilon}+2\textstyle\sum_x \Omega_{i[k}^{x}[\Omega_{|h|j]}^{x}\;{}^{*}h^{ph}\varepsilon_x+\\ &\quad +2(\delta_{j]}^{p}-{}^{*}k_{j]}{}^{p})X_x+2\,{}^{*}k_{j]x}\,X^{p}]B_p^{\alpha}+\\ &\qquad +2\textstyle\sum_x(\nabla_{[j}\Omega_{|i|k]}^{x}+2X^{h}\,{}^{*}g_{i[j}\,\Omega_{|h|k]}^{x})N_x^{\alpha}+\\ &\qquad\quad +2\textstyle\sum_{x,y}\Omega_{i[k}^{x}(B_{j]}^{\gamma}\,H_{\gamma\varepsilon_y}^{xy}+2\,{}^{*}k_{j]x}\,X^{y})N_y^{\alpha}\end{aligned} \tag{1.25}$$

The generalized Gauss equations (1.18) follow by multiplying $B_{\alpha}^{q}$ to both sides of (1.25) and rearranging the indices suitably. Similarly, the generalized Codazzi equations (1.19) may be obtained by multiplying $N_{\alpha}^{z}$ to both sides of (1.25) and rearranging the indices. In the derivation of both equations, use of the relations I-(2.23) has been made.

## 2 The generalized fundamental equations on a hypersubmanifold $X_{n-1}$ of ${}^{*}g$-MEX$_n$

In this section, we investigate the fundamental equations on a hypersubmanifold of ${}^{*}g$-MEX$_n$. On a hypersubmanifold $X_{n-1}$ of ${}^{*}g$-MEX$_n$, the theory of submanifolds assumes a particularly simple and geometrically illuminating form. This simplication is mainly due to the fact that at each point of $X_{n-1}$ there exists a unique normal $N^{\nu}$.

In this case, we may take

$$\varepsilon_x = 1 \tag{2.1}$$

without the loss of generality. Therefore, quantities intoduced in I and the previous section take the following simpler forms and values:

$$N_x^{\alpha} = N_n^{\alpha} \overset{\text{def}}{=} N^{\alpha}, \quad N_{\alpha}^{x} = N_{\alpha}^{n} \overset{\text{def}}{=} N_{\alpha} \tag{2.2a}$$

$$X_x = X^x = X_{\alpha}\,N^{\alpha} \overset{\text{def}}{=} \chi \tag{2.2b}$$

$$\Omega_{ij}^{x} = \Omega_{ij}^{n} = (D_{\beta}N_{\alpha})B_i^{\alpha}\,B_j^{\beta} \overset{\text{def}}{=} \Omega_{ij} \tag{2.2c}$$

$$\Lambda_{ij}^{x} = \Lambda_{ij}^{n} = (\nabla_{\beta}N_{\alpha})B_i^{\alpha}\,B_j^{\beta} \overset{\text{def}}{=} \Lambda_{ij} \tag{2.2d}$$

$${}^{*}k_{ix} = {}^{*}k_i{}^{x} = {}^{*}k_{in} = {}^{*}k_{\alpha\beta}\,B_i^{\alpha}\,B^{\beta} \overset{\text{def}}{=} {}^{*}k_i \tag{2.2e}$$

$${}^{*}k_{xy} = {}^{*}k_x{}^{y} = {}^{*}k_{nn} = 0 \tag{2.2f}$$

$$H_{\gamma}^{xy} = H_{\gamma}^{nn} = 0 \tag{2.2g}$$

In virtue of (2.1) and (2.2), it may be easily shown that

$$\Omega_{ij} = \Lambda_{ij} - 2\chi\,{}^{*}g_{ij} \tag{2.3a}$$

$${}^{*}k_x{}^{i} = -{}^{*}k_i \tag{2.3b}$$

**Theorem 9** *At points of a hypersubmanifold $X_{n-1}$ of $^*g$-$MEX_n$, the following generalized fundamental equations hold:*

**(The generalized Gauss formulae on $X_{n-1}$ of $^*g$-MEX$_n$)**

$$D^\circ_j B^\nu_i = (-\Lambda_{ij} + 2\chi\, {}^*g_{ij})N^\nu \qquad 2.4a$$

**(The generalized Weingarten equations on $X_{n-1}$ of $^*g$-MEX$_n$)**

$$\begin{aligned} D^\circ_j N^\nu_x &= ({}^*h^{ih}\,\Lambda_{hj} + 2X^i\,{}^*k_j)B^\nu_i + 2(X_j + \chi\,{}^*k_j)N^\nu \\ &= [{}^*h^{ih}\,\Omega_{hj} + 2X^i\,{}^*k_j + 2\chi(\delta^i_j - {}^*k_j{}^i)]B^\nu_i + \\ &\qquad\qquad + 2(X_j + \chi\,{}^*k_j)N^\nu \end{aligned} \qquad 2.4b$$

**(The generalized Gauss equations on $X_{n-1}$ of $^*g$-MEX$_n$)**

$$\begin{aligned} \bar{R}_{ijk}{}^h &= R_{\beta\gamma\varepsilon}{}^\alpha\, B^\beta_i\, B^\gamma_j\, B^\varepsilon_k\, B^h_\alpha + 2\Omega_{k[i}(\Omega_{|p|j]}\,{}^*h^{hp} + \\ &\qquad + 2\chi(\delta^h_{j]} - {}^*k_{j]}{}^h + 2\,{}^*k_{j]}\,X^h) \end{aligned} \qquad 2.4c$$

**(The generalized Codazzi equations on $X_{n-1}$ of $^*g$-MEX$_n$)**

$$2\nabla_{[k}\Omega_{|i|j]} = R_{\beta\gamma\varepsilon}{}^\alpha\, B^\beta_k\, B^\gamma_j\, B^\varepsilon_i\, N_\alpha + 4\chi\,\Omega_{i[k}{}^*k_{j]} + 4\,{}^*g_{i[j}\,\Omega_{|h|k]}X^h \qquad 2.4d)$$

*Proof.* In virtue of (2.1), (2.2), and (2.3), the identities (2.4) follow from (1.1), (1.11), (1.18), and (1.19), respectively.

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